Differential Geometry II - Smooth Manifolds
Winter Term 2023/2024
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Exam – Solutions

Exercise 1 (5 + 7 + 5 points)**:**

(a) Define the notion of a *smooth n-manifold* (without boundary). In particular, explain all the terms appearing in the definition.

(Note: you do not have to define the terms *topological space*, *basis* of a topological space, and the space $\mathbb{R}^n$ with its standard topology.)

(b) Let M be a smooth n-manifold (without boundary), let $p \in M$ and let (U, φ) be a smooth coordinate chart for M such that $p \in U$.

(i) What is the tangent space T_pM to M at p? Write down a basis for T_pM and explain how any of those basis elements acts on a function $f \in C^\infty(M)$.

(ii) What is the cotangent space T_p^*M to M at p? Write down a basis for its k-th exterior power $\Lambda^k\left(T_p^*M\right)$, where $1 \leq k \leq n$?

(c) Define the notion of a *smooth map* between smooth manifolds (without boundary). What is its *rank* (at a point) and when does one say that it has *constant rank*? Define also the notions of *smooth immersions* and *smooth submersions*. Finally, formulate the *rank theorem* (in the setting of smooth manifolds).

Solution:

(a) Let $n \in \mathbb{N}$. A *topological manifold of dimension n* is a topological space $M = (M, \mathcal{T})$ such that:

- M is *second-countable*, i.e., $\mathcal{T}$ admits a countable basis,
- M is *Hausdorff*, i.e., for all points $p, q \in M$ with $p \neq q$, there exist open subsets $U, V \subseteq M$ such that $p \in U$, $q \in V$ and $U \cap V = \emptyset$,
- M is *locally Euclidean of dimension n*, i.e., for every point $p \in M$ there exists an open subset $U \subseteq M$ containing p, an open subset $\hat{U} \subseteq \mathbb{R}^n$ and a homeomorphism $\varphi \colon U \to \hat{U}$. The pair (U, φ) is called a *(coordinate) chart* for M.

Let M be a topological manifold of dimension n. An *atlas* for M is a collection $\mathcal{A}$ of charts (U, φ) for M such that

$$M = \bigcup_{(U,\varphi)\in\mathcal{A}} U = M.$$

An atlas $\mathcal{A}$ for M is called *smooth* if for all charts $(U, \varphi), (V, \psi) \in \mathcal{A}$, the *transition map*

$$\psi \circ \varphi^{-1} \colon \varphi(U \cap V) \to \psi(U \cap V)$$

is smooth (C^∞) as a map between open subsets of $\mathbb{R}^n$ (i.e., all its partial derivatives exist and are continuous). Now, a smooth atlas $\mathcal{A}$ for M is called *maximal* if it is not strictly contained in any larger smooth atlas. Finally, an *n-dimensional smooth manifold* is a pair $(M, \mathcal{A})$, where M is a topological n-manifold and $\mathcal{A}$ is a maximal smooth atlas for M.

(b) We denote by $(x^1, \dots, x^n)$ the coordinate functions of φ.

(b)(i) Recall first that a *derivation* D at p is a map $D \colon C^\infty(M) \to \mathbb{R}$ which is $\mathbb{R}$-linear and satisfies the Leibniz rule at p, i.e., for all $f, g \in C^\infty(M)$ we have

$$D(fg) = f(p)\, D(g) + g(p)\, D(f).$$

The tangent space to M at p is defined as the set of all derivations of $C^\infty(M)$ at p, that is,

$$T_pM \coloneqq \{D \colon C^\infty(M) \to \mathbb{R} \mid D \text{ is a derivation at } p\},$$

and it is an $\mathbb{R}$-vector space. A basis for T_pM is given by the derivations $\frac{\partial}{\partial x^i}\big|_p$, $1 \le i \le n$, which are characterized by either of the following expressions:

$$\left.\frac{\partial}{\partial x^i}\right|_p = (d\varphi_p)^{-1}\left(\left.\frac{\partial}{\partial x^i}\right|_{\varphi(p)}\right) = d\big(\varphi^{-1}\big)_{\varphi(p)}\left(\left.\frac{\partial}{\partial x^i}\right|_{\varphi(p)}\right).$$

