Differential Geometry II - Smooth Manifolds
Winter Term 2024/2025
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Exam – Solutions

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Exercise 1 (2 + 3 + 2 + 3 + 2 + 6 points)**:**
Let M be a smooth n-manifold (without boundary). Let $p \in M$ and let $\big(U, \varphi = (x^i)\big)$ be a smooth coordinate chart for M such that $p \in U$.

(a) Define the *tangent space* T_pM to M at p and write down a basis for T_pM.

(b) Given $v \in T_pM$ and $f, g \in C^\infty(M)$, prove the following assertions:

(i) If f is constant, then $vf = 0$.

(ii) If $f(p) = g(p) = 0$, then $v(fg) = 0$.

(c) Define the *cotangent space* T_p^*M to M at p and write down a basis for T_p^*M.

(d) Define the notion of a *smooth vector bundle* of rank k over M. When is it said to be *(smoothly) trivial*?

(e) Write down a smooth local trivialization over U for both the tangent bundle TM and the cotangent bundle T^*M of M.

(f) Show that TM is a (smoothly) trivial vector bundle if and only if T^*M is a (smoothly) trivial vector bundle.

Solution:

(a) Recall that:

- A *derivation of $C^\infty(M)$ at $p \in M$* is an $\mathbb{R}$-linear map $v\colon C^\infty(M) \to \mathbb{R}$ which satisfies the product rule:

$$v(fg) = f(p)\, v(g) + g(p)\, v(f), \quad \forall\, f, g \in C^\infty(M).$$

- The *tangent space* T_pM to M at $p \in M$ is the set of all derivations of $C^\infty(M)$ at $p \in M$. It is easy to see that T_pM is an $\mathbb{R}$-vector space.

- The coordinate vectors $\left\{\frac{\partial}{\partial x^i}\Big|_p\right\}_{i=1}^n$, determined by the given smooth chart $\big(U,(x^i)\big)$ for M, form a basis for T_pM. Recall that $\frac{\partial}{\partial x^i}\Big|_p$ is the derivation at p that takes the i-th partial derivative of (the coordinate representation of) f at (the coordinate representation of) p.

(b)(i) By assumption and by the $\mathbb{R}$-linearity of v, we may assume that $f \equiv 1$. Then by the product rule we obtain

$$vf = v(f \cdot f) = f(p)\,vf + f(p)\,vf = 2\,vf,$$

which implies that $vf = 0$.

(b)(ii) By assumption and by the product rule we obtain

$$v(fg) = f(p)\,vg + g(p)\,vf = 0.$$

(c) Recall that:

- The *cotangent space* T_p^*M to M at $p \in M$ is defined to be the dual vector space of T_pM.
- The coordinate covectors $\left\{dx^i\big|_p\right\}_{i=1}^n$, determined by the given smooth chart $\big(U,(x^i)\big)$, form a basis for T_p^*M, which is the basis dual to the basis $\left\{\frac{\partial}{\partial x^i}\Big|_p\right\}_{i=1}^n$ for T_pM described in part (a).

(d) Recall that:

- *Topological vector bundle*: Let M be a topological space. A *(real) vector bundle of rank k over M* is a topological space E together with a continuous surjective map $\pi\colon E \to M$ satisfying the following conditions:
  (i) For each $p \in M$, the *fiber* $E_p = \pi^{-1}(p)$ over p is endowed with the structure of a k-dimensional $\mathbb{R}$-vector space.
  (ii) For each $p \in M$, there exists an open neighborhood U of p in M and a homeomorphism $\Phi\colon \pi^{-1}(U) \to U \times \mathbb{R}^k$, called a *local trivialization of E over U*, satisfying the following conditions:
    - $\pi_U \circ \Phi = \pi$, where $\pi_U\colon U \times \mathbb{R}^k \to U$ is the projection.
    - For each $q \in U$, the restriction of Φ to E_q is an $\mathbb{R}$-vector space isomorphism from E_q to $\{q\} \times \mathbb{R}^k \cong \mathbb{R}^k$.

  The space E is called *the total space* of the bundle, M is called its *base*, and π is called its *projection*.
- *Smooth vector bundle*: If, in the above definition, M and E are smooth manifolds, π is a smooth map, and the local trivializations can be chosen to be diffeomorphisms, then E is called a *smooth vector bundle over M*. In this case, any local trivialization that is a diffeomorphism onto its image is called a *smooth local trivialization*.

