



Differential Geometry II - Smooth Manifolds

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Exercise Sheet 3

Exercise 1: Let M be a smooth manifold, and suppose we have two open covers $\mathcal{V}, \mathcal{U}$ with $\mathcal{V}$ refining $\mathcal{U}$. Given a partition of unity subordinate to $\mathcal{V}$, construct a partition of unity subordinate to $\mathcal{U}$.

Exercise 2 (Exercise 2.27): Recall the statement of the extension lemma (Lemma 2.26). Suppose M is a smooth manifold with or without boundary, $A \subseteq M$ is a closed subset, and $f : A \rightarrow \mathbb{R}^k$ is a smooth function. For any open subset U containing A , there exists a smooth function $\tilde{f} : M \rightarrow \mathbb{R}^k$ such that $\tilde{f}|_A = f$ and $\text{supp } \tilde{f} \subseteq U$. Give a counterexample to show that the conclusion can be false if A is not closed.

Exercise 3 (Lemma 1.13/Exercise 1.14):

Suppose $\mathcal{X}$ is a locally finite collection of subsets of a topological space M . Show that

1. The collection $\{\overline{X} : X \in \mathcal{X}\}$ is also locally finite.
2. $\overline{\bigcup_{X \in \mathcal{X}} X} = \bigcup_{X \in \mathcal{X}} \overline{X}$.

Exercise 4 (Problem 2-6): Let $P : \mathbb{R}^{n+1} \setminus \{0\} \rightarrow \mathbb{R}^{k+1} \setminus \{0\}$ be a smooth function, and suppose that for some $d \in \mathbb{Z}$, $P(\lambda x) = \lambda^d P(x)$ for all $\lambda \in \mathbb{R} \setminus \{0\}$ and $x \in \mathbb{R}^{n+1} \setminus \{0\}$. (Such a function is said to be *homogeneous of degree d* .) Show that the map $\tilde{P} : \mathbb{RP}^n \rightarrow \mathbb{RP}^k$ defined by $\tilde{P}([x]) = [P(x)]$ is well defined and smooth.

Exercise 5 (Problem 2-10): For any topological space M , let $C(M)$ denote the algebra of continuous functions $f : M \rightarrow \mathbb{R}$. Given a continuous map $F : M \rightarrow N$, define $F^* : C(N) \rightarrow C(M)$ by $F^*(f) = f \circ F$.

1. Show that F^* is a linear map.
2. Suppose M and N are smooth manifolds. Show that $F : M \rightarrow N$ is smooth if and only if $F^*(C^\infty(N)) \subseteq C^\infty(M)$.

3. Suppose $F : M \rightarrow N$ is a homeomorphism between smooth manifolds. Show that it is a diffeomorphism if and only if F^* restricts to an isomorphism from $C^\infty(N)$ to $C^\infty(M)$.

[Remark: this result show that in a certain sense, the entire smooth structure of M is encoded in the subset $C^\infty(M) \subset C(M)$. In fact, some authors define a smooth structure on a topological manifold M to be a subalgebra of $C(M)$ with certain properties; see, e.g. Nestruiev's *Smooth manifolds and observables* (2003).]