Here, the elements $\frac{\partial}{\partial x^i}\big|_{\varphi(p)} \in T_{\varphi(p)}\varphi(U)$ correspond to the usual partial derivatives and they form a basis for $T_{\varphi(p)}\varphi(U) \cong T_{\varphi(p)}\mathbb{R}^n \cong \mathbb{R}^n$. Thus, given $f \in C^\infty(U)$, each derivation $\frac{\partial}{\partial x^i}\big|_p$ acts on f as follows:

$$\left.\frac{\partial}{\partial x^i}\right|_p f = \left.\frac{\partial}{\partial x^i}\right|_{\varphi(p)} \big(f \circ \varphi^{-1}\big) = \frac{\partial(f \circ \varphi^{-1})}{\partial x^i}\,(\varphi(p))\,;$$

in other words, the derivation $\frac{\partial}{\partial x^i}\big|_p$ takes the i-th partial derivative of the coordinate representation $f \circ \varphi^{-1}$ of f at the coordinate representation $\varphi(p)$ of $p \in U$.

(b)(ii) *The cotangent space T_p^*M to M at p* is the dual of T_pM, i.e., the vector space of $\mathbb{R}$-linear maps $T_pM \to \mathbb{R}$. A basis for T_p^*M is given by the dual basis $(dx^i)_{1\le i\le n}$ to $\left(\frac{\partial}{\partial x^i}\big|_p\right)_{1\le i\le n}$; explicitly, the i-th basis element dx^i is defined by

$$dx^i\left(\left.\frac{\partial}{\partial x^j}\right|_p\right) = \delta_{ij}.$$

More generally, a basis for $\Lambda^k(T_p^*M)$ is given as follows: for an increasing multi-index $I = (i_1, \dots, i_k)$ (i.e., $1 \leq i_1 < \cdots < i_k \leq n$), we define

$$dx^I := dx^{i_1} \wedge \ldots \wedge dx^{i_k},$$

and if $\mathfrak{I}_k$ denotes the set of increasing multi-indices of length k, then the set $(dx^I)_{I \in \mathfrak{I}_k}$ is a basis for $\Lambda^k(T_p^*M)$.

(c) A map $F\colon M \to N$ between smooth manifolds is called *smooth* if for every $p \in M$ there exist smooth charts (U, φ) for M containing p and (V, ψ) for N containing $F(p)$ such that $F(U) \subseteq V$ and the composite map

$$\psi \circ F \circ \varphi^{-1}\colon \varphi(U) \to \psi(V)$$

is smooth (as a map between open subsets of Euclidean spaces).

The *rank of F at a point* $p \in M$ is defined to be the rank of its differential at p, i.e.,

$$\operatorname{rk}_p F := \operatorname{rk} dF_p = \dim_{\mathbb{R}} \operatorname{Im} dF_p,$$

or equivalently, the rank of the Jacobian matrix of F in any smooth chart. If F has the same rank r at every point $p \in M$, then we say that F has *constant rank r*.

Smooth immersions are smooth maps whose differential is everywhere injective, and *smooth submersions* are smooth maps whose differential is everywhere surjective. (They constitute two classes of smooth maps of constant rank.) Finally, let us formulate:

The rank theorem: Let M and N be smooth manifolds (without boundary) of dimension m and n, respectively, and let $F\colon M \to N$ be a smooth map of constant rank r. For each $p \in M$ there exist smooth charts (U, φ) for M centered at p and (V, ψ) for N centered at $F(p)$ such that $F(U) \subseteq V$, in which F has a coordinate representation of the form

$$\widehat{F}\left(x^1, \dots, x^r, x^{r+1}, \dots, x^m\right) = \left(x^1, \dots, x^r, 0, \dots, 0\right).$$

In particular, if F is a smooth immersion, resp. a smooth submersion, then this becomes

$$\widehat{F}\left(x^1, \dots, x^m\right) = \left(x^1, \dots, x^m, 0, \dots, 0\right),$$

resp.

$$\widehat{F}\left(x^1, \dots, x^n, x^{n+1}, \dots, x^m\right) = \left(x^1, \dots, x^n\right).$$

Exercise 2 (1 + 9 points)**:**

(a) Let M be a smooth n-manifold and let (U, φ) be a smooth coordinate chart for M. Write down a smooth coordinate chart for TM with domain $\pi^{-1}(U)$, where $\pi\colon TM \to M$ is the tangent bundle of M.