- *Trivial topological vector bundle*: If $E \to M$ is a (topological) vector bundle and If there exists a local trivialization of E over all of M, called a *global trivialization of* E, then E is called a *trivial bundle.* In this case, E itself is homeomorphic to the product space $M \times \mathbb{R}^k$.
- *Trivial smooth vector bundle*: If $E \to M$ is a smooth vector bundle that admits a smooth global trivialization, then we say that E is *smoothly trivial.* In this case, E is diffeomorphic to $M \times \mathbb{R}^k$, not just homeomorphic (as in previous case).

(e) We have seen in *Proposition 6.4* that

$$\Phi \colon \pi^{-1}(U) \to U \times \mathbb{R}^n,\ v^i \frac{\partial}{\partial x^i}\bigg|_p \mapsto \big(p, (v^1, \ldots, v^n)\big)$$

is a smooth local trivialization of $\pi \colon TM \to M$ over U, determined by the given smooth chart $\big(U, (x^i)\big)$ for M.

We have seen in *Proposition 8.3* that

$$\Psi \colon \pi^{-1}(U) \to U \times \mathbb{R}^n,\ \xi_i\, dx^i\big|_p \mapsto \big(p, (\xi_1, \ldots, \xi_n)\big),$$

is a smooth local trivialization of $\pi \colon T^*M \to M$ over U, determined by the given smooth chart $\big(U, (x^i)\big)$ for M.

(f) To prove the claim, we will use the following two statements:

- [*Exercise Sheet* 10, *Exercise* 5]: Given a smooth vector bundle $E \to M$ of rank k, there is a correspondence between smooth local frames and smooth local trivializations.
- *Lemma 8.7*: Let M be a smooth manifold. If (E_i) is a rough local frame over an open subset $U \subseteq M$ and if (ε^i) is its dual coframe, then (E_i) is smooth if and only if (ε^i) is smooth.

Assume first that TM is a (smoothly) trivial vector bundle. Then there exists a smooth global trivialization $\Phi \colon TM \to M \times \mathbb{R}^n$, which corresponds to a smooth global frame $\big\{X_1, \ldots, X_n\big\}$. Then the global coframe $\big\{\omega_1, \ldots, \omega_n\big\}$ dual to $\big\{X_1, \ldots, X_n\big\}$ is smooth and corresponds to a smooth global trivialization $\Psi \colon T^*M \to M \times \mathbb{R}^n$. It follows that T^*M is a (smoothly) trivial vector bundle.

Assume now that T^*M is a (smoothly) trivial vector bundle. Argue similarly, we conclude that TM is a (smoothly) trivial vector bundle.

Exercise 2 (2 + 3 + 11 points)**:**

(a) Given a smooth manifold M (without boundary), define the notion of an *embedded submanifold* S of M. When is S said to be *properly embedded* in M?

(b) Formulate the *regular level set theorem* (in the setting of smooth manifolds without boundary) and explain what a *regular point* and a *regular value* of a smooth map are.

(c) Consider the smooth function

$$f\colon \mathbb{R}^2 \to \mathbb{R},\ (x, y) \mapsto x^3 + y^3 + 1.$$

(i) Determine the regular values of f.

(ii) For which $c \in \mathbb{R}$ is the level set $f^{-1}(c)$ a properly embedded submanifold of $\mathbb{R}^2$? Justify your answer.

(iii) For each $c \in \mathbb{R}$ such that the corresponding level set $S = f^{-1}(c)$ is a properly embedded submanifold of $\mathbb{R}^2$, determine the tangent space $T_pS \cong d\iota_p(T_pS) \subset T_p\mathbb{R}^2 \cong \mathbb{R}^2$, where $p \in S$ and $\iota\colon S \hookrightarrow \mathbb{R}^2$ is the inclusion map.

Solution:

(a) Let M be a smooth manifold. An *embedded submanifold of* M is a subset $S \subseteq M$ that is a topological manifold in the subspace topology, endowed with a smooth structure with respect to which the inclusion map $S \hookrightarrow M$ is a smooth embedding. If, additionally, the inclusion map $S \hookrightarrow M$ is *proper* (that is, the preimage of every compact set is again compact), then $S \subseteq M$ is said to be *properly embedded.*

(b) Recall that:

- *The regular level set theorem*: Every regular level set of a smooth map between smooth manifolds (without boundary) is a properly embedded submanifold whose codimension is equal to the dimension of the codomain.

