



Differential Geometry II - Smooth Manifolds

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Exercise Sheet 2

Exercise 1 (*Product manifolds*):

Let $M_1, \dots, M_k$ be smooth manifolds of dimensions $n_1, \dots, n_k$, respectively, where $k \geq 2$. Show that the product space $M_1 \times \dots \times M_k$ is a smooth manifold of dimension $n_1 + \dots + n_k$ by constructing a smooth manifold structure on it.

Exercise 2 (*The real projective n -space*):

Denote by $\mathbb{RP}^n$ the real projective n -space, where $n \geq 1$. For each $i \in \{0, \dots, n\}$ consider the chart (U_i, φ_i) for $\mathbb{RP}^n$ described in Appendix A, where

$$U_i := \{[x_0 : \dots : x_n] \in \mathbb{RP}^n \mid x_i \neq 0\} \subseteq \mathbb{RP}^n$$

and

$$\begin{aligned} \varphi_i: U_i &\rightarrow \mathbb{R}^n \\ [x_0 : \dots : x_n] &\mapsto \left(\frac{x_0}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \dots, \frac{x_n}{x_i} \right). \end{aligned}$$

Show that the collection

$$\mathcal{A}_{\mathbb{RP}^n} := \{(U_i, \varphi_i)\}_{i=0}^n$$

is a smooth atlas for $\mathbb{RP}^n$.

Exercise 3 (*Equivalent characterizations of smoothness*):

Let M and N be smooth manifolds and let $F: M \rightarrow N$ be a map. Show that F is smooth if and only if either of the following conditions (a), (b) is satisfied:

- (a) For every $p \in M$ there exist smooth charts (U, φ) containing p and (V, ψ) containing $F(p)$ such that $U \cap F^{-1}(V)$ is open in M and the composite map $\psi \circ F \circ \varphi^{-1}$ is smooth from $\varphi(U \cap F^{-1}(V))$ to $\psi(V)$.
- (b) F is continuous and there exist smooth atlases $\{(U_\alpha, \varphi_\alpha)\}$ and $\{(V_\beta, \psi_\beta)\}$ for M and N , respectively, such that for each α and β , $\psi_\beta \circ F \circ \varphi_\alpha^{-1}$ is a smooth map from $\varphi_\alpha(U_\alpha \cap F^{-1}(V_\beta))$ to $\psi_\beta(V_\beta)$.

- (c) Show that $\mathbb{R}$ with its standard smooth structure is diffeomorphic to $\mathbb{R}$ with atlas $\{(\mathbb{R}, x \mapsto x^3)\}$.

Exercise 4: Let M , N and P be smooth manifolds. Prove the following assertions:

- (a) If $c: M \rightarrow N$ is a constant map, then c is smooth.
- (b) The identity map $\text{Id}_M: M \rightarrow M$ is smooth.
- (c) If $U \subseteq M$ is an open submanifold, then the inclusion map $\iota: U \hookrightarrow M$ is smooth.
- (d) If $F: M \rightarrow N$ and $G: N \rightarrow P$ are smooth maps, then the composite $G \circ F: M \rightarrow P$ is also smooth.