(b) Let U be an open subset of $\mathbb{R}^n$ and let $f\colon U \to \mathbb{R}^k$ be a smooth function. Define the *graph* $\Gamma(f)$ of f as a subset of $\mathbb{R}^n \times \mathbb{R}^k$ and give it the subspace topology. Show that it can be endowed with the structure of a smooth n-manifold (without boundary) and determine its tangent bundle $T\big(\Gamma(f)\big)$ up to diffeomorphism.

Solution:

(a) Denote by $(x^1, \dots, x^n)$ the coordinate functions of φ and consider the coordinate basis

$$\left\{ \left.\frac{\partial}{\partial x^i}\right|_p \right\}_{1 \leq i \leq n}$$

for T_pM at $p \in U$. We have seen that the map

$$\begin{aligned} \widetilde{\varphi}\colon \pi^{-1}(U) &\to \mathbb{R}^{2n} \\ v^i \left.\frac{\partial}{\partial x^i}\right|_p &\mapsto \big(x^1(p), \dots, x^n(p), v^1, \dots, v^n\big) \end{aligned}$$

is a smooth coordinate chart for TM with image $\varphi(U) \times \mathbb{R}^n$.

(b) The graph $\Gamma(f)$ of f is defined as

$$\Gamma(f) \coloneqq \big\{ \big(x, f(x)\big) \in \mathbb{R}^n \times \mathbb{R}^k \mid x \in U \big\}$$

and we endow it with the subspace topology of $\mathbb{R}^n \times \mathbb{R}^k$. There are continuous maps

$$\begin{aligned} F\colon U &\to \Gamma(f) \\ x &\mapsto \big(x, f(x)\big) \end{aligned}$$

and

$$\begin{aligned} G\colon \Gamma(f) &\to U \\ \big(x, f(x)\big) &\mapsto x, \end{aligned}$$

which are mutually inverse. Indeed, F is obtained by co-restricting the product of the continuous maps $\mathrm{Id}_U\colon U \to U$ and $f\colon U \to \mathbb{R}^k$ to its image $\Gamma(f)$, and G is obtained by restricting and co-restricting the continuous map $\mathrm{pr}_1\colon \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n$ to U and $\Gamma(f)$. Therefore, $\Gamma(f)$ is homeomorphic to U; in particular, $\Gamma(f)$ is a topological n-manifold, and $\big(\Gamma(f), G\big)$ is a global coordinate chart for $\Gamma(f)$. Hence, it induces a smooth structure on $\Gamma(f)$ by *Proposition 1.6*. It follows that $\Gamma(f)$ with this smooth structure is diffeomorphic to U. Finally, by part (a) we infer that

$$T\big(\Gamma(f)\big) \cong U \times \mathbb{R}^n \cong \Gamma(f) \times \mathbb{R}^n;$$

in other words, the tangent bundle of $\Gamma(f)$ is a trivial smooth vector bundle of rank n.

Exercise 3 (8 + 8 points)**:**

(a) Let M be a non-empty, compact, smooth manifold. Show that there exists no smooth submersion $F\colon M \to \mathbb{R}^k$ for any $k \in \mathbb{Z}_{\geq 1}$.

(b) For each $\lambda \in \mathbb{R}$, consider the set

$$M_\lambda := \left\{(x, y) \in \mathbb{R}^2 \mid y^2 = x(x - \lambda)\right\}.$$

For which values of λ is M_λ a properly embedded submanifold of $\mathbb{R}^2$?

Solution:

(a) Assume by contradiction that there exist a positive integer k and a smooth submersion $F\colon M \to \mathbb{R}^k$. Since M is compact and F is continuous, the image $F(M) \subseteq \mathbb{R}^k$ is also compact, and since $\mathbb{R}^k$ is Hausdorff, we infer that $F(M)$ is a closed subset of $\mathbb{R}^k$. On the other hand, since F is a smooth submersion, it is an open map by *Proposition 4.11*, so the image $F(M)$ is an open subset of $\mathbb{R}^k$. But since $\mathbb{R}^k$ is connected and $M \neq \emptyset$, it follows that $F(M) = \mathbb{R}^k$, which implies that $\mathbb{R}^k$ is compact, a contradiction.