- *Regular point*: Let $\Phi\colon M \to N$ be a smooth map. A point $p \in M$ is called a *regular point* of Φ if the differential $d\Phi_p\colon T_pM \to T_{\Phi(p)}N$ is surjective.

- *Regular value*: Let $\Phi\colon M \to N$ be a smooth map. A point $c \in N$ is called a *regular value* of Φ if every point of the level set $\Phi^{-1}(c)$ is a regular point of Φ.

(c)(i) The gradient of f at an arbitrary point $(x, y) \in \mathbb{R}^2$ is given by

$$\operatorname{grad}(f)(x, y) = (3x^2,\, 3y^2),$$

and it is obvious that it vanishes precisely at the origin $(x, y) = (0, 0) \in \mathbb{R}^2$. Since $f(0, 0) = 1$ and since the fibers of f are disjoint, we conclude that every $c \in \mathbb{R} \setminus \{1\}$ is a regular value of f, while $c = 1$ is a critical value of f.

(c)(ii) By *Corollary 5.10* we infer that each level set $f^{-1}(c)$, where $c \neq 1$, is a properly embedded submanifold of $\mathbb{R}^2$.

Now, regarding the level set $f^{-1}(1)$, *Corollary 5.10* does *not* say that $f^{-1}(1)$ is not an embedded submanifold, so we have to argue differently in order to treat this case. Observe that

$$f^{-1}(1) = \{(x,y) \in \mathbb{R}^2 \mid x^3 + y^3 = 0\} = \{(x,-x) \mid x \in \mathbb{R}\}$$

is the line $y = -x$ in the plane $\mathbb{R}^2$, which is clearly diffeomorphic to $\mathbb{R}$. Hence, (it is straightforward to check that) the closed fiber $f^{-1}(1)$ is a properly embedded submanifold of $\mathbb{R}^2$; see also [*Exercise Sheet* 8*, Exercise* 1(*b*)].

(c)(iii) Let $c \in \mathbb{R} \setminus \{1\}$, set $S \coloneqq f^{-1}(c)$, and pick $p = (p_x, p_y) \in S$. By part (b) and by [*Exercise Sheet* 9*, Exercise* 4(*a*)] we know that $T_pS = \ker df_p$, and the differential df_p is represented by the row matrix $(3p_x^2, 3p_y^2)$. Thus, if $V = (V_x, V_y) \in T_p\mathbb{R}^2 \cong \mathbb{R}^2$, then

$$df_p(V_x, V_y) = 3\, p_x^2\, V_x + 3\, p_y^2\, V_y,$$

and hence

$$T_pS = \{V = (V_x, V_y) \in T_p\mathbb{R}^2 \mid p_x^2\, V_x + p_y^2\, V_y = 0\}.$$

Finally, when $c = 1$, we observe that

$$S \coloneqq f^{-1}(1) = \{(x,y) \in \mathbb{R}^2 \mid x + y = 0\}$$

is a linear subspace of $\mathbb{R}^2$ (for instance, S may be viewed as the kernel of the linear map $L\colon \mathbb{R}^2 \to \mathbb{R}$, $(x,y)^T \mapsto (1,1) \cdot (x,y)^T = x + y$), and hence

$$T_pS = \{V = (V_x, V_y) \in T_p\mathbb{R}^2 \mid V_x + V_y = 0\}$$

for any $p \in S$ (e.g., by applying [*Exercise Sheet* 9*, Exercise* 4(*a*)] to the (smooth) linear map L described above).

Exercise 3 (6 + 6 points)**:**

(a) Compute the Lie bracket of each of the following pairs of smooth vector fields on $\mathbb{R}^3$:

(i) $X_1 = y\,\dfrac{\partial}{\partial z} - 2xy^2\dfrac{\partial}{\partial y}$ and $Y_1 = \dfrac{\partial}{\partial y}$.

(ii) $X_2 = -y\,\dfrac{\partial}{\partial x} + x\,\dfrac{\partial}{\partial y}$ and $Y_2 = -z\,\dfrac{\partial}{\partial y} + y\,\dfrac{\partial}{\partial z}$.

(iii) $X_3 = -y\,\dfrac{\partial}{\partial x} + x\,\dfrac{\partial}{\partial y}$ and $Y_3 = y\,\dfrac{\partial}{\partial x} + x\,\dfrac{\partial}{\partial y}$.

(b) Compute the flow of $X_2 = X_3$ and examine whether it is a *complete* smooth vector field on $\mathbb{R}^3$.

