Problem Set 1

Exercise Session on September 8 — **Due: Tue, Sept 15, 10am, on Moodle**

1 The Homework

The following two problems are the core of this homework. You work out solutions and turn them in.

Problem 1: Expected Maximum of Subgaussians

Let $\{X_i\}_{i=1}^n$ be a collection of n zero-mean σ^2-subgaussian random variables, not necessarily independent of each other. Let $Y = \max_{i \in \{1,2,\cdots,n\}} X_i$. Prove that $\mathbb{E}[Y] \leq \sqrt{2\sigma^2 \log n}$. *Hint:* Recall that by Jensen, $e^{\lambda \mathbb{E}[X]} \leq \mathbb{E}[e^{\lambda X}]$.

Solution 1. Consider the MGF of Y, we have the following relations for all $\lambda \geq 0$

$$\mathbb{E}\left[e^{\lambda Y}\right] = \mathbb{E}\left[e^{\lambda \max_{i \in \{1,2,\ldots,n\}} X_i}\right] \leq \mathbb{E}\left[\sum_{i \in \{1,2,\ldots,n\}} e^{\lambda X_i}\right].$$

By the linearity of expectation (this does not require independence), we have

$$\mathbb{E}\left[\sum_{i \in \{1,2,\ldots,n\}} e^{\lambda X_i}\right] = \sum_{i \in \{1,2,\ldots,n\}} \mathbb{E}\left[e^{\lambda X_i}\right].$$

By the assumption that $\{X_i\}_{i=1}^n$ are σ^2-subgaussian random variables, we can upper bound each summand by $e^{\lambda^2\sigma^2/2}$. Combining, we thus have

$$\mathbb{E}[e^{\lambda Y}] \leq n e^{\lambda^2 \sigma^2/2}.$$

Using the hint, we have

$$e^{\lambda E[Y]} \leq \mathbb{E}[e^{\lambda Y}] \leq n e^{\lambda^2\sigma^2/2} = e^{\lambda^2\sigma^2/2 + \log n},$$

where the last rewrite is often a useful trick: Write everything into the exponent. Taking log on both sides and dividing by λ, we find

$$E[Y] \leq \lambda \frac{\sigma^2}{2} + \frac{1}{\lambda} \log n.$$

Optimizing over λ, we have the optimal $\lambda^* = \sqrt{\frac{2 \log n}{\sigma^2}}$, which gives us the desired inequality.

Problem 2: Bounded random variables are subgaussian

This problem is a guided proof of a slightly weakened version of Lemma 2.4.

(a) Prove the following inequality:

$$\cosh(x) = (e^x + e^{-x})/2 \leq e^{x^2/2}. \tag{1}$$

(b) Using the previous inequality, give an upper bound on the moment generating function of a random variable S that only takes the values $+1$ and -1, with equal probability.

Hint: The upper bound should depend on the parameter of the moment generating function.

(c) Consider any random variable X and let X' be a random variable *independent of* X, but with exactly the same distribution. Show that

$$\mathbb{E}_X[e^{\lambda(X-\mathbb{E}[X])}] \leq \mathbb{E}_{X,X'}[e^{\lambda(X-X')}]. \tag{2}$$

(d) Show that the random variables $(X - X')$ and $S(X - X')$, where S is as in Part (b) and assumed independent of X and X', have the same distribution.

(e) From the previous part, we thus know that

$$\mathbb{E}_{X,X'}[e^{\lambda(X-X')}] = \mathbb{E}_{S,X,X'}[e^{\lambda S(X-X')}]. \tag{3}$$

Now assume that X is a bounded random variable, $X \in [a, b]$. Condition on $X = x$ and $X' = x'$, and take expectation over S. Observe that $(x - x')^2 \leq (b - a)^2$. Use this and your result from Part (b) to further upper bound $\mathbb{E}_{S,X,X'}[e^{\lambda S(X-X')}]$.

(f) Combine your results to give an upper bound on the moment generating function of a centered bounded random variable $X - \mathbb{E}[X]$, where $X \in [a, b]$.

Hint: The upper bound should depend on the parameter of the moment generating function as well as a and b.

(g) Compare your result to Lemma 2.4. Discuss the differences.