(b) Fix $\lambda \in \mathbb{R}$, consider the smooth function

$$f_\lambda\colon \mathbb{R}^2 \to \mathbb{R},\ (x, y) \mapsto y^2 - x^2 + \lambda x$$

and observe that $M_\lambda = f_\lambda^{-1}(0)$. We have

$$\operatorname{grad}(f_\lambda)(x, y) = \left(\frac{\partial f_\lambda}{\partial x}(x, y),\ \frac{\partial f_\lambda}{\partial y}(x, y)\right) = (-2x + \lambda,\ 2y),$$

and thus

$$\operatorname{grad}(f_\lambda)(x, y) = (0, 0) \iff (x, y) = \left(\frac{\lambda}{2}, 0\right).$$

We now distinguish two cases.

Case I: If $\lambda \neq 0$, then $\left(\frac{\lambda}{2}, 0\right) \notin f_\lambda^{-1}(0)$, so the row matrix $\operatorname{grad}(f_\lambda)(x, y)$ has rank 1 for every $(x, y) \in f_\lambda^{-1}(0)$. This means that $0 \in \mathbb{R}$ is a regular value of f_λ, and hence $M_\lambda = f_\lambda^{-1}(0)$ is a properly embedded 1-submanifold of $\mathbb{R}^2$ according to the *regular level set theorem.*

Case II: If $\lambda = 0$, then $0 \in \mathbb{R}$ is a critical value of f_0, since $(0, 0) \in M_0 = f_0^{-1}(0)$ and $\operatorname{grad}(f_0)(0, 0) = (0, 0)$, so now the *regular level set theorem* does not give any information about $f_0^{-1}(0)$. However, we observe that the level set

$$M_0 = f_0^{-1}(0) = \left\{(x, y) \in \mathbb{R}^2 \mid y^2 = x^2\right\}$$

is the union of the lines $y = x$ and $y = -x$ in the plane $\mathbb{R}^2$, and we have already seen in part (b) of *Exercise* 3, *Sheet* 8 that M_0 is not an embedded submanifold of $\mathbb{R}^2$, since it is not even a topological manifold (with the subspace topology inherited from $\mathbb{R}^2$).

The regular level set theorem: Every regular level set of a smooth map between smooth manifolds is a properly embedded submanifold whose codimension is equal to the dimension of the codomain.

Exercise 4 (3 + 6 + 5 + 4 points)**:**
Consider the open submanifold

$$M := \{(x, y) \in \mathbb{R}^2 \mid x > 0,\ y > 0\} \subseteq \mathbb{R}^2,$$

the map

$$F\colon M \to M,\ (x, y) \mapsto \left(xy, \frac{y}{x}\right),$$

and the smooth vector fields

$$X := x\,\frac{\partial}{\partial x} + y\,\frac{\partial}{\partial y} \quad \text{and} \quad Y := y\,\frac{\partial}{\partial x}$$

on M.

(a) Show that F is a diffeomorphism, compute its Jacobian matrix $DF(x, y)$ at an arbitrary point $(x, y) \in M$, and determine its inverse F^{-1}.

(b) Compute the pushforwards F_*X and F_*Y of X and Y, respectively.

(c) Compute the Lie brackets $[X, Y]$ and $[F_*X, F_*Y]$.

(d) Find the maximal integral curve of Y starting at the point $(1, 1) \in M$ and describe its image geometrically.

Solution:

(a) Define the map

$$G\colon M \to M,\ (x, y) \mapsto \left(\sqrt{x/y}, \sqrt{xy}\right)$$

and note that it is smooth, as the functions $(x, y) \in M \mapsto x/y \in \mathbb{R}_{>0}$ and $(x, y) \in M \mapsto xy \in \mathbb{R}_{>0}$ are smooth (they are rational polynomials with non-vanishing denominator), and $u \in \mathbb{R}_{>0} \mapsto \sqrt{u} \in \mathbb{R}_{>0}$ is smooth as well. Observe now that

$$(G \circ F)(x, y) = \left(\sqrt{\frac{xy}{y/x}}, \sqrt{xy \cdot \frac{y}{x}}\right) = (x, y)$$

and

$$(F \circ G)(x, y) = \left(\sqrt{\frac{x}{y}} \cdot \sqrt{xy}, \frac{\sqrt{xy}}{\sqrt{x/y}}\right) = (x, y)$$

for all $(x, y) \in M$, so F and G are mutually inverse. Hence, F is a diffeomorphism with inverse $F^{-1} = G$. Furthermore, the Jacobian of F is given by

$$DF(x, y) = \begin{pmatrix} y & x \\ -y/x^2 & 1/x \end{pmatrix} \quad \text{for all } (x, y) \in M.$$

