



Differential Geometry II - Smooth Manifolds

Winter Term 2026/2027

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Exercise Sheet 3 – Solutions

Exercise 1: Let M be a smooth manifold, and suppose we have two open covers $\mathcal{V}, \mathcal{U}$ with $\mathcal{V}$ refining $\mathcal{U}$. Given a partition of unity subordinate to $\mathcal{V}$, construct a partition of unity subordinate to $\mathcal{U}$.

Solution: Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ and $\mathcal{V} = \{V_\beta\}_{\beta \in B}$ be the open covers in the statement, so that $\mathcal{V}$ refines $\mathcal{U}$. By definition of refinement, we may choose an indexing map $\pi : B \rightarrow A$ such that for every $\beta \in B$, $V_\beta \subseteq U_{\pi(\beta)}$. Now let $\{\psi_\beta\}_{\beta \in B}$ be a smooth partition of unity subordinate to $\mathcal{V}$. We construct a smooth partition of unity $\{\varphi_\alpha\}_{\alpha \in A}$ subordinate to $\mathcal{U}$ by

$$\varphi_\alpha(x) := \sum_{\beta \in \pi^{-1}(\alpha)} \psi_\beta(x).$$

Note that φ_α is a well-defined, nonnegative, and smooth function on U_α : since $\{\psi_\beta\}_{\beta \in B}$ is a partition of unity, the family of supports $\{\text{supp}(\psi_\beta)\}_{\beta \in B}$ is locally finite, so the sum above defining φ_α is a finite sum of nonnegative smooth functions. Furthermore, for any $x \in M$,

$$\sum_{\alpha \in A} \varphi_\alpha(x) = \sum_{\alpha \in A} \sum_{\beta \in \pi^{-1}(\alpha)} \psi_\beta(x) = \sum_{\beta \in B} \psi_\beta(x) = 1$$

so we have indeed constructed a partition of unity.

Exercise 2 (Exercise 2.27): Recall the statement of the extension lemma (Lemma 2.26). Suppose M is a smooth manifold with or without boundary, $A \subseteq M$ is a closed subset, and $f : A \rightarrow \mathbb{R}^k$ is a smooth function. For any open subset U containing A , there exists a smooth function $\tilde{f} : M \rightarrow \mathbb{R}^k$ such that $\tilde{f}|_A = f$ and $\text{supp } \tilde{f} \subseteq U$. Give a counterexample to show that the conclusion can be false if A is not closed.

Solution: Consider the open interval $A = (0, 1)$ supporting the smooth function $f(x) = 1/x$. Then for any $\varepsilon > 0$, the open set $U = (-\varepsilon, 1)$ contains A but there is no smooth extension of f to U as continuity forces its value at $0 \in U$ to be ill-defined.

Exercise 3 (Lemma 1.13/Exercise 1.14):

Suppose $\mathcal{X}$ is a locally finite collection of subsets of a topological space M . Show that

1. The collection $\{\overline{X} : X \in \mathcal{X}\}$ is also locally finite.
2. $\overline{\bigcup_{X \in \mathcal{X}} X} = \bigcup_{X \in \mathcal{X}} \overline{X}$.

Solution: Let $p \in M$ be an arbitrary point. By assumption, there is an open neighborhood $U \ni p$ such that the set

$$I = \{X \in \mathcal{X} : X \cap U \neq \emptyset\}$$

is finite. Note that if $\overline{X} \cap U \neq \emptyset$, then we can pick some point x in the intersection and choose an open neighborhood $W \subset U$ containing x . This set W , being open and containing a point of $\overline{X}$, must contain a point y of X itself. In other words, the set of $\overline{X}$'s intersecting U is contained in the finite set I , so $\{\overline{X} : X \in \mathcal{X}\}$ is locally finite.

For the second part, we will show inclusion in both directions. To show $\supseteq$, note that for any $Y \in \mathcal{X}$ we have $\overline{Y} \subset \overline{\bigcup_{X \in \mathcal{X}} X}$. Taking a union over Y of this inclusion yields $\bigcup_{Y \in \mathcal{X}} \overline{Y} \subseteq \overline{\bigcup_{X \in \mathcal{X}} X}$, without the use of local finiteness. For the reverse inclusion, we consider the contrapositive and suppose there is a point x not in $\bigcup_{X \in \mathcal{X}} \overline{X}$. By local finiteness, we can find an open neighborhood U of x meeting only finitely many of the X 's, say $X_1, \dots, X_k$. Shrink U by deleting the closures of each $X_1, \dots, X_k$ to obtain

$$V = U \setminus \bigcup_{i=1}^k \overline{X_i},$$

which is still an open neighborhood of x , which does not meet any of the $X \in \mathcal{X}$. Thus, V is an open neighborhood of x disjoint from $\bigcup_{X \in \mathcal{X}} X$, demonstrating that x is not in the closure $\overline{\bigcup_{X \in \mathcal{X}} X}$.

Exercise 4 (Problem 2-6): Let $P : \mathbb{R}^{n+1} \setminus \{0\} \rightarrow \mathbb{R}^{k+1} \setminus \{0\}$ be a smooth function, and suppose that for some $d \in \mathbb{Z}$, $P(\lambda x) = \lambda^d P(x)$ for all $\lambda \in \mathbb{R} \setminus \{0\}$ and $x \in \mathbb{R}^{n+1} \setminus \{0\}$. (Such a function is said to be *homogeneous of degree d* .) Show that the map $\tilde{P} : \mathbb{RP}^n \rightarrow \mathbb{RP}^k$ defined by $\tilde{P}([x]) = [P(x)]$ is well defined and smooth.

