Differential Geometry II - Smooth Manifolds
Winter Term 2025/2026
Lecturer: Dr. N. Tsakanikas
Assistant: L. E. Rösler

Exam – Solutions

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Exercise 1 (5 + 10 + 5 + 2 points)**:** Let n be a positive integer.

(a) Define the notion of a *smooth n-manifold* (without boundary). In particular, explain all the terms appearing in the definition.

(Note: you do not have to define the terms *topological space*, *basis* of a topological space, and the space $\mathbb{R}^n$ with its standard topology.)

(b) Let M be a smooth n-manifold (without boundary). Let $p \in M$ and let $\big(U, \varphi = (x^i)\big)$ be a smooth coordinate chart for M such that $p \in U$.

(i) Define the *tangent space* T_pM to M at p, write down a basis for T_pM and explain how these basis elements are defined.

(ii) Define the *cotangent space* T_p^*M to M at p, write down a basis for T_p^*M and explain how these basis elements are defined.

(iii) Define the notion of a *smooth vector bundle* of rank k over M.

(iv) Describe the projection map and the fibers of the tangent bundle TM of M. Subsequently, write down a smooth local trivialization for TM over U and construct a smooth local frame for TM over U.

(c) Define the notion of a *smooth map* between smooth manifolds (without boundary) and explain how one defines the *differential* of a smooth map at every point of its domain. Define also the notions of *smooth immersions*, *smooth submersions* and *smooth embeddings*.

(d) Define the notion of a *regular value* of a smooth map and formulate the *regular level set theorem* (in the setting of smooth manifolds without boundary).

Solution:

(a) First, we define a *topological n-manifold* (without boundary). This is a topological space M satisfying the following three conditions:

(i) M is *Hausdorff*: For any two distinct points $p, q \in M$, there exist disjoint open sets $U, V \subset M$ such that $p \in U$ and $q \in V$.

(ii) M is *second-countable*: There exists a countable basis for the topology of M.

(iii) M is *locally Euclidean of dimension n*: For every $p \in M$, there exists an open neighborhood $U \subseteq M$ of p and a homeomorphism $\varphi\colon U \to \varphi(U)$, where $\varphi(U)$ is an open subset of $\mathbb{R}^n$ with its standard topology.

The pair (U, φ) is called a *chart* (or *coordinate chart*) for M.

A *smooth atlas* on a topological manifold M is a collection of charts $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in A}$ such that:

(1) The domains U_α cover M, i.e., $\bigcup_{\alpha\in A} U_\alpha = M$.

(2) For any $\alpha, \beta \in A$, the *transition map*

$$\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$$

is smooth as a map between open subsets of $\mathbb{R}^n$.

A smooth atlas which is maximal with respect to inclusion is called a *smooth structure*. Finally, a *smooth n-manifold* (without boundary) is a topological n-manifold M equipped with a *smooth structure*.

(b)

(i) The *tangent space* T_pM at $p \in M$ is the set of all derivations at p, i.e., $\mathbb{R}$-linear maps $v\colon C^\infty(M) \to \mathbb{R}$ satisfying the product rule:

$$v(fg) = f(p)\,v(g) + g(p)\,v(f).$$

Given coordinates (x^i) around p, a basis for T_pM is the set

$$\left\{ \frac{\partial}{\partial x^1}\bigg|_p, \ldots, \frac{\partial}{\partial x^n}\bigg|_p \right\},$$

where

$$\begin{aligned} \frac{\partial}{\partial x^i}\bigg|_p &: C^\infty(M) \to \mathbb{R} \\ & f \mapsto \frac{\partial(f\circ\varphi^{-1})}{\partial x_i}\big(\varphi(p)\big). \end{aligned}$$

(ii) The *cotangent space* T_p^*M at $p \in M$ is the dual space of T_pM, i.e., the $\mathbb{R}$-vector space of linear functionals on T_pM. The dual basis to $\left\{ \frac{\partial}{\partial x^i}\big|_p \right\}$ is $\{dx^1|_p, \ldots, dx^n|_p\}$, and it is defined by the rule

$$dx^i|_p \left(\frac{\partial}{\partial x^j}\bigg|_p \right) = \delta_{ij}.$$

(iii) A *smooth vector bundle of rank k* over M is a smooth manifold E together with a smooth surjection $\pi : E \to M$ such that:

(1) For each $p \in M$, the fiber $E_p = \pi^{-1}(p)$ is a k-dimensional real vector space.

(2) For each $p \in M$, there exists a neighborhood U of p and a diffeomorphism $\Phi\colon \pi^{-1}(U) \to U \times \mathbb{R}^k$ over U, called a *smooth local trivialization*, that maps each fiber E_q, $q \in U$, linearly onto $\{q\} \times \mathbb{R}^k \simeq \mathbb{R}^k$.

(iv) The tangent bundle is defined as $TM = \bigsqcup_{p\in M} T_pM$ with projection $\pi\colon TM \to M$, $\pi(v) = p$ for $v \in T_pM$, i.e., the fiber over $p \in M$ is $\pi^{-1}(p) = T_pM$. Over the chart $\big(U, \varphi = (x^i)\big)$, a smooth local trivialization is

$$\Phi\colon \pi^{-1}(U) \to U \times \mathbb{R}^n, \quad v = v^i \left.\frac{\partial}{\partial x^i}\right|_p \mapsto \big(p, (v^1, \dots, v^n)\big).$$

Hence, a smooth local frame over U is

$$\left\{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\right\}.$$

(c) A map $F\colon M \to N$ between smooth manifolds is called *smooth* if for every $p \in M$, there exist charts (U, φ) around p and (V, ψ) around $F(p)$ such that $F(U) \subset V$ and $\psi \circ F \circ \varphi^{-1}\colon U \to \psi(V)$ is smooth as a map between open subsets of Euclidean spaces. The *differential* of F at p is the linear map $dF_p\colon T_pM \to T_{F(p)}N$ defined by $dF_p(v)(f) = v(f \circ F)$ for $f \in C^\infty(N)$ and $v \in T_pM$. Morever, the given smooth map F is called:

- a *smooth immersion* if dF_p is injective for all $p \in M$;
- a *smooth submersion* if dF_p is surjective for all $p \in M$; and
- a *smooth embedding* if it is a smooth immersion and a topological embedding (i.e., a homeomorphism onto its image, endowed with the subspace topology).

(d) For a smooth map $F\colon M \to N$, a point $c \in N$ is called a *regular value* of F if for every $p \in F^{-1}(c)$, the differential dF_p is surjective. The *regular level set theorem* asserts that if $c \in N$ is a regular value of $F\colon M \to N$, then $F^{-1}(c)$ is a properly embedded submanifold of M of dimension $\dim M - \dim N$.