Solution:

(a) According to *Proposition 7.12* = [*Exercise Sheet* 11, *Exercise* 4(*a*)], we have the following coordinate formula for the Lie bracket: If

$$X = \sum_{i=1}^{n} X^i\,\frac{\partial}{\partial x^i} \qquad \text{and} \qquad Y = \sum_{j=1}^{n} Y^j\,\frac{\partial}{\partial x^j}$$

are the coordinate expressions for the smooth vector fields X and Y on the smooth n-manifold M in terms of some smooth local coordinates (x^i) for M, then the Lie bracket $[X, Y]$ has the following coordinate expression:

$$[X, Y] = \sum_{j=1}^{n}\sum_{i=1}^{n}\left(X^i\,\frac{\partial Y^j}{\partial x^i} - Y^i\,\frac{\partial X^j}{\partial x^i}\right)\frac{\partial}{\partial x^j}.$$

Therefore, in case (i), writing

$$X_1 = 0\,\frac{\partial}{\partial x} - 2xy^2\,\frac{\partial}{\partial y} + y\,\frac{\partial}{\partial z} \qquad \text{and} \qquad Y_1 = 0\,\frac{\partial}{\partial x} + 1\,\frac{\partial}{\partial y} + 0\,\frac{\partial}{\partial z},$$

we compute that

$$[X_1, Y_1] = 4xy\,\frac{\partial}{\partial y} - \frac{\partial}{\partial z},$$

Similarly, in case (ii) we obtain

$$[X_2, Y_2] = -z\,\frac{\partial}{\partial x} + x\,\frac{\partial}{\partial z}.$$

while in case (iii) we compute that

$$[X_3, Y_3] = 2x\,\frac{\partial}{\partial x} - 2y\,\frac{\partial}{\partial y}.$$

(b) To compute the integral curves of X_2, we have to solve the following system of ODEs:

$$\begin{cases} \dot\gamma^1(t) &= & -\gamma^2(t), \\ \dot\gamma^2(t) &= & \gamma^1(t), \\ \dot\gamma^3(t) &= & 0. \end{cases}$$

Combining the first two equations we obtain $\ddot{\gamma}^1 = -\gamma^1$, and thus

$$\gamma^1(t) = a\cos(t) - b\sin(t), \ t \in \mathbb{R}$$

for some $a, b \in \mathbb{R}$. Using the first equation, this then also gives

$$\gamma^2(t) = a\sin(t) + b\cos(t), \ t \in \mathbb{R}.$$

On the other hand, the third equation above gives $\gamma^3(t) \equiv c, \ t \in \mathbb{R}$, for some $c \in \mathbb{R}$. One then verifies that the curve

$$\begin{aligned} \gamma\colon \mathbb{R} &\to \mathbb{R}^3 \\ t &\mapsto \big(a\cos(t) - b\sin(t),\, a\sin(t) + b\cos(t),\, c\big) \end{aligned}$$

is indeed a solution to the above system of equations. Notice that $\gamma(0) = (a, b, c)$, so the parameters are uniquely determined by the starting value.

In conclusion, the maximal flow domain of X_2 is $\mathcal{D} = \mathbb{R} \times \mathbb{R}^3$ and the maximal flow of X_2 is given by

$$\begin{aligned} \theta\colon \mathbb{R} \times \mathbb{R}^3 &\to \mathbb{R}^3 \\ \big(t,\, (a, b, c)\big) &\mapsto \big(a\cos(t) - b\sin(t),\, a\sin(t) + b\cos(t),\, c\big). \end{aligned}$$

In particular, X_2 is complete.

Exercise 4 (5 + 8 + 3 points)**:**

(a) Let M be a smooth n-manifold (without boundary) and let $\omega \in \Omega^1(M)$.

(i) Let $\big(U, (x^i)\big)$ be a smooth coordinate chart for M, and write $\omega|_U = \sum_{i=1}^n \omega_i \, dx^i$ in this chart. Find an expression for the *exterior derivative* $d\omega \in \Omega^2(M)$ of ω in this chart (that is, an expression of $d\omega$ in terms of the natural basis induced in each fiber of $\Lambda^2(T^*M)$ by the given chart).