(b) The push-forward F_*X of X is the unique vector field on M that is F-related to X, i.e.,

$$dF_{(x,y)}\left(X_{(x,y)}\right) = (F_*X)_{F(x,y)} \quad \text{for all } (x, y) \in M.$$

The matrix of $dF_{(x,y)}$ with respect to the bases provided by $\partial/\partial x$ and $\partial/\partial y$ is precisely $DF(x,y)$. Since X is given with respect to this basis by $(x,y)^T$ and since

$$\begin{pmatrix} y & x \\ -y/x^2 & 1/x \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2xy \\ 0 \end{pmatrix},$$

we obtain that

$$(F_*X)_{F(x,y)} = 2xy \left.\frac{\partial}{\partial x}\right|_{F(x,y)}.$$

Replacing (x,y) by $G(x,y)$ yields

$$(F_*X)_{(x,y)} = 2x \left.\frac{\partial}{\partial x}\right|_{(x,y)},$$

and thus $F_*X = 2x\,\frac{\partial}{\partial x}$.

Similarly, since Y is given by $(y,0)^T$ with respect to the standard basis and since

$$\begin{pmatrix} y & x \\ -y/x^2 & 1/x \end{pmatrix} \cdot \begin{pmatrix} y \\ 0 \end{pmatrix} = \begin{pmatrix} y^2 \\ -y^2/x^2 \end{pmatrix},$$

we obtain

$$(F_*Y)_{F(x,y)} = y^2 \left.\frac{\partial}{\partial x}\right|_{F(x,y)} - \frac{y^2}{x^2} \left.\frac{\partial}{\partial y}\right|_{F(x,y)}.$$

Replacing (x,y) by $G(x,y)$ yields

$$(F_*Y)_{(x,y)} = xy \left.\frac{\partial}{\partial x}\right|_{(x,y)} - y^2 \left.\frac{\partial}{\partial y}\right|_{(x,y)},$$

and thus $F_*Y = xy\,\frac{\partial}{\partial x} - y^2\,\frac{\partial}{\partial y}$.

Alternatively, working as in the solution of part (d)(ii) of *Exercise* 4, *Sheet* 11 and using coordinates (u,v) in the codomain, we compute that

$$DF\big(G(u,v)\big) = \begin{pmatrix} \sqrt{uv} & \sqrt{\frac{u}{v}} \\ -\frac{\sqrt{uv}}{\frac{u}{v}} & \sqrt{\frac{v}{u}} \end{pmatrix},$$

as well as

$$X_{G(u,v)} = \sqrt{\frac{u}{v}} \left.\frac{\partial}{\partial x}\right|_{G(u,v)} + \sqrt{uv} \left.\frac{\partial}{\partial y}\right|_{G(u,v)} \quad \text{and} \quad Y_{G(u,v)} = \sqrt{uv} \left.\frac{\partial}{\partial x}\right|_{G(u,v)},$$

whence

$$(F_*X)_{(u,v)} = 2u \left.\frac{\partial}{\partial u}\right|_{(u,v)} \quad \text{and} \quad (F_*Y)_{(u,v)} = uv \left.\frac{\partial}{\partial u}\right|_{(u,v)} - v^2 \left.\frac{\partial}{\partial v}\right|_{(u,v)}.$$

(c) By part (a) of *Exercise* 5, *Sheet* 11 we have

$$\begin{aligned}
[X,Y] &= \left(\underbrace{\left(x\frac{\partial y}{\partial x} - y\frac{\partial x}{\partial x}\right)}_{=-y} + \underbrace{\left(y\frac{\partial y}{\partial y} - 0\cdot\frac{\partial x}{\partial y}\right)}_{=y}\right)\frac{\partial}{\partial x} + \\
&\quad + \left(\underbrace{\left(x\frac{\partial 0}{\partial x} - y\frac{\partial y}{\partial x}\right)}_{=0} + \underbrace{\left(y\frac{\partial 0}{\partial y} - 0\cdot\frac{\partial y}{\partial y}\right)}_{=0}\right)\frac{\partial}{\partial y} \\
&= 0.
\end{aligned}$$