Exercise 2 (4 + 5 + 5 + 3 points)**:**
Consider the smooth function

$$f\colon \mathbb{R}^2 \to \mathbb{R},\ (x,y) \mapsto y^2 - x^3 - x^2.$$

(a) Determine the regular values of f.

(b) Deduce that the level set $S := f^{-1}(4)$ is a properly embedded submanifold of $\mathbb{R}^2$ and describe its tangent space $T_pS \subset T_p\mathbb{R}^2 \simeq \mathbb{R}^2$ at the point $p \in \{(-2,0),(0,2)\}$.

(c) Determine the function $g := df(V)\colon \mathbb{R}^2 \to \mathbb{R}$, where

$$V = y\,\frac{\partial}{\partial x} + x\,\frac{\partial}{\partial y} \in \mathfrak{X}(\mathbb{R}^2),$$

and examine whether the level set $g^{-1}(0)$ is an embedded submanifold of $\mathbb{R}^2$.

(d) Compute the line integrals

$$\int_{\widetilde{\alpha}} df \quad \text{and} \quad \int_{\widetilde{\alpha}} dg,$$

where $\widetilde{\alpha}$ denotes the restriction of the smooth plane curve α from *Exercise 3*(a) below to the closed interval $[-1,1] \subset \mathbb{R}$, i.e., $\widetilde{\alpha} = \alpha|_{[-1,1]}$.

Solution:

(a) We compute the differential of f:

$$df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy = (-3x^2 - 2x)\,dx + 2y\,dy.$$

The critical points of f occur where both its partial derivatives vanish:

$$-3x^2 - 2x = x(-3x-2) = 0, \quad 2y = 0.$$

Thus, the critical points are $(0,0)$ and $(-\frac{2}{3},0)$. We then compute the critical values:

$$f(0,0) = 0$$

and

$$f(-\tfrac{2}{3},0) = -(-\tfrac{2}{3})^3 - (-\tfrac{2}{3})^2 = \tfrac{8}{27} - \tfrac{4}{9} = \tfrac{8-12}{27} = -\tfrac{4}{27}.$$

Therefore, the regular values of f are all $c \in \mathbb{R} \setminus \{0, -\frac{4}{27}\}$.

(b) Since 4 is a regular value of f, by the regular level set theorem we conclude that $S = f^{-1}(4)$ is a properly embedded 1-dimensional submanifold of $\mathbb{R}^2$. The tangent space T_pS at any point $p \in S$ is the kernel of df_p. Specifically:

- At $p = (-2,0)$ we have

  $$df_{(-2,0)} = \big(-3(-2)^2 - 2(-2)\big)\,dx_{(-2,0)} + 0 \cdot dy_{(-2,0)} = (-12+4)\,dx = -8\,dx_{(-2,0)}$$

  which yields

  $$T_{(-2,0)}S = \ker(-8\,dx_{(-2,0)}) = \operatorname{span}\left\{ \left.\frac{\partial}{\partial y}\right|_{(-2,0)} \right\}$$

  (vertical line).

- At $p = (0, 2)$ we have
$$df_{(0,2)} = 0 \cdot dx_{(0,2)} + 4\, dy_{(0,2)} = 4\, dy_{(0,2)}.$$
Thus,
$$T_{(0,2)}S = \ker(4\, dy_{(0,2)}) = \operatorname{span}\left\{ \left.\frac{\partial}{\partial x}\right|_{(0,2)} \right\}$$
(horizontal line).

(c) We first determine $df(V)$:
$$df(V) = (-3x^2 - 2x) \cdot y + 2y \cdot x = -3x^2y - 2xy + 2xy = -3x^2y.$$
So $g = df(V)$ is the function
$$\mathbb{R}^2 \to \mathbb{R},\ (x, y) \mapsto -3x^2y.$$
The set $g^{-1}(0)$ consists of points $(x, y) \in \mathbb{R}^2$ where $x = 0$ or $y = 0$. This is the union of the coordinate axes in $\mathbb{R}^2$. This is not a manifold at the origin (it is a cross).

(d) We have $\widetilde{\alpha}(t) = (t^3, t^2)$ for $t \in [-1, 1]$. Note that $\alpha(-1) = (-1, 1)$ and $\alpha(1) = (1, 1)$. By the Fundamental Theorem for line integrals we obtain
$$\int_{\widetilde{\alpha}} df = f(1, 1) - f(-1, 1) = -1 - 1 = -2$$
and
$$\int_{\widetilde{\alpha}} dg = g(1, 1) - g(-1, 1) = -3 - (-3) = 0.$$
$\rightsquigarrow$ **Alternatively**, since
$$df = (-3x^2 - 2x)\, dx + 2y\, dy,$$
pulling back df along α gives
$$\begin{aligned}\alpha^*(df) &= \left[(-3(t^3)^2 - 2t^3) \cdot 3t^2 + 2t^2 \cdot 2t\right] dt = \left[(-3t^6 - 2t^3) \cdot 3t^2 + 4t^3\right] dt \\ &= (-9t^8 - 6t^5 + 4t^3)\, dt.\end{aligned}$$
Thus,
$$\int_{\widetilde{\alpha}} df = \int_{-1}^{1} (-9t^8 - 6t^5 + 4t^3)\, dt,$$
and since $t \mapsto t^3$ and $t \mapsto t^5$ are odd functions, we obtain
$$\int_{\widetilde{\alpha}} df = \int_{-1}^{1} (-9t^8)\, dt = [-t^9]_{-1}^{1} = -2.$$
Similarly, for g, we compute that
$$dg = -6xy\, dx - 3x^2\, dy,$$

and pulling it back along α gives

$$\alpha^*(dg) = \big[-6(t^3)(t^2)\cdot 3t^2 - 3(t^3)^2 \cdot 2t\big]\,dt = \big[-18t^7 - 6t^7\big]\,dt = -24t^7\,dt.$$

Therefore,

$$\int_{\widetilde{\alpha}} dg = \int_{-1}^{1} (-24t^7)\,dt = 0$$

(odd function over symmetric interval).

Exercise 3 (5 + 10 points)**:**

(a) Examine whether the following plane curves are smooth immersions:

(i) $\alpha\colon \mathbb{R} \to \mathbb{R}^2,\ t \mapsto (t^3,\, t^2)$.

(ii) $\beta\colon \mathbb{R} \to \mathbb{R}^2,\ t \mapsto (t^3 - 4t,\, t^2 - 4)$.

If so, then examine subsequently whether they are smooth embeddings.

