



Differential Geometry II - Smooth Manifolds

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Exercise Sheet 2 – Solutions

Exercise 1 (*The real projective n -space*):

Denote by $\mathbb{RP}^n$ the real projective n -space, where $n \geq 1$. For each $i \in \{0, \dots, n\}$ consider the chart (U_i, φ_i) for $\mathbb{RP}^n$ described in Appendix A, where

$$U_i := \{[x_0 : \dots : x_n] \in \mathbb{RP}^n \mid x_i \neq 0\} \subseteq \mathbb{RP}^n$$

and

$$\begin{aligned} \varphi_i : U_i &\rightarrow \mathbb{R}^n \\ [x_0 : \dots : x_n] &\mapsto \left(\frac{x_0}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \dots, \frac{x_n}{x_i} \right). \end{aligned}$$

Show that the collection

$$\mathcal{A}_{\mathbb{RP}^n} := \{(U_i, \varphi_i)\}_{i=0}^n$$

is a smooth atlas for $\mathbb{RP}^n$.

Solution: First we observe that the U_i 's indeed cover $\mathbb{RP}^n$, since every point in $\mathbb{RP}^n$ can be given by its homogeneous coordinates with at least one nonzero entry. The U_i 's are mapped homeomorphically to $\mathbb{R}^n$ by φ_i , with inverse

$$\varphi_i^{-1} : (a_0, \dots, \widehat{a_i}, \dots, a_n) \mapsto [a_0, \dots, a_{i-1}, 1, a_i, \dots, a_n].$$

It remains to show that the transition maps $\varphi_j \circ \varphi_i^{-1}$ are smooth. Note that the domain of this transition map is exactly

$$\{(a_0, \dots, \widehat{a_i}, \dots, a_n) \in \mathbb{R}^n : a_j \neq 0\}.$$

Computing $\varphi_j \circ \varphi_i^{-1}$ directly, we obtain

$$(a_0, \dots, \widehat{a_i}, \dots, a_n) \mapsto \left(\frac{a_0}{a_j}, \dots, \frac{a_{i-1}}{a_j}, \frac{1}{a_j}, \frac{a_{i+1}}{a_j}, \dots, \frac{a_n}{a_j} \right) \quad (\text{omitting the } j\text{th coordinate})$$

whenever $i \neq j$, and simply the identity map when $i = j$, so we see that all transition maps are smooth.

Exercise 2 (*Equivalent characterizations of smoothness*): Let M and N be smooth manifolds and let $F: M \rightarrow N$ be a map. Show that F is smooth if and only if either of the following conditions is satisfied:

- (a) For every $p \in M$ there exist smooth charts (U, φ) containing p and (V, ψ) containing $F(p)$ such that $U \cap F^{-1}(V)$ is open in M and the composite map $\psi \circ F \circ \varphi^{-1}$ is smooth from $\varphi(U \cap F^{-1}(V))$ to $\psi(V)$.
- (b) F is continuous and there exist smooth atlases $\{(U_\alpha, \varphi_\alpha)\}$ and $\{(V_\beta, \psi_\beta)\}$ for M and N , respectively, such that for each α and β , $\psi_\beta \circ F \circ \varphi_\alpha^{-1}$ is a smooth map from $\varphi_\alpha(U_\alpha \cap F^{-1}(V_\beta))$ to $\psi_\beta(V_\beta)$.

Solution:

(a) We prove the two directions:

- ($\Rightarrow$) Suppose F is smooth and let $p \in M$. Then there exist smooth charts (U, φ) containing p and (V, ψ) containing $F(p)$ such that $F(U) \subseteq V$ and such that $\psi \circ F \circ \varphi^{-1}$ is smooth from $\varphi(U)$ to $\psi(V)$. Then $U \cap F^{-1}(V) = U$, and thus the charts (U, φ) and (V, ψ) satisfy the conditions specified in (a).
- ($\Leftarrow$) Assume that (a) holds and let $p \in M$. Let (U, φ) resp. (V, ψ) be the charts given by (a). Then, setting $U' := U \cap F^{-1}(V)$ and $\varphi' := \varphi|_{U'}$, we infer that (U', φ') is a smooth chart containing p such that $F(U') \subseteq V$ and such that $\psi \circ F \circ (\varphi')^{-1}: \varphi'(U') \rightarrow \psi(V)$ is smooth.