(ii) Deduce that ω is closed if and only if for every point $p \in M$ there exists a smooth coordinate chart $\big(U, (x^i)\big)$ such that $p \in U$ and

$$\frac{\partial \omega_j}{\partial x^i} = \frac{\partial \omega_i}{\partial x^j} \quad \text{for all } 1 \leq i, j \leq n,$$

where $\omega|_U = \sum_{i=1}^n \omega_i \, dx^i$ in this chart.

(b) Consider the smooth 1-forms

$$\omega = y \cos(xy) \, dx + x \cos(xy) \, dy \in \Omega^1(\mathbb{R}^2)$$

and

$$\eta = x \cos(xy) \, dx + y \cos(xy) \, dy \in \Omega^1(\mathbb{R}^2).$$

(i) Show that ω is closed and exact.

(ii) Show that η is neither closed nor exact.

(iii) Compute $\omega \wedge \eta$.

(c) Evaluate the line integral

$$\int_\gamma \omega,$$

where γ is the straight line segment from $(0,0)$ to $(\sqrt{\pi}, \sqrt{\pi})$.

Solution:

(a)(i) By definition of the exterior derivative, we have

$$d\omega = \sum_{i=1}^n d\omega_i \wedge dx^i,$$

and since

$$d\omega_i = \sum_{j=1}^n \frac{\partial \omega_i}{\partial \omega_j} \, dx^j \quad \text{for every } 1 \leq i \leq n,$$

we obtain

$$\begin{aligned} d\omega &= \sum_i \left(\sum_j \frac{\partial \omega_i}{\partial \omega_j} dx^j \right) \wedge dx^i = \sum_{i,j} \frac{\partial \omega_i}{\partial \omega_j} \, dx^j \wedge dx^i \\ &= \sum_{j<i} \frac{\partial \omega_i}{\partial \omega_j} \, dx^j \wedge dx^i + \sum_{j=i} \frac{\partial \omega_i}{\partial \omega_j} \underbrace{dx^j \wedge dx^i}_{=0} + \sum_{j>i} \frac{\partial \omega_i}{\partial \omega_j} \underbrace{dx^j \wedge dx^i}_{=-dx^i \wedge dx^j} \\ &= \sum_{j<i} \left(\frac{\partial \omega_i}{\partial x^j} - \frac{\partial \omega_j}{\partial x^i} \right) dx^j \wedge dx^i. \end{aligned}$$

(a)(ii) Assume first that ω is closed, and let $\big(U,(x^i)\big)$ be an arbitrary smooth chart for M. By the above computation we have

$$0 = d\omega|_U = \sum_{j<i} \left(\frac{\partial \omega_i}{\partial x^j} - \frac{\partial \omega_j}{\partial x^i} \right) dx^j \wedge dx^i,$$

and since $\{dx^j \wedge dx^i\}_{j<i}$ gives a basis in each fiber of $\Lambda^2(T^*M)$, we have

$$\frac{\partial \omega_i}{\partial x^j} = \frac{\partial \omega_j}{\partial x^i}$$

for all $j < i$. By symmetry, the equation holds in fact for all i, j, so we deduce the forward direction.

For the reverse direction, let $p \in M$ be arbitrary and let $\big(U,(x^i)\big)$ be a smooth chart around $p \in M$ such that on U we have

$$\frac{\partial \omega_j}{\partial x^i} = \frac{\partial \omega_i}{\partial x^j} \quad \text{for all } 1 \leq i, j \leq n.$$

By part (a)(i), we obtain

$$d\omega = \sum_{j<i} \underbrace{\left(\frac{\partial \omega_i}{\partial x^j} - \frac{\partial \omega_j}{\partial x^i} \right)}_{=0} dx^j \wedge dx^i = 0,$$

and thus $d\omega_p = 0$. As $p \in M$ was arbitrary, we conclude that $d\omega = 0$, so ω is closed.

(b)(i) Consider the function

$$f \colon \mathbb{R}^2 \to \mathbb{R}, \ (x, y) \mapsto \sin(xy)$$

and observe that $df = \omega$; in other words, ω is exact. Hence, ω is closed. (This can also be verified with a direct computation).

(b)(ii) We have

$$\begin{aligned}
d\eta &= d\big(x\cos(xy)\big) \wedge dx + d\big(y\cos(xy)\big) \wedge dy \\
&= \Big(\big(\cos(xy) - xy\sin(xy)\big)dx + \big(-x^2\sin(xy)\big)\,dy \Big) \wedge dx \,+ \\
&\quad + \Big(-y^2\sin(xy)\,dx + \big(\cos(xy) - xy\sin(xy)\big)dy \Big) \wedge dy \\
&= -x^2\sin(xy)\,dy \wedge dx - y^2\sin(xy)\,dx \wedge dy \\
&= (x^2 - y^2)\sin(xy)\,dx \wedge dy,
\end{aligned}$$

which does not vanish identically on $\mathbb{R}^2$. Therefore, η is not closed (see also part (a)(ii)), and thus η cannot be exact either.