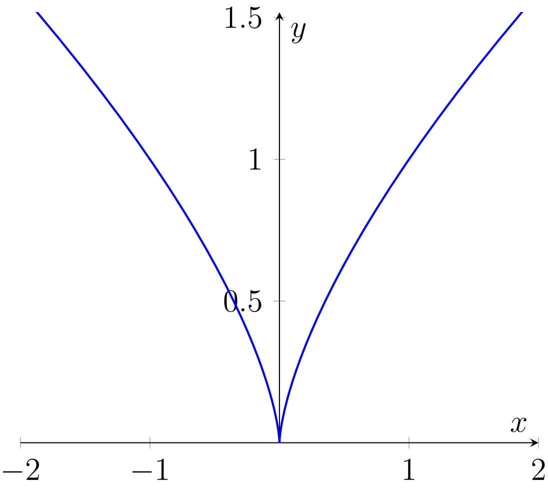
(b) Let $\pi\colon M \to N$ be a smooth submersion. Prove the following assertions:

(i) Every point of M is in the image of a smooth local section of π.

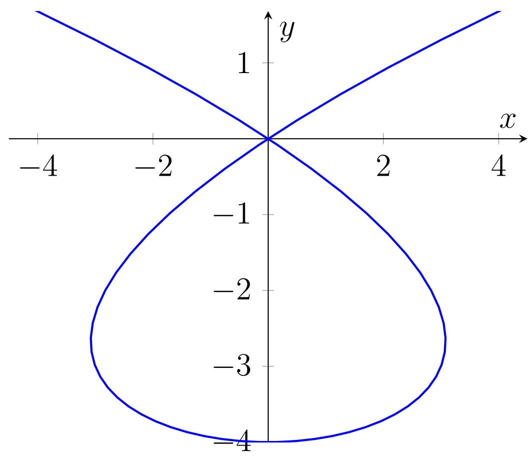
(ii) π is an open map.

Solution:

(a) We first deal with (i). The map $\alpha(t) = (t^3, t^2)$, $t \in \mathbb{R}$, is clearly smooth, but it is not an immersion, since its velocity vector $\alpha'(t) = (3t^2, 2t)$ vanishes at the point $t = 0$. Thus, α cannot be an embedding either.



We now deal with (ii). The map $\beta(t) = (t^3 - 4t, t^2 - 4)$, $t \in \mathbb{R}$, is clearly smooth and its velocity vector $\beta'(t) = (3t^2 - 4, 2t)$, $t \in \mathbb{R}$, is nowhere vanishing, so β is an immersion. However, the image curve $\beta(\mathbb{R})$ has a self-intersection for $t = -2$, $t = 2$, that is, $\beta(-2) = (0, 0) = \beta(2)$, and hence β cannot be an embedding.



(b)

(i) Let $p \in M$ and set $q = \pi(p) \in N$. Since π is a submersion, by the rank theorem there exist smooth charts (U, φ) centered at p and (V, ψ) centered at q such that $\pi(U) \subseteq V$ and π has the coordinate representation

$$\begin{aligned} \widehat{\pi} = \psi \circ \pi \circ \varphi^{-1} \colon \varphi(U) &\to \psi(V) \\ (x_1, \dots, x_m) &\mapsto (x_1, \dots, x_n), \end{aligned}$$

where $m = \dim M$ and $n = \dim N$ with $m \geq n$. Denote by $\iota \colon \mathbb{R}^n \to \mathbb{R}^m$ the map sending the n-tuple $(x_1, \dots, x_n)$ to the m-tuple $(x_1, \dots, x_n, 0, \dots, 0)$. Also, let $\varepsilon > 0$ be small enough so that $B^m(0, \varepsilon) \subseteq \varphi(U)$ and $B^n(0, \varepsilon) \subseteq \psi(V)$. We then define

$$\begin{aligned} \sigma \colon \psi^{-1}(B^n(0, \varepsilon)) &\to M \\ q' &\mapsto \varphi^{-1}(\iota(\psi(q'))). \end{aligned}$$

Note that this is well-defined: if $q' \in \psi^{-1}(B^n(0, \varepsilon))$, then $\psi(q') \in B^n(0, \varepsilon)$, and thus $\iota(\psi(q')) \in B^m(0, \varepsilon)$. Furthermore, we have

$$\pi \circ \sigma(q') = \psi^{-1} \circ \widehat{\pi} \circ \iota(\psi(q')) = \psi^{-1}(\psi(q')) = q',$$

where we used that $\widehat{\pi} \circ \iota = \mathrm{id}$.

(ii) Let $U \subset M$ be open. We need to show $\pi(U)$ is open in N. Take any $q \in \pi(U)$ and choose $p \in U$ with $\pi(p) = q$. By part (i) there exists a local section $\sigma \colon V \to M$ with $\sigma(q) = p$. Since σ is continuous, $\sigma^{-1}(U)$ is an open neighborhood of q in V. Then $\pi(U) \supset \pi(\sigma(\sigma^{-1}(U))) = \sigma^{-1}(U)$, which contains q. Thus, $\pi(U)$ contains a neighborhood of each of its points, so it is an open subset of N, as desired.

Exercise 4 (5 + 5 + 4 + 6 points)**:**

Consider the following three smooth vector fields on $\mathbb{R}^2$:

$$X = x\,\frac{\partial}{\partial x} + y\,\frac{\partial}{\partial y}\,, \quad Y = -y\,\frac{\partial}{\partial x} + x\,\frac{\partial}{\partial y}\,, \quad Z = (x^2+y^2)\,\frac{\partial}{\partial x} \in \mathfrak{X}(\mathbb{R}^2).$$

(a) Show that the map

$$\begin{aligned} \Phi\colon (0,+\infty)\times(-\pi,\pi) &\to \mathbb{R}^2 \\ (r,\theta) &\mapsto (r\cos\theta,\, r\sin\theta) \end{aligned}$$

is a diffeomorphism onto its image $U := \Phi(V)$, where $V := (0,+\infty)\times(-\pi,\pi)$, and compute the inverse Ψ of the restriction of Φ to the subset $(0,+\infty)\times\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ of V.

(b) Compute the coordinate representation of Z in polar coordinates on the right half-plane $M = \big\{(x,y)\in\mathbb{R}^2 \mid x>0\big\}$, that is, determine the pushforward $\Psi_*\big(Z|_M\big)$.

[You may use (without proof) the formula $\dfrac{d}{du}(\arctan u) = \dfrac{1}{1+u^2}$.]

(c) Compute the Lie brackets $[X,Z]$ and $[Y,Z]$.

(d) Compute the flow of

$$W = -x\,\frac{\partial}{\partial x} - y\,\frac{\partial}{\partial y} \in \mathfrak{X}(\mathbb{R}^2)$$

and examine whether it is a *complete* vector field on $\mathbb{R}^2$. Describe also geometrically the image of the maximal integral curve of W starting at the point $(-1,-1)\in\mathbb{R}^2$.