Solution 2. (a) To prove this inequality, we can proceed via the expansions of the exponential function. Specifically,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \dots$$
$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \frac{x^6}{6!} - \dots$$

Adding up and dividing by 2,

$$\cosh(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} \dots + \frac{x^{2n}}{(2n)!} + \dots$$

Expanding $e^{x^2/2}$ via the standard expansion for e^y,

$$e^{x^2/2} = 1 + \frac{x^2}{2} + \frac{x^4}{4 \cdot 2!} + \frac{x^6}{8 \cdot 3!} \dots + \frac{x^{2n}}{2^n n!} + \dots$$

A term-by-term comparison and noting that $2^n n! \leq (2n)!$ gives the claimed bound.

(b) Simply write out

$$\mathbb{E}\left[e^{\lambda S}\right] = \sum_s p_S(s) e^{\lambda s} = \frac{1}{2} e^{\lambda} + \frac{1}{2} e^{-\lambda} \leq e^{\lambda^2/2}.$$

(c) Write out using the independence of X and X'

$$\mathbb{E}_{X,X'}[e^{\lambda(X-X')}] = \mathbb{E}_X\left[\mathbb{E}_{X'}[e^{\lambda(X-X')}]\right].$$

Now, for the inner expectation, we apply Jensen's inequality, noting that the exponential function is convex:

$$\mathbb{E}_{X'}[e^{\lambda(X-X')}] \geq e^{\mathbb{E}_{X'}[\lambda(X-X')]} = e^{\lambda(X-\mathbb{E}_{X'}[X'])} = e^{\lambda(X-\mathbb{E}[X])},$$

where we have used the linearity of expectation and the fact that X and X' have the same distribution.

(d) For example, we can argue via the CDF. First, we observe that since X and X' are indistinguishable, we must have for any real number y

$$\mathbb{P}(X - X' \leq y) = \mathbb{P}(X' - X \leq y)$$

But then, by conditioning, we must have for any real number y

$$\begin{aligned}\mathbb{P}(S(X - X') \leq y) &= \mathbb{P}(S = 1)\mathbb{P}((X - X') \leq y) + \mathbb{P}(S = -1)\mathbb{P}(-(X - X') \leq y) \\ &= \mathbb{P}(X - X' \leq y),\end{aligned}$$

which proves the claim.

(e) Following the instruction, we write

$$\mathbb{E}_{S,X,X'}[e^{\lambda S(X-X')}] = \mathbb{E}_{X,X'}\left[\mathbb{E}_S[e^{\lambda S(X-X')}] | X, X'\right].$$

Now, for any fixed values $X = x$ and $X' = x'$, we have, using Part (b),

$$\mathbb{E}_S[e^{\lambda S(x-x')}] \leq e^{\lambda^2 (x-x')^2/2} \leq e^{\lambda^2 (b-a)^2/2}.$$

Hence,

$$\mathbb{E}_{X,X'}[e^{\lambda(X-X')}] = \mathbb{E}_{S,X,X'}[e^{\lambda S(X-X')}] \leq e^{\lambda^2 (b-a)^2/2}.$$

(f) Combining everything, we have

$$\begin{aligned}\mathbb{E}_X[e^{\lambda(X-\mathbb{E}[X])}] &\le \mathbb{E}_{X,X'}[e^{\lambda(X-X')}] \\ &= \mathbb{E}_{S,X,X'}[e^{\lambda S(X-X')}] \\ &\le e^{\lambda^2(b-a)^2/2}.\end{aligned}$$

(g) In the lecture notes, we have shown that for a bounded random variable X, the moment generating function satisfies

$$\mathbb{E}_X[e^{\lambda(X-\mathbb{E}[X])}] \le e^{\frac{\lambda^2(b-a)^2}{8}}.$$

So, the proof above establishes also that bounded random variables are subgaussian, but with a suboptimal parameter: the argument developed here says that bounded random variables are $(b-a)^2$-subgaussian, where with the more intricate argument leading to the Lemma in the lecture notes, you can actually show that they are $(b-a)^2/4$-subgaussian. Needless to say, for many applications, these two results are equally interesting, and there is only a small gain to be had from the factor of 4 improvement in the exponent.

2 Additional Problems

Problem 3 is the proof of Lemma 2.4 (a stronger, but less insightful version of Problem 2 above). You do not need to know this proof for the exam. Hence, if you prefer, you can safely skip Problem 3. By contrast, Problems 4 and 5 concern material that we assume you are already familiar with from earlier classes you followed. If this is not the case, then by all means, do Problems 4 and 5. Needless to say, you will receive solutions for all problems.