(b)(iii) We compute that

$$\begin{aligned}
\omega \wedge \eta &= \big(y\cos(xy)\,dx + x\cos(xy)\,dy\big) \wedge \big(x\cos(xy)\,dx + y\cos(xy)\,dy\big) \\
&= y^2\cos^2(xy)\,dx \wedge dy + x^2\cos^2(xy)\,dy \wedge dx \\
&= (y^2 - x^2)\cos^2(xy)\,dx \wedge dy.
\end{aligned}$$

(c) The straight line segment from $(0,0)$ to $(\sqrt{\pi}, \sqrt{\pi})$ can be parametrized by the smooth curve segment

$$\gamma\colon \left[0, \sqrt{\pi}\right] \to \mathbb{R}^2,\ t \mapsto (t,t).$$

Since $\omega = df$ is exact, by the fundamental theorem of line integrals we obtain

$$\int_\gamma \omega = \int_\gamma df = (f \circ \gamma)\big(\sqrt{\pi}\big) - (f \circ \gamma)(0) = \sin(\pi) - \sin(0) = 0.$$

Exercise 5 (3 + 2 + 6 + 5 points)**:**
Consider the smooth map

$$F\colon \mathbb{R}^2 \to \mathbb{R}^2,\ (s,t) \mapsto (st, e^t)$$

and the smooth 1-form

$$\omega = x\,dy \in \Omega^1(\mathbb{R}^2).$$

Note that we denote the standard coordinates on the source by (s,t) and the standard coordinates on the target by (x,y).

(a) Compute $d\omega$, $F^*\omega$ and $F^*\omega \wedge F^*\omega$.

(b) Verify by direct computation that $d(F^*\omega) = F^*(d\omega)$.

(c) Consider the set

$$D = \big\{(s,t) \in \mathbb{R}^2 \mid s > 0,\ t > 0\big\}$$

and determine its image under F. Show that the restriction $G := F|_D\colon D \to F(D)$ is a diffeomorphism, compute its Jacobian matrix $DG(s,t)$ at an arbitrary point $(s,t) \in D$, and determine the inverse G^{-1} of the diffeomorphism G.

(d) Compute the pushforward G_*V of the smooth vector field

$$V_{(s,t)} = st\,\frac{\partial}{\partial s}\bigg|_{(s,t)} + t^2e^t\,\frac{\partial}{\partial t}\bigg|_{(s,t)} \in \mathfrak{X}(D).$$

Solution:

(a) We have

$$d\omega = dx \wedge dy$$

and

$$F^*\omega = (st)\,d(e^t) = ste^t\,dt,$$

and hence

$$F^*\omega \wedge F^*\omega = 0 = F^*(\omega \wedge \omega).$$

(b) We compute

$$d(F^*\omega) = d(ste^t) \wedge dt = \big(te^t\,ds + s(1+t)e^t\,dt\big) \wedge dt = te^t\,ds \wedge dt$$

and

$$F^*(d\omega) = d(st) \wedge d(e^t) = (t\,ds + s\,dt) \wedge (e^t\,dt) = te^t\,ds \wedge dt,$$

whence $d(F^*\omega) = F^*(d\omega)$, as claimed.

(c) Looking at the formula $F(s,t) = (st, e^t)$, we see that the second coordinate can take any value strictly greater than 1 as t runs through $\mathbb{R}_{>0}$, and then, independently of t, the first coordinate can take any value strictly greater than 0 as s runs through $\mathbb{R}_{>0}$. Hence, we set

$$E := \big\{(x,y) \in \mathbb{R}^2 \mid x > 0,\ y > 1\big\}$$

and we will show that this is indeed the image of D under F. Note first that for every $(s,t) \in D$ we have $st > 0$ and $e^t > 1$, so $F(D) \subseteq E$. For the reverse inclusion, consider the function

$$\begin{aligned} H\colon E &\to D \\ (x,y) &\mapsto \left(\frac{x}{\log y}, \log y\right). \end{aligned}$$

We now claim that $G \circ H = \mathrm{Id}_E$. Indeed, for every $(x,y) \in E$, we have

$$G \circ H(x,y) = \left(\frac{x}{\log y} \log y, e^{\log y}\right) = (x,y).$$

Therefore, $E \subseteq G(D) = F(D)$, so that in fact $E = F(D)$.