Solution:

(a) Geometrically, $r\in(0,+\infty)$ is the distance from the origin, and $\theta\in(-\pi,\pi)$ is the angle from the positive x-axis. Observe now that the image of Φ is the plane without the non-positive x-axis, that is,

$$U = \Phi(V) = \mathbb{R}^2 \setminus \big\{(x,y)\in\mathbb{R}^2 \mid x\le 0,\ y=0\big\}.$$

Indeed, if $(x,y)\in\mathbb{R}^2\setminus\{0\}$ is arbitrary, then for $r := \sqrt{x^2+y^2}$, the point $\frac{1}{r}(x,y)$ is on the unit circle $\mathbb{S}^1\subseteq\mathbb{R}^2$. Hence, there exists a unique $\theta\in(-\pi,\pi]$ such that $\frac{1}{r}(x,y) = \big(\cos(\theta),\, \sin(\theta)\big)$, so that $(x,y) = \big(r\cos(\theta),\, r\sin(\theta)\big)$. That is, there is a bijection between $(0,\infty)\times(-\pi,\pi]$ and $\mathbb{R}^2\setminus\{0\}$, given by the same formula as Φ. If we remove $(0,\infty)\times\{\pi\}$ from the domain (to obtain $(0,\infty)\times(-\pi,\pi)$), then we have to remove the set

$$\Big\{\big(r\cos(\pi),\, r\sin(\pi)\big) \mid r\in(0,\infty)\Big\} = \big\{(x,0) \mid x<0\big\}$$

from the target. Thus, the image of Φ is the set U described above, as claimed. Now, since $\Phi\colon V\to U$ is bijective and smooth with Jacobian determinant

$$\det\big(D\Phi(r,\theta)\big) = \det\begin{pmatrix}\cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta\end{pmatrix} = r\big(\cos^2\theta+\sin^2\theta\big) = r\neq 0,$$

by the *inverse function theorem* and *Proposition 4.9*(f) we conclude that Φ is a diffeomorphism.

Regarding now the restriction of Φ to $V' := (0, +\infty) \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \subset V$, note that its image is the right half-plane $M = \{(x, y) \in \mathbb{R}^2 \mid x > 0\}$. The inverse $\Psi\colon M \to V'$ of $\Phi|_{V'}$ is given by

$$\Psi(x, y) = \left(\sqrt{x^2 + y^2},\, \arctan\left(\frac{y}{x}\right)\right),$$

where arctan takes values in $(-\frac{\pi}{2}, \frac{\pi}{2})$, since $x > 0$.

(b) To compute $\Psi_*(Z|_M)$ directly using the definition of the pushforward, we need to compute $d\Psi_{(x,y)}(Z_{(x,y)})$ for each $(x, y) \in M$. Since $\Psi\colon M \to V'$ is a diffeomorphism, and $\Psi(x, y) = (r, \theta) = \left(\sqrt{x^2 + y^2}, \arctan(y/x)\right)$, we compute the differential $d\Psi_{(x,y)}$.

First, we compute the Jacobian matrix of Ψ:

$$d\Psi_{(x,y)} = \begin{pmatrix} \dfrac{\partial r}{\partial x}(x, y) & \dfrac{\partial r}{\partial y}(x, y) \\ \dfrac{\partial \theta}{\partial x}(x, y) & \dfrac{\partial \theta}{\partial y}(x, y) \end{pmatrix}.$$

Since $r = \sqrt{x^2 + y^2}$ and $\theta = \arctan(y/x)$ (for $x > 0$), we have:

$$\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}} = \cos\theta, \quad \frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}} = \sin\theta,$$

and using that $\frac{d}{du} \arctan u = \frac{1}{1+u^2}$ we also obtain:

$$\frac{\partial \theta}{\partial x} = \frac{1}{1 + (y/x)^2} \cdot \left(-\frac{y}{x^2}\right) = -\frac{y}{x^2 + y^2} = -\frac{\sin\theta}{r}$$

and

$$\frac{\partial \theta}{\partial y} = \frac{1}{1 + (y/x)^2} \cdot \frac{1}{x} = \frac{x}{x^2 + y^2} = \frac{\cos\theta}{r},$$

as $(x, y) = \Phi(r, \theta) = (r\cos\theta, r\sin\theta)$. Thus,

$$d\Psi_{(r\cos\theta, r\sin\theta)} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\dfrac{\sin\theta}{r} & \dfrac{\cos\theta}{r} \end{pmatrix}.$$

Now,

$$Z_{(x,y)} = (x^2 + y^2)\, \frac{\partial}{\partial x}\bigg|_{(x,y)} = r^2\, \frac{\partial}{\partial x}\bigg|_{(x,y)}.$$

To compute $d\Psi_{(x,y)}(Z_{(x,y)})$, we express $Z_{(x,y)}$ in terms of the basis $\left\{ \frac{\partial}{\partial x}\big|_{(x,y)}, \frac{\partial}{\partial y}\big|_{(x,y)} \right\}$ and apply the matrix:

$$d\Psi_{(x,y)}(Z_{(x,y)}) = d\Psi_{(x,y)}\left(r^2\, \frac{\partial}{\partial x}\bigg|_{(x,y)}\right) = r^2 \cdot d\Psi_{(x,y)}\left(\frac{\partial}{\partial x}\bigg|_{(x,y)}\right).$$

Since $d\Psi_{(x,y)}\left(\frac{\partial}{\partial x}\Big|_{(x,y)}\right)$ is the first column of the Jacobian matrix (as it gives the components with respect to $\left\{\frac{\partial}{\partial r}\Big|_{(r,\theta)}, \frac{\partial}{\partial \theta}\Big|_{(r,\theta)}\right\}$), we have:

$$d\Psi_{(x,y)}\left(\frac{\partial}{\partial x}\bigg|_{(x,y)}\right) = \cos\theta\,\frac{\partial}{\partial r}\bigg|_{(r,\theta)} - \frac{\sin\theta}{r}\,\frac{\partial}{\partial\theta}\bigg|_{(r,\theta)}.$$

Therefore,

$$d\Psi_{(x,y)}(Z_{(x,y)}) = r^2\left(\cos\theta\,\frac{\partial}{\partial r}\bigg|_{(r,\theta)} - \frac{\sin\theta}{r}\,\frac{\partial}{\partial\theta}\bigg|_{(r,\theta)}\right) = r^2\cos\theta\,\frac{\partial}{\partial r}\bigg|_{(r,\theta)} - r\sin\theta\,\frac{\partial}{\partial\theta}\bigg|_{(r,\theta)}.$$