Another way to see this is to compute $X \circ Y$ and $Y \circ X$ and to subtract their expressions (that is, to repeat the proof of the aforementioned formula). We have

$$\begin{aligned}
X \circ Y &= \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)\left(y\frac{\partial}{\partial x}\right) \\
&= x\frac{\partial}{\partial x}\left(y\frac{\partial}{\partial x}\right) + y\frac{\partial}{\partial y}\left(y\frac{\partial}{\partial x}\right) \\
&= xy\frac{\partial^2}{\partial x^2} + y\frac{\partial}{\partial x} + y^2\frac{\partial^2}{\partial x \partial y}
\end{aligned}$$

and

$$\begin{aligned}
Y \circ X &= \left(y\frac{\partial}{\partial x}\right)\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) \\
&= y\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}\right) + y\frac{\partial}{\partial x}\left(y\frac{\partial}{\partial y}\right) \\
&= y\frac{\partial}{\partial x} + xy\frac{\partial^2}{\partial x^2} + y^2\frac{\partial^2}{\partial x \partial y},
\end{aligned}$$

so $X \circ Y = Y \circ X$, and hence $[X,Y] = 0$.

Now, as $[F_*X, F_*Y] = F_*[X,Y]$ by part (b) of *Exercise* 7, *Sheet* 11, we also conclude that

$$[F_*X, F_*Y] = 0.$$

One can also see this with a direct calculation, using part (b) and part (a) of *Exercise* 5, *Sheet* 11; namely, we have

$$\begin{aligned}
[F_*X, F_*Y] &= \left(\underbrace{\left(2x\,\frac{\partial(xy)}{\partial x} - xy\,\frac{\partial(2x)}{\partial x}\right)}_{=0} + \underbrace{\left(0\cdot\frac{\partial(xy)}{\partial y} + y^2\,\frac{\partial(2x)}{\partial y}\right)}_{=0}\right)\frac{\partial}{\partial x} \\
&\quad + \left(\underbrace{\left(2x\,\frac{\partial(-y^2)}{\partial x} - xy\,\frac{\partial 0}{\partial x}\right)}_{=0} + \underbrace{\left(0\cdot\frac{\partial(-y^2)}{\partial y} + y^2\cdot\frac{\partial 0}{\partial y}\right)}_{=0}\right)\frac{\partial}{\partial y} \\
&= 0.
\end{aligned}$$

(d) If $\gamma\colon J \to \mathbb{R}^2$ is a smooth curve in M, written in coordinates as $\gamma(t) = \big(\gamma^1(t), \gamma^2(t)\big)$, then the condition $\gamma'(t) = Y_{\gamma(t)}$ for γ to be an integral curve of Y translates to the system

$$\frac{d\gamma^1}{dt}(t) = \gamma^2(t) \quad \text{and} \quad \frac{d\gamma^2}{dt}(t) = 0 \quad \text{for } t \in J,$$

whence γ^2 is a constant function and γ^1 is an affine function. Since we also require that $\gamma(0) = (1,1)$, we infer that the unique maximal integral curve of Y starting at $(1,1) \in M$ is the smooth curve

$$\gamma\colon J = (-1, +\infty) \to M \subseteq \mathbb{R}^2,\ t \mapsto \big(t+1, 1\big)$$

whose image is the straight line segment $\{(x,1) \in \mathbb{R}^2 \mid x > 0\}$.

Exercise 5 (6 + 4 + 4 + 4 + 3 points)**:**
Consider the vector field
$$X = -x\,\frac{\partial}{\partial x} + y\,\frac{\partial}{\partial y} \in \mathfrak{X}(\mathbb{R}^2)$$
and the covector field
$$\omega = 2xy^3\,dx + 3x^2y^2\,dy \in \mathfrak{X}^*(\mathbb{R}^2) = \Omega^1(\mathbb{R}^2).$$

(a) Compute the flow of X. Is it a complete vector field?

(b) Show that ω is closed and examine whether it is exact.

(c) Compute the line integral $\int_\gamma \omega$, where γ is the arc of the parabola $y = x^2$ from $(0,0)$ to $(1,1)$.