Next, notice that $H \circ G = \mathrm{Id}_D$, since

$$H \circ G(s,t) = \left(\frac{st}{\log e^t}, \log e^t\right) = (s,t)$$

for every $(s,t) \in D$.

As both G and H are clearly smooth, we conclude that G is a diffeomorphism and $G^{-1} = H$.

Finally, we compute that

$$DG(s,t) = \begin{pmatrix} \dfrac{\partial(st)}{\partial s} & \dfrac{\partial(st)}{\partial t} \\ \dfrac{\partial(e^t)}{\partial s} & \dfrac{\partial(e^t)}{\partial t} \end{pmatrix} = \begin{pmatrix} t & s \\ 0 & e^t \end{pmatrix} \quad \text{for every } (s,t) \in D.$$

(d) By definition of the pushforward G_*V, we have

$$(G_*V)_{G(s,t)} = dG_{(s,t)}\big(V_{(s,t)}\big) \quad \text{for all } (s,t) \in D.$$

Expressing everything in terms of the standard bases provided by the standard coordinates, we obtain

$$(G_*V)_{G(s,t)} = dG_{(s,t)}\big(V_{(s,t)}\big) = \begin{pmatrix} t & s \\ 0 & e^t \end{pmatrix} \begin{pmatrix} st \\ t^2 e^t \end{pmatrix} = \begin{pmatrix} st^2(1+e^t) \\ t^2 e^{2t} \end{pmatrix}.$$

Now, to compute $(G_*V)_{(x,y)}$, we substitute (s,t) by $G^{-1}(x,y) = (x/\log y, \log y)$ on both sides:

$$(G_*V)_{(x,y)} = \begin{pmatrix} \frac{x}{\log y}(\log y)^2(1+e^{\log y}) \\ (\log y)^2 e^{2\log y} \end{pmatrix} = \begin{pmatrix} x(1+y)\log y \\ (y \log y)^2 \end{pmatrix};$$

in other words, we have

$$\big(G_*V\big)_{(x,y)} = x(1+y)\log y \left.\frac{\partial}{\partial x}\right|_{(x,y)} + (y\log y)^2 \left.\frac{\partial}{\partial y}\right|_{(x,y)}.$$

Exercise 6 (3 + 1 + 2 + 4 + 6 + 6 points)**:**

(a) Define the *integral* $\int_M \omega$ of an n-form ω on a smooth n-manifold M, where $n \geq 1$.

(You do not have to recall the definition of the integral of a (continuous) n-form on $\mathbb{R}^n$ or $\mathbb{H}^n$, but you should write explicitly the conditions that M and ω must satisfy so that the integral $\int_M \omega$ makes sense.)

(b) Formulate *Stokes' theorem*.

(c) State and prove *Green's theorem*.

(d) Using Green's theorem, compute the area of the disc

$$D_r := \left\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq r^2\right\},$$

where $r > 0$.

(e) Let M be an oriented, smooth n-manifold with or without boundary, where $n \geq 1$. Prove the following assertion: If η is a positively oriented orientation form on M, then

$$\int_M \eta > 0.$$

(f) Let M be a compact, connected, oriented, smooth n-manifold without boundary (i.e., $\partial M = \emptyset$), where $n \geq 1$, and let $\omega \in \Omega^{n-1}(M)$. Show that there exists a point $p \in M$ such that $(d\omega)_p = 0 \in \Lambda^n(T_p^* M)$.

Solution:

(a) We assume that M is an oriented, smooth n-manifold with or without boundary and that ω is a compactly supported (continuous) n-form on M, where $n \geq 1$. To define $\int_M \omega$, we now proceed in two steps:

- *1st step*: Suppose first that ω is compactly supported in the domain of a single smooth chart (U, φ) for M that is either positively or negatively oriented. We define *the integral of* ω *over* M to be

$$\int_M \omega = \pm \int_{\varphi(U)} (\varphi^{-1})^* \omega \tag{1}$$

with the positive sign for a positively oriented chart, and the negative sign otherwise.