This holds for all $(x,y)\in M$ with corresponding $(r,\theta)=\Psi(x,y)$. Hence,

$$\Psi_*(Z|_M) = r^2\cos\theta\,\frac{\partial}{\partial r} - r\sin\theta\,\frac{\partial}{\partial\theta}.$$

(c) We compute the Lie brackets using the explicit component formula; see *Proposition 7.13*. In coordinates (x,y) with $x^1 = x, x^2 = y$, we have:

$$X = X^1\frac{\partial}{\partial x} + X^2\frac{\partial}{\partial y} = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y},$$

$$Y = Y^1\frac{\partial}{\partial x} + Y^2\frac{\partial}{\partial y} = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y},$$

$$Z = Z^1\frac{\partial}{\partial x} + Z^2\frac{\partial}{\partial y} = (x^2+y^2)\frac{\partial}{\partial x} + 0\,\frac{\partial}{\partial y}.$$

$\rightsquigarrow$ *Computation of* $[X,Z]$: For each $j=1,2$, we compute the coefficient:

$$[X,Z]^j = \sum_{i=1}^{2}\left(X^i\frac{\partial Z^j}{\partial x^i} - Z^i\frac{\partial X^j}{\partial x^i}\right).$$

For $j=1$ (the $\frac{\partial}{\partial x}$ component):

$$\begin{aligned}
[X,Z]^1 &= X^1\frac{\partial Z^1}{\partial x} + X^2\frac{\partial Z^1}{\partial y} - Z^1\frac{\partial X^1}{\partial x} - Z^2\frac{\partial X^1}{\partial y}\\
&= x\frac{\partial}{\partial x}(x^2+y^2) + y\frac{\partial}{\partial y}(x^2+y^2) - (x^2+y^2)\frac{\partial}{\partial x}(x) - 0\cdot\frac{\partial}{\partial y}(x)\\
&= x(2x) + y(2y) - (x^2+y^2)(1) - 0\\
&= 2x^2 + 2y^2 - x^2 - y^2 = x^2+y^2.
\end{aligned}$$

For $j=2$ (the $\frac{\partial}{\partial y}$ component):

$$\begin{aligned}
[X,Z]^2 &= X^1\frac{\partial Z^2}{\partial x} + X^2\frac{\partial Z^2}{\partial y} - Z^1\frac{\partial X^2}{\partial x} - Z^2\frac{\partial X^2}{\partial y}\\
&= x\frac{\partial}{\partial x}(0) + y\frac{\partial}{\partial y}(0) - (x^2+y^2)\frac{\partial}{\partial x}(y) - 0\cdot\frac{\partial}{\partial y}(y)\\
&= 0 + 0 - (x^2+y^2)(0) - 0 = 0.
\end{aligned}$$

Thus,

$$[X, Z] = (x^2 + y^2)\,\frac{\partial}{\partial x} = Z.$$

⇝ *Computation of* $[Y, Z]$:

For $j = 1$:

$$\begin{aligned}
[Y, Z]^1 &= Y^1\,\frac{\partial Z^1}{\partial x} + Y^2\,\frac{\partial Z^1}{\partial y} - Z^1\,\frac{\partial Y^1}{\partial x} - Z^2\,\frac{\partial Y^1}{\partial y} \\
&= (-y)\,\frac{\partial}{\partial x}(x^2 + y^2) + x\,\frac{\partial}{\partial y}(x^2 + y^2) - (x^2 + y^2)\,\frac{\partial}{\partial x}(-y) - 0 \cdot \frac{\partial}{\partial y}(-y) \\
&= (-y)(2x) + x(2y) - (x^2 + y^2)(0) - 0 \\
&= -2xy + 2xy = 0.
\end{aligned}$$

For $j = 2$:

$$\begin{aligned}
[Y, Z]^2 &= Y^1\,\frac{\partial Z^2}{\partial x} + Y^2\,\frac{\partial Z^2}{\partial y} - Z^1\,\frac{\partial Y^2}{\partial x} - Z^2\,\frac{\partial Y^2}{\partial y} \\
&= (-y)\,\frac{\partial}{\partial x}(0) + x\,\frac{\partial}{\partial y}(0) - (x^2 + y^2)\,\frac{\partial}{\partial x}(x) - 0 \cdot \frac{\partial}{\partial y}(x) \\
&= 0 + 0 - (x^2 + y^2)(1) - 0 = -(x^2 + y^2).
\end{aligned}$$

Thus,

$$[Y, Z] = 0 \cdot \frac{\partial}{\partial x} - (x^2 + y^2)\,\frac{\partial}{\partial y} = -(x^2 + y^2)\,\frac{\partial}{\partial y}.$$

⇝ **Alternatively**, we compute the Lie brackets using their very definition $[V, W] = VW - WV$ on smooth functions.

First, for $[X, Z]$, we plug in a test function f:

$$X(Z(f)) - Z(X(f)) = X\left((x^2 + y^2)\frac{\partial f}{\partial x}\right) - Z\left(x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}\right).$$

First term:

$$\begin{aligned}
X\left((x^2 + y^2)\frac{\partial f}{\partial x}\right) &= \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)\left((x^2 + y^2)\frac{\partial f}{\partial x}\right) \\
&= x\left[2x\frac{\partial f}{\partial x} + (x^2 + y^2)\frac{\partial^2 f}{\partial x^2}\right] + y\left[2y\frac{\partial f}{\partial x} + (x^2 + y^2)\frac{\partial^2 f}{\partial x \partial y}\right] \\
&= 2x^2\frac{\partial f}{\partial x} + x(x^2 + y^2)\frac{\partial^2 f}{\partial x^2} + 2y^2\frac{\partial f}{\partial x} + y(x^2 + y^2)\frac{\partial^2 f}{\partial x \partial y}.
\end{aligned}$$

Second term:

$$\begin{aligned}
Z\left(x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}\right) &= (x^2 + y^2)\frac{\partial}{\partial x}\left(x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}\right) \\
&= (x^2 + y^2)\left[\frac{\partial f}{\partial x} + x\frac{\partial^2 f}{\partial x^2} + y\frac{\partial^2 f}{\partial x \partial y}\right].
\end{aligned}$$

Subtract:

$$[X, Z](f) = (2x^2 + 2y^2)\frac{\partial f}{\partial x} - (x^2 + y^2)\frac{\partial f}{\partial x} = (x^2 + y^2)\frac{\partial f}{\partial x} = Z(f).$$

Thus, $[X, Z] = Z$.