Problem 3: Hoeffding's Lemma

Prove Lemma 2.4 in the lecture notes. In other words, prove that if X is a zero-mean random variable taking values in $[a,b]$ then

$$\mathbb{E}[e^{\lambda X}] \le e^{\frac{\lambda^2}{2}[(a-b)^2/4]}.$$

Expressed differently, X is $[(a-b)^2/4]$-subgaussian.

Hint: You can use the following steps to prove the lemma:

1. Let $\lambda > 0$. Let X be a random variable such that $a \le X \le b$ and $\mathbb{E}[X] = 0$. By considering the convex function $x \to e^{\lambda x}$, show that

$$\mathbb{E}[e^{\lambda X}] \le \frac{b}{b-a}e^{\lambda a} - \frac{a}{b-a}e^{\lambda b}. \tag{4}$$

2. Let $p = -a/(b-a)$ and $h = \lambda(b-a)$. Verify that the right-hand side of (4) equals $e^{L(h)}$ where

$$L(h) = -hp + \log(1-p+pe^h).$$

3. By Taylor's theorem, there exists $\xi \in (0,h)$ such that

$$L(h) = L(0) + hL'(0) + \frac{h^2}{2}L''(\xi).$$

Show that $L(h) \le h^2/8$ and hence $\mathbb{E}[e^{\lambda X}] \le e^{\lambda^2(b-a)^2/8}$.

Solution 3. Since $e^{\lambda x}$ is convex in x we have for all $a \leq x \leq b$,

$$e^{\lambda x} \leq \frac{b-x}{b-a} e^{\lambda a} + \frac{x-a}{b-a} e^{\lambda b}.$$

If we take the expected value of this wrt X and recall that $\mathbb{E}[X] = 0$ then it follows that

$$\mathbb{E}[e^{\lambda X}] \leq \frac{b}{b-a} e^{\lambda a} - \frac{a}{b-a} e^{\lambda b}.$$

Consider the right-hand side. Note that we must have $a < 0$ and $b > 0$ since $\mathbb{E}[X] = 0$. Following the hint further, consider now

$$L(h) = L(0) + hL'(0) + \frac{h^2}{2} L''(\xi).$$

First we have $L(0) = 0$. Next,

$$L'(h) = -p + \frac{pe^h}{1 - p + pe^h},$$

hence,

$$L'(h = 0) = 0.$$

Finally,

$$L''(\xi) = \frac{pe^{\xi}(1 - p + pe^{\xi}) - p^2 e^2 \xi}{(1 - p + pe^{\xi})^2} = \frac{pe^{\xi}(1 - p)}{(1 - p + pe^{\xi})^2}.$$

It thus remains to show that this expression is bounded by $1/4$ for all $0 \leq \xi \leq h$. Thus, we can define $a = pe^{\xi}$ and $b = 1 - p$, with which we can write

$$L''(\xi) = \frac{pe^{\xi}(1 - p)}{(1 - p + pe^{\xi})^2} = \frac{ab}{(a + b)^2}.$$

and use the inequality $\frac{ab}{(a+b)^2} \leq \frac{1}{4}, \forall a, b \in \mathbb{R}$ to conclude.

An alternative way to solve this problem could be define $\phi(\lambda) = \ln \mathbb{E}[e^{\lambda X}]$.

$$\phi'(\lambda) = \frac{d}{d\lambda} \ln \mathbb{E}[e^{\lambda X}] = \frac{\mathbb{E}[Xe^{\lambda X}]}{\mathbb{E}[e^{\lambda X}]}$$

So $\phi'(0) = \frac{0}{1} = 0$.

$$\phi''(\lambda) = \frac{d}{d\lambda} \phi'(\lambda) = \frac{d}{d\lambda} \frac{\mathbb{E}[Xe^{\lambda X}]}{\mathbb{E}[e^{\lambda X}]} = \frac{\mathbb{E}[X^2 e^{\lambda X}]\mathbb{E}[e^{\lambda X}] - \mathbb{E}[Xe^{\lambda X}]\mathbb{E}[Xe^{\lambda X}]}{\mathbb{E}[e^{\lambda X}]^2}$$

For $\lambda = 0$, we have

$$\phi''(0) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \mathrm{Var}(X)$$

Also, we have $\phi(\lambda) \leq \phi(0) + \phi'(0)\lambda + \phi''(0)\frac{\lambda^2}{2} = \frac{\lambda^2}{2} \mathrm{Var}(X)$. As X is random variable taking values in $[a, b]$. The largest variance is achieved when $\Pr\{X = a\} = \frac{b}{b-a}$ $\Pr\{X = b\} = \frac{-a}{b-a}$.

$$\mathrm{Var}(X) \leq \frac{(b-a)^2}{4} \tag{5}$$

Therefore we have

$$\mathbb{E}[e^{\lambda X}] \leq e^{\frac{\lambda^2}{2}\frac{(b-a)^2}{4}}$$

X is $[(b-a)^2/4]$-subgaussian.