- *2nd step*: Let $\{U_i\}$ be a finite open cover of $\operatorname{supp} \omega$ by domains of positively or negatively oriented smooth charts, and let $\{\psi_i\}$ be a smooth partition of unity subordinate to this cover. We define *the integral of* ω *over* M to be

$$\int_M \omega = \sum_i \int_M \psi_i \, \omega \, . \tag{2}$$

Since for each i the n-form $\psi_i \, \omega$ is compactly supported in U_i, each of the terms in this (finite) sum is well defined according to (1).

(b) *Stokes' theorem*: Let M be an oriented, smooth n-manifold with boundary and let ω be a compactly supported, smooth $(n-1)$-form on M, where $n \geq 1$. Then

$$\int_M d\omega = \int_{\partial M} \omega \,.$$

Here, ∂M is understood to have the induced (Stokes) orientation, and the ω on the right-hand side is to be interpreted as $\iota^*_{\partial M}\omega$, where $\iota \colon \partial M \hookrightarrow M$. If $\partial M = \varnothing$, then the right-hand side is to be interpreted as 0.

(c) *Green's theorem*: Let $D \subseteq \mathbb{R}^2$ be a compact regular domain (i.e., properly embedded codimension-0 submanifold with boundary), and let P, Q be smooth real-valued functions on D. Then

$$\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy = \int_{\partial D} P\, dx + Q\, dy.$$

Proof of Green's theorem: Apply *Stokes' theorem* to the 1-form $P\, dx + Q\, dy$ on D.

(d) In general, applying *Green's theorem* with $P(x,y) = -y$ and $Q(x,y) = x$, we can compute the *area* of a compact regular domain $D \subseteq \mathbb{R}^2$ by the formula

$$A(D) = \frac{1}{2} \int_{\partial D} (x\, dy - y\, dx)\,.$$

Now, since the disc

$$D_r = \big\{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq r^2\big\}$$

is a certainly a compact regular domain with boundary the circle

$$S_r = \big\{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = r^2\big\},$$

which can be parametrized by the smooth curve

$$\gamma \colon [0, 2\pi] \to \mathbb{R}^2,\ t \mapsto \big(r\cos(t), r\sin(t)\big),$$

we obtain

$$\begin{aligned}
A(D_r) &= \frac{1}{2} \int_\gamma (x\, dy - y\, dx) \\
&= \frac{1}{2} \int_{[0,2\pi]} \gamma^*(x\, dy - y\, dx) \\
&= \frac{1}{2} \int_{[0,2\pi]} r\cos(t)\, d\big(r\sin(t)\big) - r\sin(t)\, d\big(r\cos(t)\big) \\
&= \frac{1}{2} \int_0^{2\pi} \big[r^2\cos^2(t) + r^2\sin^2(t)\big]\, dt \\
&= \frac{1}{2} r^2\, t\Big|_0^{2\pi} \\
&= \pi r^2.
\end{aligned}$$

(e) Given any smooth chart $\big(U, (x^i)\big)$ for M, we may write

$$\eta = f\, dx^1 \wedge \ldots \wedge dx^n$$

for some continuous function $f\colon U \to \mathbb{R}$, which satisfies

$$f(x^1, \dots, x^n) = \eta_x \left(\left.\frac{\partial}{\partial x^1}\right|_x, \dots, \left.\frac{\partial}{\partial x^n}\right|_x \right),$$

where $x = (x^1, \dots, x^n) \in U$.

In view of the above, since η is a positively oriented orientation form on M, if (U, φ) is a positively oriented smooth chart for M (with connected domain U), then $(\varphi^{-1})^*\eta$ is a positive function times $dy^1 \wedge \ldots \wedge dy^n$ (while if (U, φ) is negatively oriented, then it is a negative function times the same form), where $(y^1, \dots, y^n)$ are coordinates in $\varphi(U) \subseteq \mathbb{R}^n$. Therefore, each term in (2) defining $\int_M \eta$ is non-negative, with at least one strictly positive term, proving thus the assertion that $\int_M \eta > 0$.

(f) Assume on the contrary that $d\omega \in \Omega^n(M)$ is an orientation form on M. Since M is connected, $d\omega$ must be either positively or negatively oriented, and hence

$$\int_M d\omega \neq 0.$$

On the other hand, by the Stokes' theorem, together with the fact that $\partial M = \emptyset$, we obtain

$$\int_M d\omega = \int_{\partial M} \omega = 0,$$

a contradiction. In conclusion, there is a point $p \in M$ such that $(d\omega)_p = 0 \in \Lambda^n(T_p^* M)$.