Second, for $[Y, Z]$, we plug in a test function f:

$$Y(Z(f)) - Z(Y(f)) = Y\left((x^2+y^2)\frac{\partial f}{\partial x}\right) - Z\left(-y\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial y}\right).$$

First term:

$$\begin{aligned} Y\left((x^2+y^2)\frac{\partial f}{\partial x}\right) &= \left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right)\left((x^2+y^2)\frac{\partial f}{\partial x}\right) \\ &= -y\left[2x\frac{\partial f}{\partial x} + (x^2+y^2)\frac{\partial^2 f}{\partial x^2}\right] + x\left[2y\frac{\partial f}{\partial x} + (x^2+y^2)\frac{\partial^2 f}{\partial x \partial y}\right] \\ &= -2xy\frac{\partial f}{\partial x} - y(x^2+y^2)\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial f}{\partial x} + x(x^2+y^2)\frac{\partial^2 f}{\partial x \partial y} \\ &= x(x^2+y^2)\frac{\partial^2 f}{\partial x \partial y} - y(x^2+y^2)\frac{\partial^2 f}{\partial x^2}. \end{aligned}$$

Second term:

$$\begin{aligned} Z\left(-y\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial y}\right) &= (x^2+y^2)\frac{\partial}{\partial x}\left(-y\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial y}\right) \\ &= (x^2+y^2)\left[-y\frac{\partial^2 f}{\partial x^2} + \frac{\partial f}{\partial y} + x\frac{\partial^2 f}{\partial x \partial y}\right]. \end{aligned}$$

Subtract:

$$[Y, Z](f) = (x^2+y^2)\left(-\frac{\partial f}{\partial y}\right) = -(x^2+y^2)\frac{\partial f}{\partial y}.$$

Thus, $[Y, Z] = -(x^2+y^2)\frac{\partial}{\partial y}$.

(d) For the computation of its integral curves, the vector field

$$W = -x\,\frac{\partial}{\partial x} - y\,\frac{\partial}{\partial y} \in \mathfrak{X}(\mathbb{R}^2)$$

yields the following system of ODEs:

$$\gamma_1'(t) = -\gamma_1(t), \quad \gamma_2'(t) = -\gamma_2(t).$$

The solution with initial condition (x_0, y_0) at $t = 0$ is thus

$$\gamma(t) = \left(x_0 e^{-t},\, y_0 e^{-t}\right),\ t \in \mathbb{R}.$$

Thus, the flow of W is described by the map $\theta_t(x, y) = (xe^{-t},\, ye^{-t})$, defined for all $t \in \mathbb{R}$ and all $(x, y) \in \mathbb{R}^2$. Hence, W is *complete*.

For the initial point $(-1, -1) \in \mathbb{R}^2$, the unique maximal integral curve of W is

$$\gamma(t) = (-e^{-t}, -e^{-t}), \quad t \in \mathbb{R}.$$

Geometrically, it is the ray from the origin $(0, 0)$ through $(-1, -1)$, but parametrized backwards (approaching but not reaching the origin as t increases).

Exercise 5 (5 + 4 + 4 points):

Consider the smooth map

$$F\colon \mathbb{R}^2 \to \mathbb{R}^3,\ (\theta,\varphi) \mapsto \big((\cos\varphi + 2)\cos\theta,\ (\cos\varphi+2)\sin\theta,\ \sin\varphi\big),$$

the smooth 1-form

$$\eta = z^2\, dx \in \Omega^1(\mathbb{R}^3),$$

and the smooth 2-form

$$\omega = y\, dz \wedge dx \in \Omega^2(\mathbb{R}^3).$$

Note that we denote the standard coordinates on the source by (θ,φ) and the standard coordinates on the target by (x, y, z).

(a) Compute the Jacobian matrix $J_F(\theta,\varphi)$ of F and determine the rank of F at every point $(\theta,\varphi) \in \mathbb{R}^2$.

(b) Compute $d\eta$ and $F^*\eta$, and verify by direct computation that $d(F^*\eta) = F^*(d\eta)$.

(c) Compute $d\omega$ and $F^*\omega$, and verify by direct computation that $d(F^*\omega) = F^*(d\omega)$.

Solution:

(a) The Jacobian matrix of F is:

$$J_F(\theta,\varphi) = \begin{pmatrix} -(\cos\varphi+2)\sin\theta & -\sin\varphi\cos\theta \\ (\cos\varphi+2)\cos\theta & -\sin\varphi\sin\theta \\ 0 & \cos\varphi \end{pmatrix}.$$

We determine the rank by checking the linear independence of the two column vectors:

$$v_1 = \frac{\partial F}{\partial\theta} = \begin{pmatrix} -(\cos\varphi+2)\sin\theta \\ (\cos\varphi+2)\cos\theta \\ 0 \end{pmatrix}, \quad v_2 = \frac{\partial F}{\partial\varphi} = \begin{pmatrix} -\sin\varphi\cos\theta \\ -\sin\varphi\sin\theta \\ \cos\varphi \end{pmatrix}.$$

The rank of J_F is 2 if v_1 and v_2 are linearly independent. Suppose that there exist scalars $\alpha, \beta \in \mathbb{R}$ such that $\alpha v_1 + \beta v_2 = 0$. This gives the system:

$$\alpha\big[-(\cos\varphi+2)\sin\theta\big] + \beta\big[-\sin\varphi\cos\theta\big] = 0, \tag{1}$$
$$\alpha\big[(\cos\varphi+2)\cos\theta\big] + \beta\big[-\sin\varphi\sin\theta\big] = 0, \tag{2}$$
$$\alpha\cdot 0 + \beta\cos\varphi = 0. \tag{3}$$

From equation (3), we have $\beta\cos\varphi = 0$. We consider two cases:

Case 1: $\cos\varphi \neq 0$. Then $\beta = 0$. Substituting into (1) and (2) gives:

$$\alpha(\cos\varphi+2)\sin\theta = 0, \qquad \alpha(\cos\varphi+2)\cos\theta = 0.$$

Since $\cos\varphi + 2 > 0$ (as $\cos\varphi \geq -1$), we have $\alpha\sin\theta = 0$ and $\alpha\cos\theta = 0$. But $\sin\theta$ and $\cos\theta$ cannot both vanish for any $\theta\in\mathbb{R}$, so $\alpha = 0$. Hence, v_1 and v_2 are linearly independent in this case.

