Problem Set 1

Exercise Session on September 8 — **Due: Tue, Sept 15, 10am, on Moodle**

1 The Homework

The following two problems are the core of this homework. You work out solutions and turn them in.

Problem 1: Expected Maximum of Subgaussians

Let $\{X_i\}_{i=1}^n$ be a collection of n zero-mean σ^2-subgaussian random variables, not necessarily independent of each other. Let $Y = \max_{i\in\{1,2,\cdots,n\}} X_i$. Prove that $\mathbb{E}[Y] \leq \sqrt{2\sigma^2 \log n}$. *Hint:* Recall that by Jensen, $e^{\lambda\mathbb{E}[X]} \leq \mathbb{E}[e^{\lambda X}]$.

Problem 2: Bounded random variables are subgaussian

This problem is a guided proof of a slightly weakened version of Lemma 2.4.

(a) Prove the following inequality:

$$\cosh(x) = (e^x + e^{-x})/2 \leq e^{x^2/2}. \tag{1}$$

(b) Using the previous inequality, give an upper bound on the moment generating function of a random variable S that only takes the values $+1$ and -1, with equal probability.

Hint: The upper bound should depend on the parameter of the moment generating function.

(c) Consider any random variable X and let X' be a random variable *independent of* X, but with exactly the same distribution. Show that

$$\mathbb{E}_X[e^{\lambda(X-\mathbb{E}[X])}] \leq \mathbb{E}_{X,X'}[e^{\lambda(X-X')}]. \tag{2}$$

(d) Show that the random variables $(X - X')$ and $S(X - X')$, where S is as in Part (b) and assumed independent of X and X', have the same distribution.

(e) From the previous part, we thus know that

$$\mathbb{E}_{X,X'}[e^{\lambda(X-X')}] = \mathbb{E}_{S,X,X'}[e^{\lambda S(X-X')}]. \tag{3}$$

Now assume that X is a bounded random variable, $X \in [a, b]$. Condition on $X = x$ and $X' = x'$, and take expectation over S. Observe that $(x - x')^2 \leq (b - a)^2$. Use this and your result from Part (b) to further upper bound $\mathbb{E}_{S,X,X'}[e^{\lambda S(X-X')}]$.

(f) Combine your results to give an upper bound on the moment generating function of a centered bounded random variable $X - \mathbb{E}[X]$, where $X \in [a, b]$.

Hint: The upper bound should depend on the parameter of the moment generating function as well as a and b.

(g) Compare your result to Lemma 2.4. Discuss the differences.

2 Additional Problems

Problem 3 is the proof of Lemma 2.4 (a stronger, but less insightful version of Problem 2 above). You do not need to know this proof for the exam. Hence, if you prefer, you can safely skip Problem 3. By contrast, Problems 4 and 5 concern material that we assume you are already familiar with from earlier classes you followed. If this is not the case, then by all means, do Problems 4 and 5. Needless to say, you will receive solutions for all problems.

Problem 3: Hoeffding's Lemma

Prove Lemma 2.4 in the lecture notes. In other words, prove that if X is a zero-mean random variable taking values in $[a, b]$ then

$$\mathbb{E}[e^{\lambda X}] \leq e^{\frac{\lambda^2}{2}[(a-b)^2/4]}.$$

Expressed differently, X is $[(a-b)^2/4]$-subgaussian.

Hint: You can use the following steps to prove the lemma:

1. Let $\lambda > 0$. Let X be a random variable such that $a \leq X \leq b$ and $\mathbb{E}[X] = 0$. By considering the convex function $x \to e^{\lambda x}$, show that

$$\mathbb{E}[e^{\lambda X}] \leq \frac{b}{b-a}e^{\lambda a} - \frac{a}{b-a}e^{\lambda b}. \tag{4}$$

2. Let $p = -a/(b-a)$ and $h = \lambda(b-a)$. Verify that the right-hand side of (4) equals $e^{L(h)}$ where

$$L(h) = -hp + \log(1 - p + pe^h).$$

3. By Taylor's theorem, there exists $\xi \in (0, h)$ such that

$$L(h) = L(0) + hL'(0) + \frac{h^2}{2}L''(\xi).$$

   Show that $L(h) \leq h^2/8$ and hence $\mathbb{E}[e^{\lambda X}] \leq e^{\lambda^2(b-a)^2/8}$.

Problem 4: Review of Random Variables

Let X and Y be discrete random variables defined on some probability space with a joint pmf $p_{XY}(x, y)$. Let $a, b \in \mathbb{R}$ be fixed.

(a) Prove that $\mathbb{E}[aX + bY] = a\mathbb{E}[X] + b\mathbb{E}[Y]$. Do not assume independence.

(b) Prove that if X and Y are independent random variables, then $\mathbb{E}[X \cdot Y] = \mathbb{E}[X] \cdot \mathbb{E}[Y]$.

(c) Assume that X and Y are not independent. Find an example where $\mathbb{E}[X \cdot Y] \neq \mathbb{E}[X] \cdot \mathbb{E}[Y]$, and another example where $\mathbb{E}[X \cdot Y] = \mathbb{E}[X] \cdot \mathbb{E}[Y]$.

(d) Prove that if X and Y are independent, then they are also uncorrelated, i.e.,

$$Cov(X, Y) := \mathbb{E}\left[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])\right] = 0. \tag{5}$$

(e) Find an example where X and Y are uncorrelated but dependent.

(f) Assume that X and Y are uncorrelated and let σ_X^2 and σ_Y^2 be the variances of X and Y, respectively. Find the variance of $aX + bY$ and express it in terms of $\sigma_X^2, \sigma_Y^2, a, b$.
Hint: First show that $Cov(X, Y) = \mathbb{E}[X \cdot Y] - \mathbb{E}[X] \cdot \mathbb{E}[Y]$.

Problem 5: Review of Gaussian Random Variables

A random variable X with probability density function

$$p_X(x) \quad = \quad \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-m)^2}{2\sigma^2}} \tag{6}$$

is called a *Gaussian* random variable.

(a) Explicitly calculate the mean $\mathbb{E}[X]$, the second moment $\mathbb{E}[X^2]$, and the variance $Var[X]$ of the random variable X.

(b) Let us now consider events of the following kind:

$$\Pr(X < \alpha). \tag{7}$$

Unfortunately for Gaussian random variables this cannot be calculated in closed form. Instead, we will rewrite it in terms of the standard Q-function:

$$Q(x) \quad = \quad \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}}\, du \tag{8}$$

Express $\Pr(X < \alpha)$ in terms of the Q-function and the parameters m and σ^2 of the Gaussian pdf.

Like we said, the Q-function cannot be calculated in closed form. Therefore, it is important to have *bounds* on the Q-function. In the next 3 subproblems, you derive the most important of these bounds, learning some very general and powerful tools along the way:

(c) Derive the Markov inequality, which says that for any non-negative random variable X and positive a, we have

$$\Pr(X \geq a) \quad \leq \quad \frac{\mathbb{E}[X]}{a}. \tag{9}$$

(d) Use the Markov inequality to derive the Chernoff bound: the probability that a real random variable Z exceeds b is given by

$$\Pr(Z \geq b) \leq \mathbb{E}\left[e^{s(Z-b)}\right], \qquad s \geq 0. \tag{10}$$

(e) Use the Chernoff bound to show that

$$Q(x) \leq e^{-\frac{x^2}{2}} \quad \text{for } x \geq 0. \tag{11}$$