(d) Compute the smooth functions $\omega(X)\colon \mathbb{R}^2 \to \mathbb{R}$ and $X\big(\omega(X)\big)\colon \mathbb{R}^2 \to \mathbb{R}$.

(e) Denote by f the function $\omega(X)$ from (d). Determine the locus of points $(x,y) \in \mathbb{R}^2$ where $\operatorname{grad}(f)(x,y) = (0,0)$ and show that if $c \in \mathbb{R} \setminus \{0\}$, then the level set $f^{-1}(c)$ is a properly embedded submanifold of $\mathbb{R}^2$.

Solution:
(a) If $\gamma\colon J \to \mathbb{R}^2$ is a smooth curve, written as $\gamma(t) = \big(\gamma^1(t), \gamma^2(t)\big)$, then the condition $\gamma'(t) = X_{\gamma(t)}$ for γ to be an integral curve of X yields the following system of ODEs:
$$\frac{d\gamma^1}{dt}(t) = -\gamma^1(t), \quad \frac{d\gamma^2}{dt}(t) = \gamma^2(t), \quad t \in J,$$
with solution
$$\gamma^1(t) = c_1 e^{-t} \text{ and } \gamma^2(t) = c_2 e^t \text{ for } t \in J = \mathbb{R}$$
for some constants $c_1, c_2 \in \mathbb{R}$. Therefore, the unique maximal integral curve of X starting at the point $(x,y) \in \mathbb{R}^2$ is the smooth curve
$$\gamma\colon \mathbb{R} \to \mathbb{R}^2,\ t \mapsto \big(xe^{-t}, ye^t\big)\,.$$
In conclusion, X is a complete smooth vector field on $\mathbb{R}^2$, whose flow is the map
$$\theta_X\colon \mathbb{R} \times \mathbb{R}^2,\ \big(t, (x,y)\big) \mapsto \big(xe^{-t}, ye^t\big)\,.$$

(b) Consider the function
$$f\colon \mathbb{R}^2 \to \mathbb{R},\ (x,y) \mapsto x^2y^3$$
and note that $\omega = df$; in other words, ω is exact, and hence it is also closed, i.e., $d\omega = 0$. The latter can also be verified by a direct computation:
$$\begin{aligned} d\omega &= \left(\frac{\partial(2xy^3)}{\partial x}\,dx + \frac{\partial(2xy^3)}{\partial y}\,dy\right) \wedge dx + \left(\frac{\partial(3x^2y^2)}{\partial x}\,dx + \frac{\partial(3x^2y^2)}{\partial y}\,dy\right) \wedge dy \\ &= 6xy^2\,dy \wedge dx + 6xy^2\,dx \wedge dy \\ &= 0. \end{aligned}$$

(c) The arc of the parabola $y = x^2$ from $(0,0)$ to $(1,1)$ can be parametrized by the smooth curve segment

$$\gamma\colon [0,1] \to \mathbb{R}^2,\ t \mapsto (t, t^2).$$

We have

$$\gamma^*\omega = 2t(t^2)^3\, dt + 3t^2(t^2)^2\, d(t^2) = 2t^7\, dt + 6t^7\, dt = 8t^7\, dt,$$

and hence

$$\int_\gamma \omega = \int_{[0,1]} \gamma^*\omega = \int_0^1 8t^7\, dt = \left[t^8\right]_0^1 = 1.$$

(d) For any $(x,y) \in \mathbb{R}^2$ we have

$$X_{(x,y)} = -x\, \frac{\partial}{\partial x}\bigg|_{(x,y)} + y\, \frac{\partial}{\partial y}\bigg|_{(x,y)}$$

and

$$\omega_{(x,y)} = 2xy^3\, dx_{(x,y)} + 3x^2y^2\, dy_{(x,y)}.$$

Since

$$dx_{(x,y)}\left(\frac{\partial}{\partial x}\bigg|_{(x,y)}\right) = 1 = dy_{(x,y)}\left(\frac{\partial}{\partial y}\bigg|_{(x,y)}\right)$$

and

$$dx_{(x,y)}\left(\frac{\partial}{\partial y}\bigg|_{(x,y)}\right) = 0 = dy_{(x,y)}\left(\frac{\partial}{\partial x}\bigg|_{(x,y)}\right),$$

we compute

$$\omega(X)(x,y) = \omega_{(x,y)}\big(X_{(x,y)}\big) = -2x^2y^3 + 3x^2y^3 = x^2y^3 = f(x,y)$$

and

$$\begin{aligned}\Big(X\big(\omega(X)\big)\Big)(x,y) = X_{(x,y)}f &= -x\,\frac{\partial f}{\partial x}(x,y) + y\,\frac{\partial f}{\partial y}(x,y) = -2x^2y^3 + 3x^2y^3 \\ &= x^2y^3 = f(x,y).\end{aligned}$$