Case 2: $\cos\varphi = 0$. Then $\sin\varphi = \pm 1$. Equation (3) gives no constraint on β. With $\cos\varphi = 0$, we have $\cos\varphi + 2 = 2$. System (1)–(2) becomes:

$$\begin{aligned} -2\alpha\sin\theta \mp \beta\cos\theta &= 0, \\ 2\alpha\cos\theta \mp \beta\sin\theta &= 0, \end{aligned}$$

where the sign $\mp$ corresponds to $\sin\varphi = \pm 1$. We treat this as a linear system in α, β. The determinant of the coefficient matrix is:

$$\begin{aligned} \det\begin{pmatrix} -2\sin\theta & \mp\cos\theta \\ 2\cos\theta & \mp\sin\theta \end{pmatrix} &= (-2\sin\theta)(\mp\sin\theta) - (\mp\cos\theta)(2\cos\theta) = \pm(2\sin^2\theta + 2\cos^2\theta) \\ &= \pm 2 \neq 0. \end{aligned}$$

Thus, the only solution is $\alpha = \beta = 0$, so v_1 and v_2 are again linearly independent.

In both cases, the columns are linearly independent. Therefore, the rank of F is 2 at every point $(\theta, \varphi) \in \mathbb{R}^2$.

(b) First, we have

$$d\eta = d(z^2\, dx) = 2z\, dz \wedge dx = -2z\, dx \wedge dz.$$

Now, we compute $F^*\eta$. We have

$$F^*\eta = \sin^2\varphi\, d\left((\cos\varphi + 2)\cos\theta\right),$$

where

$$\begin{aligned} d\left((\cos\varphi + 2)\cos\theta\right) &= \frac{\partial}{\partial\theta}\left((\cos\varphi + 2)\cos\theta\right) d\theta + \frac{\partial}{\partial\varphi}\left((\cos\varphi + 2)\cos\theta\right) d\varphi \\ &= -(\cos\varphi + 2)\sin\theta\, d\theta - \cos\theta\sin\varphi\, d\varphi. \end{aligned}$$

Thus,

$$\begin{aligned} F^*\eta &= \sin^2\varphi\left[-(\cos\varphi + 2)\sin\theta\, d\theta - \cos\theta\sin\varphi\, d\varphi\right] \\ &= -(\cos\varphi + 2)\sin^2\varphi\sin\theta\, d\theta - \sin^3\varphi\cos\theta\, d\varphi. \end{aligned}$$

Next, we compute $d(F^*\eta)$. Writing $F^*\eta = A\, d\theta + B\, d\varphi$ with

$$A = -(\cos\varphi + 2)\sin^2\varphi\sin\theta \quad \text{and} \quad B = -\sin^3\varphi\cos\theta,$$

we have

$$d(F^*\eta) = \left(\frac{\partial B}{\partial\theta} - \frac{\partial A}{\partial\varphi}\right) d\theta \wedge d\varphi.$$

We compute:

$$\frac{\partial B}{\partial\theta} = \frac{\partial}{\partial\theta}\left(-\sin^3\varphi\cos\theta\right) = \sin^3\varphi\sin\theta,$$

and

$$\begin{aligned} \frac{\partial A}{\partial\varphi} &= \frac{\partial}{\partial\varphi}\left[-(\cos\varphi + 2)\sin^2\varphi\sin\theta\right] \\ &= -\sin\theta\frac{\partial}{\partial\varphi}\left[(\cos\varphi + 2)\sin^2\varphi\right] \\ &= -\sin\theta\left[(-\sin\varphi)\sin^2\varphi + (\cos\varphi + 2)\cdot 2\sin\varphi\cos\varphi\right] \\ &= -\sin\theta\left[-\sin^3\varphi + 2(\cos\varphi + 2)\sin\varphi\cos\varphi\right] \\ &= \sin\theta\sin^3\varphi - 2(\cos\varphi + 2)\sin\varphi\cos\varphi\sin\theta. \end{aligned}$$

Hence,

$$\begin{aligned}\frac{\partial B}{\partial \theta} - \frac{\partial A}{\partial \varphi} &= \sin^3\varphi \sin\theta - \big(\sin\theta \sin^3\varphi - 2(\cos\varphi + 2)\sin\varphi\cos\varphi\sin\theta\big) \\ &= 2(\cos\varphi + 2)\sin\varphi\cos\varphi\sin\theta,\end{aligned}$$

which yields

$$d(F^*\eta) = 2(\cos\varphi + 2)\sin\varphi\cos\varphi\sin\theta\, d\theta \wedge d\varphi.$$

Finally, we compute $F^*(d\eta) = F^*(2z\, dz \wedge dx) = 2\sin\varphi\, F^*(dz \wedge dx)$. We have

$$F^* dz = d(\sin\varphi) = \cos\varphi\, d\varphi$$

and

$$F^* dx = d\left((\cos\varphi + 2)\cos\theta\right) = -(\cos\varphi + 2)\sin\theta\, d\theta - \cos\theta\sin\varphi\, d\varphi.$$

Then

$$\begin{aligned}F^*(dz \wedge dx) &= (F^* dz) \wedge (F^* dx) = (\cos\varphi\, d\varphi) \wedge (-(\cos\varphi + 2)\sin\theta\, d\theta - \cos\theta\sin\varphi\, d\varphi) \\ &= -\cos\varphi(\cos\varphi + 2)\sin\theta\, d\varphi \wedge d\theta - \cos\varphi\cos\theta\sin\varphi\, d\varphi \wedge d\varphi \\ &= -\cos\varphi(\cos\varphi + 2)\sin\theta\, d\varphi \wedge d\theta \\ &= \cos\varphi(\cos\varphi + 2)\sin\theta\, d\theta \wedge d\varphi \quad (\text{since } d\varphi \wedge d\theta = -d\theta \wedge d\varphi).\end{aligned}$$

Therefore,

$$\begin{aligned}F^*(d\eta) &= 2\sin\varphi \cdot \cos\varphi(\cos\varphi + 2)\sin\theta\, d\theta \wedge d\varphi \\ &= 2(\cos\varphi + 2)\sin\varphi\cos\varphi\sin\theta\, d\theta \wedge d\varphi.\end{aligned}$$

Comparing the above formulas, we see that $d(F^*\eta) = F^*(d\eta)$, as required by the commutation of the pullback with the exterior derivative.

(c) First, we have

$$d\omega = d(y\, dz \wedge dx) = dy \wedge dz \wedge dx = dx \wedge dy \wedge dz.$$

Now, we compute $F^*\omega = F^*(y\, dz \wedge dx)$. We have

$$F^*\omega = (\cos\varphi + 2)\sin\theta\, d(\sin\varphi) \wedge d\left((\cos\varphi + 2)\cos\theta\right).$$

We already computed

$$d(\sin\varphi) = \cos\varphi\, d\varphi,$$

and

$$d\left((\cos\varphi + 2)\cos\theta\right) = -\cos\theta\sin\varphi\, d\varphi - (\cos\varphi + 2)\sin\theta\, d\theta,$$

so we obtain

$$\begin{aligned}F^*\omega &= (\cos\varphi + 2)\sin\theta\, (\cos\varphi\, d\varphi) \wedge (-\cos\theta\sin\varphi\, d\varphi - (\cos\varphi + 2)\sin\theta\, d\theta) \\ &= (\cos\varphi + 2)\sin\theta\, [-(\cos\varphi + 2)\sin\theta\cos\varphi\, d\varphi \wedge d\theta] \\ &= (\cos\varphi + 2)^2\sin^2\theta\cos\varphi\, d\theta \wedge d\varphi.\end{aligned}$$

To verify that $d(F^*\omega)$ and $F^*(d\omega)$ agree, observe simply that both are 3-forms on $\mathbb{R}^2$, and therefore we have

$$d(F^*\omega) = 0 = F^*(d\omega).$$

Exercise 6 (6 + 2 + 5 points)**:** Let n be a positive integer.

