Homework 8
Introduction to Quantum Information Processing

Exercise 1 *Dynamics of 1-qubit density matrix*

In class we showed that the general form of a 1-qubit density matrix is

$$\rho = \frac{1}{2}(I + \vec{a} \cdot \vec{\sigma})$$

where $\vec{a} = a_x, a_y, a_z)$ is a vector in the unit three dimensional ball (the Bloch *ball*) $\|\vec{a}\| \leq 1$ and $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ are the three usual Pauli matrices.

In a later class we will study the dynamics of qubits driven by magnetic fields. This is an important subject as it at the basis of NMR but also modern quantum information processing technologies. We will see that the time-evolution for a single qubit in a static plus rotating magnetic field in the rotating frame is given by the Hamiltonian

$$H = \frac{\hbar\delta}{2}\sigma_z - \frac{\hbar\omega_1}{2}\sigma_x$$

Here $\omega_1 \propto$ is proportional to the strength of the rotating field and and $\delta = \omega - \omega_0$ is the so-called detuning between the Larmor and rotating magnetic field frequencies.

For the moment you dont have to know where this comes from (as we will see this in later classes). We recall that for a pure state the time evolution is given by:

$$|\Psi(t)\rangle = U(t)|\Psi(0)\rangle = e^{-\frac{i}{\hbar}tH}|\Psi(0)\rangle$$

From the general definition of a density matrix $\rho = \sum_{i=1}^{K} p_i|\varphi_i\rangle\langle\varphi_i|$ you can infer that the time-evolution is given by (say why?)

$$\rho_t = U_t \rho U_t^\dagger$$

with U_t the evolution operator.

a) Show that the density matrix at time t is of the form

$$\rho_t = \frac{1}{2}(I + \vec{a}(t) \cdot \vec{\sigma})$$

and compute the vector $\vec{a}(t)$.

Hint: We recall the following Euler-type formula seen in previous homeworks:

$$\exp(i\alpha\vec{\sigma} \cdot \vec{n}) = (\cos\alpha)I + i(\sin\alpha)\vec{\sigma} \cdot \vec{n}$$

where α is any angle and $\vec{n}$ is a unit vector.

b) Check that $\|\vec{a}(t)\| = \|\vec{a}\|$. So the vector $\vec{a}(t)$ evolves on a sphere (inside the Bloch ball) of radius given by the initial vector.

c) Find a simple proof of the last statement without ever computing $\vec{a}(t)$.

Exercise 2 *Density matrix: a decoherence model*

In the following, we will study a model of decoherence of one qubit interacting with the environment. The whole system is defined in the hilbert space $\mathcal{H} = \mathcal{H}_\mathcal{E} \otimes \mathcal{H}_b$ where $\mathcal{H}_\mathcal{E}$ is the Hilbert space describing the possible states of the environment and $\mathcal{H}_b = \mathbb{C}^2$ is the Hilbert space describing the possible states of the qubit.

Let $|\phi_0\rangle = \alpha |0\rangle + \beta |1\rangle \in \mathcal{H}_b$ be the initial state of the qubit and $|\mathcal{E}\rangle \in \mathcal{H}_b$ that of the environment (or sometimes called *heat-bath*). Let $(|i\rangle)_{i\geq 1} \in \mathcal{H}_\mathcal{E}$ be an "infinite" orthonormal basis of the environment $\mathcal{H}_\mathcal{E}$. We define the evolution operator $U = \sum_{i=1}^{+\infty} |i\rangle\langle i| \otimes \mathcal{D}(\theta_i)$ for some distinct angles $\theta_i \in \mathbb{R}$, and the dephasing operator: $\mathcal{D}(\theta_i) = |0\rangle\langle 0| + e^{i\theta_i} |1\rangle\langle 1|$.

If the environment makes a transition from state $|\mathcal{E}\rangle$ to $|i\rangle$, we let $\mu(\theta_i) = P(|\mathcal{E}\rangle \to |i\rangle)$ the probability of such a transition. Note that $\langle i|\mathcal{E}\rangle = e^{i \arg\langle i|\mathcal{E}\rangle}\sqrt{\mu(\theta_i)}$.

a) What is the initial global state $|\psi_0\rangle$ of the whole system?

b) Show that U is a unitary operator (describe your steps).

c) The state of the system evolves (in discrete time steps say) with a power $n \in \mathbb{N}$ of the operator U as $|\psi_n\rangle = U^n |\psi_0\rangle$. Show that $\mathcal{D}(\theta_i)^n = \mathcal{D}(n\theta_i)$ and deduce that

$$|\psi_n\rangle = \sum_{i=1}^{+\infty} e^{i \arg\langle i|\mathcal{E}\rangle}\sqrt{\mu(\theta_i)}\,|i\rangle \otimes (\mathcal{D}(n\theta_i)\,|\phi_0\rangle)$$

d) Now let's consider the density matrix of the qubit itself: $\rho_n = \mathrm{tr}_{\mathcal{H}_\mathcal{E}}[|\psi_n\rangle\langle\psi_n|]$. First, using only the result of question (a), show that we have initially:

$$\rho_0 = \begin{pmatrix} |\alpha|^2 & \alpha\beta^* \\ \alpha^*\beta & |\beta|^2 \end{pmatrix}$$

e) For any angle $\theta \in \mathbb{R}$, show that we have:

$$\mathcal{D}(\theta)\rho_0\mathcal{D}(\theta)^\dagger = \begin{pmatrix} |\alpha|^2 & \alpha\beta^* e^{-i\theta} \\ \alpha^*\beta e^{i\theta} & |\beta|^2 \end{pmatrix}$$

f) Now let's consider $\hat{\theta}$ a random variable taking values $\theta_i in \mathbb{R}$ with probability partial $\mu(\theta_i)$. Use the result of question (c) and (e) to show that the density matrix of the qubit coincide with the following expression:

$$\rho_n = \begin{pmatrix} |\alpha|^2 & \alpha\beta^*\mathbb{E}[e^{-in\hat{\theta}}] \\ \alpha^*\beta\mathbb{E}[e^{in\hat{\theta}}] & |\beta|^2 \end{pmatrix}$$

g) Now say that the values θ_i form a quasicontinuum and that μ is the PDF of a gaussian distribution of mean 0 and variance σ^2. Show that the density matrix of the qubit evolves as:

$$\rho_n = \begin{pmatrix} |\alpha|^2 & \alpha\beta^* e^{-\frac{1}{2}\sigma^2 n^2} \\ \alpha^*\beta e^{-\frac{1}{2}\sigma^2 n^2} & |\beta|^2 \end{pmatrix}$$

Calculate $\rho_\infty = \lim_{n\to\infty} \rho_n$.

h) How does the von Neumann entropy of the qubit evolve from initial time $n = 0$ to final time $n \to +\infty$?