Solution: First we check that $\tilde{P}$ is well-defined. Given two points $x, y \in \mathbb{R}^{n+1}$ representing the same point in $\mathbb{RP}^n$, we have some scaling factor λ so that $\lambda x = y$. Thus by homogeneity of the function P ,

$$P(y) = P(\lambda x) = \lambda^d P(x)$$

which define equivalent points in $\mathbb{RP}^k$.

For smoothness, we will check on the canonical coordinate charts of $\mathbb{RP}^n$ and $\mathbb{RP}^k$. Recall that for each $i \in \{0, \dots, n\}$ (resp. for each $j \in \{0, \dots, k\}$) we can consider an open subset $\mathbb{R}^n \simeq U_i \subset \mathbb{RP}^n$ (resp. $\mathbb{R}^k \simeq V_j \subset \mathbb{RP}^k$) where the i th homogeneous coordinate (resp. j th homogeneous coordinate) is nonvanishing.

Write $P = (P_0, \dots, P_k)$ be the components of the function P , and let $x \in \mathbb{RP}^n$ be an arbitrary point, belonging to some chart U_i . We pick some j so that the j th coordinate

of $P(x)$, that is $P_j(x)$, is nonvanishing. Then in the local chart U_i and V_j , we can express P as the function $\mathbb{R}^n \rightarrow \mathbb{R}^k$ given by

$$(u_1, \dots, u_n) \mapsto \left(\frac{P_0(u')}{P_j(u')}, \dots, \frac{P_{j-1}(u')}{P_j(u')}, \frac{P_{j+1}(u')}{P_j(u')}, \dots, \frac{P_k(u')}{P_j(u')} \right)$$

where $u' = (u_0, \dots, 1, \dots, u_n)$ with an extra coordinate 1 in the i th coordinate. Since the preceding formula is a nonsingular rational polynomial expression in each coordinate, we have verified that P is smooth.

Exercise 5 (Problem 2-10): For any topological space M , let $C(M)$ denote the algebra of continuous functions $f : M \rightarrow \mathbb{R}$. Given a continuous map $F : M \rightarrow N$, define $F^* : C(N) \rightarrow C(M)$ by $F^*(f) = f \circ F$.

1. Show that F^* is a linear map.
2. Suppose M and N are smooth manifolds. Show that $F : M \rightarrow N$ is smooth if and only if $F^*(C^\infty(N)) \subseteq C^\infty(M)$.
3. Suppose $F : M \rightarrow N$ is a homeomorphism between smooth manifolds. Show that it is a diffeomorphism if and only if F^* restricts to an isomorphism from $C^\infty(N)$ to $C^\infty(M)$.

[Remark: this result show that in a certain sense, the entire smooth structure of M is encoded in the subset $C^\infty(M) \subset C(M)$. In fact, some authors define a smooth structure on a topological manifold M to be a subalgebra of $C(M)$ with certain properties; see, e.g. Nestruev's *Smooth manifolds and observables* (2003).]

Solution: For 1, consider a linear combination $af + bg$ of continuous functions, where $a, b \in \mathbb{R}$ and $f, g \in C(N)$. Then

$$F^*(af + bg) = (af + bg) \circ F = a(f \circ F) + b(g \circ F) = aF^*(f) + bF^*(g).$$

For 2, we consider a smooth map F . Since composition of smooth maps of smooth, we see that for every $f \in C^\infty(N)$, the map $F^*(f) = f \circ F \in C^\infty(M)$ is smooth, hence $F^*(C^\infty(N)) \subseteq C^\infty(M)$. Conversely, we assume the latter and show that F is smooth. To do this, it suffices to check locally at every point. For a point $x \in M$ we pick some Euclidean local chart (V, ψ) on N of $F(x)$. Let $y_1, \dots, y_n$ be the coordinate functions on V . Shrinking V if necessary, we may extend $y_1, \dots, y_n$ to global smooth functions $\tilde{y}_1, \dots, \tilde{y}_n$ on all of N using smooth bump functions, and by assumption,

$$F^*(\tilde{y}_i) = \tilde{y}_i \circ F \text{ is smooth for } i = 1, \dots, n.$$

On a Euclidean chart (U, φ) around x contained in $F^{-1}(V)$, we see therefore that $(\psi \circ F \circ \varphi^{-1})_i = (\tilde{y}_i \circ F) \circ \varphi^{-1}$ is a smooth function for each coordinate.

For 3, supposing that F is a diffeomorphism, it has a smooth inverse F^{-1} . By 2, we conclude that F^* induces an isomorphism on smooth functions, with inverse given by $(F^{-1})^*$. Conversely, suppose F^* induces an isomorphism on smooth functions. By 2, we

know that F is a smooth function. Now let F^{-1} be the inverse of F as a homeomorphism, and let $(F^*)^{-1}$ be the inverse of the isomorphism F^* . We may verify directly that in fact the pullback map $(F^{-1})^*$ is equal to $(F^*)^{-1}$ on $C^\infty(M)$. Thus by 2 again, F^{-1} is smooth, and is a smooth inverse to F . Hence F is a diffeomorphism.