Homework 7
Introduction to Quantum Information Processing

Exercise 1 *Partial density matrices: illustrative examples*

- Compute the partial density matrices corresponding to the following pure state:

$$|\Psi\rangle_{AB} = \epsilon|00\rangle + \sqrt{1-\epsilon^2}|11\rangle$$

with $0 \leq \epsilon \leq 1$. Discuss what happens as ϵ varies and draw the representation in the Bloch ball.

- Now compute the two partial density matrices associated to:

$$|\Phi\rangle_{AB} = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$$

What is the rank of the partial density matrices ? Why ? Draw the representation on the Bloch ball.

- Now we change one sign in the above:

$$|\Phi'\rangle_{AB} = \frac{1}{2}(|00\rangle + |01\rangle - |10\rangle + |11\rangle)$$

Again compute the two partial reduced density matrices and their rank. Is the rank different or same as above ? Why ?

- Now consider the three qubit state:

$$|W\rangle_{ABC} = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)$$

Consider the "bipartition" $A \mid BC$; compute the partial density matrices ρ_A and ρ_{BC}. Do the calculations with the Dirac notation. Then write also the final results in tableau (array) representation.

Same question for the other bipartitions $B \mid AC$ and $C \mid AB$ (hint: no extra calculation is necessary).

Exercise 2 *Tableau computation of partial density matrices*

Sometimes its useful to compute partial traces directly from the tableau (array) representation of the (full) matrix instead of using the Dirac notation (but note that beyond 4×4 matrices this tends to be a bit tedious and Dirac notation is better). Take a 4×4 matrix index as follows $M_{ij;kl}$ with $i, j, k, l \in \{0, 1\}$ (as in class) with lines and columns ordered according to 00, 01, 10, 11. The two partial traces are:

$$M^{(1)}_{jl} = \sum_{m=0,1} M_{mj;ml} \quad \text{and} \quad M^{(1)}_{ik} = \sum_{m=0,1} M_{im;km}$$

What do these formulae amount to do when you are given the tableau (array):

$$\begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{pmatrix}$$

Exercise 3 *The difference between a Bell state and a statistical mixture of* $|00\rangle$ *and* $|11\rangle$

We consider a source that distributes to A and B either:

- An *EPR pair* in the perfect Bell state $|B_{00}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$;
- Or distributes a pair of qubits in a *statistical mixture* of states $|00\rangle$, $|11\rangle$ with uniform probabilities 1/2.

This exercise illustrates in many ways that the two kind of situations are completely different.

a) Write down the density matrix ρ_{Bell} associated to the Bell state in Dirac notation as well as in matrix array form (in the computational basis).

b) Write down the density matrix ρ_{stat} associated to the statistical mixture above in Dirac notation as well as in matrix array form (in the computational basis).

c) In a Bell/CSHS experiment one measures the observable

$$\mathcal{B} = A \otimes B + A \otimes B' - A' \otimes B + A' \otimes B'$$

What is the theoretical average if the state when the state is ρ_{Bell} ? (Use results proven in class and no need to reproduce calculations). And now compute the theoretical average if the state is ρ_{stat}. What are the values of the of these two averages for the optimal CSHS-angles $\alpha = 0$, $\alpha' = -\frac{\pi}{4}$, $\beta = \frac{\pi}{8}$, $\beta' = -\frac{\pi}{8}$?