
Homework 5
Introduction to Quantum Information Processing

Exercise 1 *Corrupted dense coding*

Alice and Bob share an entangled pair in the state :

$$|\Psi\rangle = (1 + \delta^2)^{-1/2} \left\{ |B_{00}\rangle + \delta e^{i\gamma} |01\rangle \right\}$$

where $\delta \in \mathbb{R}_+$ and $\gamma \in [0, 2\pi]$. Here $|B_{00}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ is the perfect Bell state and we view the additional term proportional to δ as a "corruption". Alice wants to transmit two classical bits of information to Bob. They do not know that the entangled pair is corrupted and they use the *usual dense coding protocol* (seen in class).

- (a) Suppose Alice sends message (00). Calculate the probabilities that at the end of the protocol Bob gets $P(00)$, $P(01)$, $P(10)$, $P(11)$.
- (b) Suppose Alice sends message (10). Calculate first the global state received by Bob and the 4 probabilities above.

Exercise 2 *Corrupted teleportation*

Consider the same state as above

$$|\Psi\rangle = (1 + \delta^2)^{-1/2} \left\{ |B_{00}\rangle + \delta e^{i\gamma} |01\rangle \right\}$$

shared between Alice and Bob and suppose Alice has the extra state $|\varphi\rangle = \alpha|0\rangle + \beta|1\rangle$. We want to analyze the teleportation protocol under the corrupted entanglement link.

- (a) Alice does a measurement in the *perfect* Bell basis in her lab. Compute the possible outcomes for the global state shared by Alice and Bob and the respective probabilities.
- (b) Describe the next steps of the protocol and explain what are the teleported states that Bob gets when the protocol is completed. In particular compare the teleported states with $|\varphi\rangle$.

Exercise 3 *An entanglement criterion for 2 qubits*

The general state of two qubits is of the form

$$|\Psi\rangle = a_{00}|0\rangle \otimes |0\rangle + a_{01}|0\rangle \otimes |1\rangle + a_{10}|1\rangle \otimes |0\rangle + a_{11}|1\rangle \otimes |1\rangle,$$

(a) Show that $|\Psi\rangle$ is a product state *if and only if* $\det A = 0$, where A is the matrix

$$A = \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix}.$$

(b) Use this criterion to determine when the state (with $\delta \geq 0$, $\epsilon \geq 0$ reals and $\gamma \in [0, 2\pi]$)

$$|\Psi_1\rangle = \frac{1}{\sqrt{1 + \delta^2 + \epsilon^2}} (|B_{00}\rangle + \delta e^{i\gamma}|1\rangle \otimes |0\rangle + \epsilon|0\rangle \otimes |1\rangle)$$

is entangled.

Exercise 4 $|W\rangle$ state

Consider the following state called W in quantum information,

$$|W\rangle = \frac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle)$$

Show that the three qubits are "totally" entangled in the sense :

- (a) It is impossible to write the state as a product of three one-qubit states :
 $|W\rangle \neq |\psi_1\rangle \otimes |\psi_2\rangle \otimes |\psi_3\rangle$ avec $|\psi_i\rangle \in \mathbb{C}^2$.
- (b) It is impossible to write the state as a product of a one-qubit state and a two-qubit state (which might itself be entangled) : $|W\rangle \neq |\psi_1\rangle \otimes |\psi_{23}\rangle$ with $|\psi_1\rangle \in \mathbb{C}^2$ et $|\psi_{23}\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^2$.