Problem 4: Review of Random Variables

Let X and Y be discrete random variables defined on some probability space with a joint pmf $p_{XY}(x,y)$. Let $a,b \in \mathbb{R}$ be fixed.

(a) Prove that $\mathbb{E}[aX+bY] = a\mathbb{E}[X] + b\mathbb{E}[Y]$. Do not assume independence.

(b) Prove that if X and Y are independent random variables, then $\mathbb{E}[X \cdot Y] = \mathbb{E}[X] \cdot \mathbb{E}[Y]$.

(c) Assume that X and Y are not independent. Find an example where $\mathbb{E}[X \cdot Y] \neq \mathbb{E}[X] \cdot \mathbb{E}[Y]$, and another example where $\mathbb{E}[X \cdot Y] = \mathbb{E}[X] \cdot \mathbb{E}[Y]$.

(d) Prove that if X and Y are independent, then they are also uncorrelated, i.e.,

$$Cov(X,Y) := \mathbb{E}\left[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])\right] = 0. \tag{6}$$

(e) Find an example where X and Y are uncorrelated but dependent.

(f) Assume that X and Y are uncorrelated and let σ_X^2 and σ_Y^2 be the variances of X and Y, respectively. Find the variance of $aX+bY$ and express it in terms of $\sigma_X^2, \sigma_Y^2, a, b$.
Hint: First show that $Cov(X,Y) = \mathbb{E}[X \cdot Y] - \mathbb{E}[X] \cdot \mathbb{E}[Y]$.

Solution 4. *(a)*

$$\begin{aligned}
\mathbb{E}[aX+bY] &= \sum_x \sum_y (ax+by) p_{XY}(x,y) \\
&= \sum_x ax \sum_y p_{XY}(x,y) + \sum_y by \sum_x p_{XY}(x,y) \\
&= a \sum_x x\, p_X(x) + b \sum_y y\, p_Y(y) \\
&= a\mathbb{E}[X] + b\mathbb{E}[Y].
\end{aligned}$$

(b) If X and Y are independent, we have $p_{XY}(x,y) = p_X(x)p_Y(y)$, then

$$\begin{aligned}
\mathbb{E}[X \cdot Y] &= \sum_X \sum_Y xy p_{XY}(x,y) \\
&= \sum_X \sum_Y x p_X(x) y p_Y(y) \\
&= \sum_X x p_X(x) \sum_Y y p_Y(y) \\
&= \mathbb{E}[X] \cdot \mathbb{E}[Y]
\end{aligned}$$

(c) For the first example, suppose $Pr(X=0, Y=1) = Pr(X=1, Y=0) = \frac{1}{2}$, and $Pr(X=0, Y=0) = Pr(X=1, Y=1) = 0$. X, Y are dependent, and we have $\mathbb{E}[X \cdot Y] = 0$ while $\mathbb{E}[X]\mathbb{E}[Y] = \frac{1}{4}$

For the second example, suppose $Pr(X=-1, Y=0) = Pr(X=0, Y=1) = Pr(X=1, Y=0) = \frac{1}{3}$. X, Y are dependent. Obviously we have $\mathbb{E}[X \cdot Y] = 0$, and furthermore $\mathbb{E}[X] = 0$, hence $\mathbb{E}[X]\mathbb{E}[Y] = 0$.

(d) If X and Y are independent, we have $p_{XY}(x,y) = p_X(x)p_Y(y)$, then

$$\begin{aligned}
\mathbb{E}[(X-\mathbb{E}[X])(Y-\mathbb{E}[Y])] &= \sum_x \sum_y (x-\mathbb{E}[X])(y-\mathbb{E}[Y])\, p_{XY}(x,y) \\
&= \sum_x \sum_y (x-\mathbb{E}[X])(y-\mathbb{E}[Y])\, p_X(x)p_Y(y) \\
&= \sum_x (x-\mathbb{E}[X])\, p_X(x) \sum_y (y-\mathbb{E}[Y])\, p_Y(y) \\
&= (\mathbb{E}[X]-\mathbb{E}[X])(\mathbb{E}[Y]-\mathbb{E}[Y]) = 0.
\end{aligned}$$

Thus, X and Y are uncorrelated.