(b) We prove the two directions:

- ($\Rightarrow$) Suppose that F is smooth. By *Proposition 2.5*, F is continuous. Now, let (U, φ) and (V, ψ) be any smooth chart for M and N , respectively. We would like to show that the map $\hat{F} := \psi \circ F \circ \varphi^{-1}$ is smooth from $\varphi(U \cap F^{-1}(V))$ to $\psi(V)$. If $U \cap F^{-1}(V)$ is empty, then there is nothing to prove. Otherwise, let $p \in U \cap F^{-1}(V)$ be arbitrary. By smoothness of F , there exist charts (W, η) containing p and (Z, θ) containing $F(p)$ such that $F(W) \subseteq Z$ and such that $\theta \circ F \circ \eta^{-1}$ is smooth from $\eta(W)$ to $\theta(Z)$. In particular, we have

$$\hat{F} = \psi \circ (\theta^{-1} \circ \theta) \circ F \circ (\eta^{-1} \circ \eta) \circ \varphi^{-1} = (\psi \circ \theta^{-1}) \circ (\theta \circ F \circ \eta^{-1}) \circ (\eta \circ \varphi^{-1})$$

on the open neighborhood $\varphi(U \cap W \cap F^{-1}(V))$ containing $\varphi(p)$. As this is a composition of smooth functions between open subsets of Euclidean spaces, it follows that the function $\hat{F}$ is smooth in a neighborhood of $\varphi(p)$. As $p \in U \cap F^{-1}(V)$ was arbitrary, we conclude that $\hat{F}$ is smooth. Hence, the maximal smooth atlases of M and N satisfy (b).

- ($\Leftarrow$) Let $p \in M$. Let $(U_\alpha, \varphi_\alpha)$ be a smooth chart containing p and let (V_β, ψ_β) be a smooth chart containing $F(p)$. By hypothesis, $\psi_\beta \circ F \circ \varphi_\alpha^{-1}$ is smooth from $\varphi_\alpha(U_\alpha \cap F^{-1}(V_\beta))$ to $\psi_\beta(V_\beta)$. As $p \in M$ was arbitrary and since F is continuous, we infer that (a) is satisfied, and thus F is smooth.

Exercise 3: Let M , N and P be smooth manifolds. Prove the following assertions:

- (a) If $c: M \rightarrow N$ is a constant map, then c is smooth.
- (b) The identity map $\text{Id}_M: M \rightarrow M$ is smooth.
- (c) If $U \subseteq M$ is an open submanifold, then the inclusion map $\iota: U \hookrightarrow M$ is smooth.
- (d) If $F: M \rightarrow N$ and $G: N \rightarrow P$ are smooth maps, then the composite $G \circ F: M \rightarrow P$ is also smooth.

Solution:

(a) Since c is constant, there exists a point $q \in N$ such that $c(x) = q$ for all $x \in M$. Fix $p \in M$, pick smooth charts (U, φ) containing p and (V, ψ) containing $q = c(p)$, and observe that $\{q\} = c(U) \subseteq V$. Since the composite map $\psi \circ c \circ \varphi^{-1}: \varphi(U) \rightarrow \psi(V)$ is clearly a constant map (with value $\psi(q)$) between open subsets of Euclidean spaces, it is certainly smooth. Therefore, the given constant map c is smooth.

(b) The identity map $\text{Id}_M: M \rightarrow M$ of M clearly has an identity map between open subsets of Euclidean spaces as a coordinate representation, so it is smooth.

(c) Fix $p \in U \subseteq M$. Recall that a smooth chart for U containing p is simply a smooth chart (V, ψ) for M such that $p \in V \subseteq U$, and clearly it holds that $\iota(V) = V$. Since the coordinate representation of ι with respect to such a smooth chart is the identity map $\text{Id}_{\psi(V)}: \psi(V) \rightarrow \psi(V)$, we deduce that $\iota: U \hookrightarrow M$ is smooth.

(d) Fix $p \in M$. Since G is smooth, there exist smooth charts (V, ψ) containing $F(p)$ and (W, θ) containing $G(F(p)) = (G \circ F)(p)$ such that $G(V) \subseteq W$ and the composite map

$$\theta \circ G \circ \psi^{-1}: \psi(V) \rightarrow \theta(W)$$

is smooth. Since F is smooth, it is continuous by *Proposition 2.5*, so $F^{-1}(V)$ is an open neighborhood of p in M , and thus there exists a smooth chart (U, φ) for M such that $p \in U \subseteq F^{-1}(V)$. In addition, the composite map

$$\psi \circ F \circ \varphi^{-1}: \varphi(U) \rightarrow \psi(V)$$

is smooth by *Remark 2.7*, and we also have $(G \circ F)(U) \subseteq G(V) \subseteq W$. Now, observe that

$$\theta \circ (G \circ F) \circ \varphi^{-1} = (\theta \circ G \circ \psi^{-1}) \circ (\psi \circ F \circ \varphi^{-1}): \varphi(U) \rightarrow \theta(W)$$

is smooth as a composition of smooth maps between open subsets of Euclidean spaces. Hence, the composite map $G \circ F: M \rightarrow P$ is smooth, as claimed.