(a) Recall that the *length* of a smooth curve segment $\gamma\colon [a,b] \to \mathbb{R}^n$, where $n > 0$, is defined to be the value of the (ordinary) integral

$$L(\gamma) := \int_a^b \left|\gamma'(t)\right| dt.$$

Show that there is no smooth covector field ω on $\mathbb{R}^n$ with the property that

$$\int_\gamma \omega = L(\gamma)$$

for every smooth curve segment γ in $\mathbb{R}^n$.

(b) Show that the integral of an exact n-form over a compact, oriented, smooth n-manifold without boundary is zero.

(c) Examine whether the smooth covector field

$$\eta = -\frac{4xz}{(x^2+1)^2}\,dx + \frac{2y}{y^2+1}\,dy + \frac{2}{x^2+1}\,dz \in \Omega^1(\mathbb{R}^3)$$

is exact. If so, then find a potential function for η. In any case, compute the line integral along the straight line segment from $(0,0,0)$ to $(1,1,1)$.

Solution:

(a) *First approach*: Suppose that such an ω exists. Fix $p \in \mathbb{R}^n$ and $v \in \mathbb{R}^n$. Consider the straight line segment

$$\gamma_\varepsilon\colon [0,\varepsilon] \to \mathbb{R}^n,\ \ \gamma_\varepsilon(t) = p + tv.$$

Then

$$L(\gamma_\varepsilon) = \int_0^\varepsilon |v|\,dt = \varepsilon|v|.$$

On the other hand,

$$\int_{\gamma_\varepsilon} \omega = \int_0^\varepsilon \omega_{\gamma_\varepsilon(t)}(\gamma_\varepsilon'(t))\,dt = \int_0^\varepsilon \omega_{p+tv}(v)\,dt,$$

where we used the natural identification $\mathbb{R}^n \cong T_{p+tv}\mathbb{R}^n$. By assumption, the above two quantities are equal for all $\varepsilon > 0$. Differentiating both with respect to ε, we obtain

$$|v| = \omega_{p+tv}(v)$$

for all $t > 0$. By continuity of ω, we can take the limit $t \to 0$ and obtain

$$|v| = \omega_p(v).$$

This must hold for all $v \in \mathbb{R}^n$. But ω_p is linear in v, while $|v|$ is not (for example, $|-v| = |v|$, but $\omega_p(-v) = -\omega_p(v)$). This is a contradiction. Hence, no such ω exists.

Second approach: Assume on the contrary that there exists such a smooth covector field ω on $\mathbb{R}^n$ with the above property. Consider a (non-constant) smooth curve segment

$$\gamma\colon [a,b] \to \mathbb{R}^n,$$

the decreasing diffeomorphism

$$\varphi\colon [a,b] \to [a,b],\ t \mapsto b-t+a,$$

and the backward reparametrization $\widetilde{\gamma} := \gamma \circ \varphi$ of γ. Note that

$$L(\gamma) = L(\widetilde{\gamma}) > 0,$$

so by assumption and by *Proposition 11.7* we obtain

$$L(\widetilde{\gamma}) = \int_{\widetilde{\gamma}} \omega = -\int_{\gamma} \omega = -L(\gamma) < 0,$$

which is absurd.

(b) Let M be a compact, oriented, smooth n-manifold without boundary, and let $\omega = d\eta$ be an exact n-form on M. By Stokes' theorem, we have

$$\int_M d\eta = \int_{\partial M} \eta = 0,$$

since $\partial M = \emptyset$.

(c) We examine whether η is exact by attempting to find a smooth function $f\colon \mathbb{R}^3 \to \mathbb{R}$ such that $df = \eta$. If such an f exists, then η is exact. To this end, we need to solve the system:

$$\frac{\partial f}{\partial x} = -\frac{4xz}{(x^2+1)^2}, \qquad \frac{\partial f}{\partial y} = \frac{2y}{y^2+1}, \qquad \frac{\partial f}{\partial z} = \frac{2}{x^2+1}.$$

Start with the z–component:

$$\frac{\partial f}{\partial z} = \frac{2}{x^2+1}.$$

Integrating with respect to z (treating x and y as constants), we obtain:

$$f(x,y,z) = \frac{2z}{x^2+1} + g(x,y),$$

where g is a smooth function of x and y. Now use the y–component:

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial y} = \frac{2y}{y^2+1}.$$

Integrating thus with respect to y, we obtain

$$g(x,y) = \int \frac{2y}{y^2+1}\,dy = \ln(y^2+1) + h(x),$$

where h is a smooth function of x. Thus,

$$f(x,y,z) = \frac{2z}{x^2+1} + \ln(y^2+1) + h(x).$$

Finally, use the x–component:

$$\frac{\partial f}{\partial x} = 2z \cdot \frac{d}{dx}\left(\frac{1}{x^2+1}\right) + h'(x) = 2z \cdot \left(-\frac{2x}{(x^2+1)^2}\right) + h'(x) = -\frac{4xz}{(x^2+1)^2} + h'(x).$$

We require that this be equal to $-\frac{4xz}{(x^2+1)^2}$, hence

$$h'(x) = 0 \quad \Longrightarrow \quad h(x) = C,$$

where $C \in \mathbb{R}$ is a constant. Choosing $C = 0$, we obtain the following potential function for η:

$$f(x, y, z) = \frac{2z}{x^2+1} + \ln(y^2+1).$$

In conclusion, η is exact.

The line integral along the straight line segment $\gamma(t) = (t, t, t)$, $t \in [0, 1]$, from $(0, 0, 0)$ to $(1, 1, 1)$ equals

$$\int_\gamma \eta = f(1,1,1) - f(0,0,0) = \left(\frac{2}{1^2+1} + \ln(1^2+1)\right) - \left(\frac{2 \cdot 0}{0^2+1} + \ln(0^2+1)\right) = 1 + \ln 2,$$

by the Fundamental Theorem for line integrals.