(e) One example where X and Y are uncorrelated but dependent is

$$\mathbb{P}_{XY}(x,y) = \begin{cases} \frac{1}{3} & \text{if } (x,y) \in \{(-1,0),(1,0),(0,1)\}, \\ 0 & \text{otherwise.} \end{cases}$$

First, it can be easily checked that $\mathbb{E}[X \cdot Y] = 0 = \mathbb{E}[X] \cdot \mathbb{E}[Y]$ (note that $\mathbb{E}[X] = 0$). Second, X and Y are dependent since $\mathbb{P}_{XY}(1,0) = \frac{1}{3}$ but $\mathbb{P}_X(1)\mathbb{P}_Y(0) = \frac{1}{3} \times \frac{2}{3}$.

(f) First, we have

$$\begin{aligned}
Cov(X,Y) &= \mathbb{E}\left[(X-\mathbb{E}[X])(Y-\mathbb{E}[Y])\right] \\
&= \mathbb{E}\left[XY - X\mathbb{E}[Y] - \mathbb{E}[X]Y + \mathbb{E}[X]\mathbb{E}[Y]\right] \\
&= \mathbb{E}[X \cdot Y] - \mathbb{E}[X] \cdot \mathbb{E}[Y].
\end{aligned}$$

Thus, $Cov(X,Y) = 0$ if and only if $\mathbb{E}[X \cdot Y] = \mathbb{E}[X] \cdot \mathbb{E}[Y]$.

Then,

$$\begin{aligned}
\sigma^2_{aX+bY} &= \mathbb{E}[aX+bY-\mathbb{E}[aX+bY]]^2 \\
&= \mathbb{E}[(aX+bY)^2] - (\mathbb{E}[aX+bY])^2 \\
&= a^2\mathbb{E}[X^2] + 2ab\mathbb{E}[X \cdot Y] + b^2\mathbb{E}[Y^2] - a^2\mathbb{E}[X]^2 - 2ab\mathbb{E}[X]\mathbb{E}[Y] - b^2\mathbb{E}[Y]^2 \\
&= a^2(\mathbb{E}[X^2]-\mathbb{E}[X]^2) + b^2(\mathbb{E}[Y^2]-\mathbb{E}[Y]^2) \\
&= a^2\sigma_X^2 + b^2\sigma_Y^2.
\end{aligned}$$

We remark that since the independence of X and Y implies $Cov(X,Y) = 0$, we also have $\sigma^2_{aX+bY} = a^2\sigma_X^2 + b^2\sigma_Y^2$ if X and Y are independent.

Problem 5: Review of Gaussian Random Variables

A random variable X with probability density function

$$p_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-m)^2}{2\sigma^2}} \tag{7}$$

is called a *Gaussian* random variable.

(a) Explicitly calculate the mean $\mathbb{E}[X]$, the second moment $\mathbb{E}[X^2]$, and the variance $Var[X]$ of the random variable X.

(b) Let us now consider events of the following kind:

$$\Pr(X < \alpha). \tag{8}$$

Unfortunately for Gaussian random variables this cannot be calculated in closed form. Instead, we will rewrite it in terms of the standard Q-function:

$$Q(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}}\, du \tag{9}$$

Express $\Pr(X < \alpha)$ in terms of the Q-function and the parameters m and σ^2 of the Gaussian pdf.

Like we said, the Q-function cannot be calculated in closed form. Therefore, it is important to have *bounds* on the Q-function. In the next 3 subproblems, you derive the most important of these bounds, learning some very general and powerful tools along the way:

(c) Derive the Markov inequality, which says that for any non-negative random variable X and positive a, we have

$$\Pr(X \geq a) \quad \leq \quad \frac{\mathbb{E}[X]}{a}. \tag{10}$$

(d) Use the Markov inequality to derive the Chernoff bound: the probability that a real random variable Z exceeds b is given by

$$\Pr(Z \geq b) \leq \mathbb{E}\big[e^{s(Z-b)}\big], \qquad s \geq 0. \tag{11}$$

(e) Use the Chernoff bound to show that

$$Q(x) \leq e^{-\frac{x^2}{2}} \quad \text{for } x \geq 0. \tag{12}$$

Solution 5. *(a)* First,

$$\begin{aligned}
\mathbb{E}[X] &= \int_{-\infty}^{\infty} x p_X(x)\,dx \\
&= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} x e^{-\frac{(x-m)^2}{2\sigma^2}}\,dx \\
&\overset{(*)}{=} \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} u e^{-\frac{u^2}{2\sigma^2}}\,du + m \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{u^2}{2\sigma^2}}\,du \qquad (13)\\
&\overset{(\dagger)}{=} 0 + m \\
&= m,
\end{aligned}$$

where $(*)$ follows by a change of variable $u = x - m$ and $(\dagger)$ follows since the first integrand in (13) is an odd function and the second integrand in (13) is a probability density function. We remark that the integral

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx$$

known as *Gaussian integral*, can be evaluated explicitly to be $\sqrt{\pi}$. Second,

$$\begin{aligned}
\mathbb{E}[X^2] &= \int_{-\infty}^{\infty} x^2 p_X(x)\,dx \\
&= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} x^2 e^{-\frac{(x-m)^2}{2\sigma^2}}\,dx \\
&\overset{(*)}{=} \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} u^2 e^{-\frac{u^2}{2\sigma^2}}\,du + \frac{2m}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} u e^{-\frac{u^2}{2\sigma^2}}\,du + m^2 \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{u^2}{2\sigma^2}}\,du \qquad (14)\\
&\overset{(\dagger)}{=} \sigma^2 + 0 + m^2 \\
&= \sigma^2 + m^2,
\end{aligned}$$

where $(*)$ follows by a change of variable $u = x - m$ and $(\dagger)$ follows from the same arguments in the evaluation of $\mathbb{E}[X]$ and an integration by parts to the first integral in (14):

$$\begin{aligned}
\frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} u^2 e^{-\frac{u^2}{2\sigma^2}}\,du &= -\frac{\sigma^2}{\sqrt{2\pi\sigma^2}} \left(u e^{-\frac{u^2}{2\sigma^2}} \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} e^{-\frac{u^2}{2\sigma^2}}\,du \right) \\
&= 0 + \sigma^2.
\end{aligned}$$

Therefore,

$$\begin{aligned} Var[X] &= \mathbb{E}[X - \mathbb{E}[X]]^2 \\ &= \mathbb{E}[X^2] - \mathbb{E}[X]^2 \\ &= \sigma^2 + m^2 - m^2 \\ &= \sigma^2. \end{aligned}$$

(b)

$$\begin{aligned} \mathbb{P}(X < \alpha) &= \int_{-\infty}^{\alpha} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-m)^2}{2\sigma^2}}\, dx \\ &\overset{(*)}{=} \int_{-\infty}^{\frac{\alpha-m}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}}\, du \\ &= 1 - Q\left(\frac{\alpha - m}{\sigma}\right), \end{aligned}$$

where $(*)$ follows by a change of variable $u = \frac{x-m}{\sigma}$.

(c)

$$\begin{aligned} \mathbb{E}[X] &= \int_0^a x p_X(x)\, dx + \int_a^\infty x p_X(x)\, dx \\ &\geq 0 + a \int_a^\infty p_X(x)\, dx \\ &= a\mathbb{P}(X \geq a). \end{aligned}$$

(d) Fix $s \geq 0$, then we have

$$\begin{aligned} \mathbb{P}(Z \geq b) &\leq \mathbb{P}(s(Z-b) \geq 0) \\ &= \mathbb{P}(e^{s(Z-b)} \geq e^0) \\ &\overset{(*)}{\leq} \mathbb{E}\big[e^{s(Z-b)}\big], \end{aligned}$$

where $(*)$ follows from the Markov inequality.

(e) Let X be a Gaussian random variable with mean zero and unit variance, then we have

$$\begin{aligned} Q(x) &= \mathbb{P}(X \geq x) \\ &\overset{(*)}{\leq} \mathbb{E}\left[e^{s(X-x)}\right] \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{s(u-x)} e^{-\frac{u^2}{2}}\, du \\ &= e^{-sx+\frac{s^2}{2}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(u-s)^2}{2}}\, du \\ &= e^{-sx+\frac{s^2}{2}}, \end{aligned}$$

where $(*)$ follows from the Chernoff bound. In order to get the tightest bound, we need to minimize $-sx + s^2/2$ which gives $s = x$ and then the desired bound is established.