(In particular, we see that $Xf = f \in C^\infty(\mathbb{R}^2)$.)

(e) By part (b) we know that

$$\operatorname{grad}(f)(x,y) = \big(2xy^3, 3x^2y^2\big)\,,\ (x,y) \in \mathbb{R}^2,$$

so the locus $\mathcal{L}$ of points where the gradient of f vanishes is the union of the two axes, that is,

$$\begin{aligned}\mathcal{L} = \big\{(x,y) \in \mathbb{R}^2 \mid \operatorname{grad}(f)(x,y) = 0\big\} &= \big\{(x,y) \in \mathbb{R}^2 \mid x = 0 \text{ or } y = 0\big\} \\ &= \{(0,y) \mid y \in \mathbb{R}\} \cup \{(x,0) \mid x \in \mathbb{R}\}\,.\end{aligned}$$

Given $c \in \mathbb{R} \setminus \{0\}$, observe that

$$f^{-1}(c) \cap \mathcal{L} = \big\{(x,y) \in \mathbb{R}^2 \mid x^2y^3 = c\big\} \cap \mathcal{L} = \emptyset,$$

and thus $f^{-1}(c)$ is a regular level set of f. It follows from the *regular level set theorem* that $f^{-1}(c)$ is a properly embedded submanifold of $\mathbb{R}^2$.

Exercise 6 (6 + 12 points)**:**

(a) Let M be a smooth n-manifold. Let $\omega \in \Omega^k(M)$ and $\eta \in \Omega^\ell(M)$. Assume that ω is closed and that η is exact. Show that $\omega \wedge \eta$ is closed and exact.

(b) Consider the smooth manifolds

$$M = \{(u, v) \in \mathbb{R}^2 \mid u^2 + v^2 < 1\} \qquad \text{and} \qquad N = \mathbb{R}^3 \setminus \{0\},$$

the smooth map

$$F \colon M \to N, \ (u, v) \mapsto \left(u, v, \sqrt{1 - u^2 - v^2}\right),$$

and the differential forms

$$\omega = \frac{x\, dy \wedge dz + y\, dz \wedge dx + z\, dx \wedge dy}{(x^2 + y^2 + z^2)^{3/2}} \in \Omega^2(N)$$

and

$$\eta = \frac{x\, dx + y\, dy + z\, dz}{(x^2 + y^2 + z^2)^{1/2}} \in \Omega^1(N).$$

(i) Compute $d\omega$ and $d\eta$.

(ii) Compute $\omega \wedge \eta$ and $\eta \wedge d\eta$.

(iii) Compute $F^*\omega$ and $F^*(d\eta)$.

(iv) Verify that $d(F^*\omega) = F^*(d\omega)$.

Solution:

(a) Since ω is a closed k-form, we have $d\omega = 0$. Since η is an exact ℓ-form, there exists $\theta \in \Omega^{\ell-1}(M)$ such that $\eta = d\theta$, and thus $d\eta = 0$. Hence, we have

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta = 0;$$

in other words, $\omega \wedge \eta$ is a closed $(k + \ell)$-form. Moreover, since

$$d(\omega \wedge \theta) = d\omega \wedge \theta + (-1)^k \omega \wedge d\theta = (-1)^k \omega \wedge \eta,$$

we infer that

$$\omega \wedge \eta = d\big((-1)^k \omega \wedge \theta\big);$$

in other words, $\omega \wedge \eta$ is an exact $(k + \ell)$-form.

(b)(i) Using the facts that

$$dx \wedge dx = dy \wedge dy = dz \wedge dz = 0$$

and

$$dx \wedge dy \wedge dz = dy \wedge dz \wedge dx = dz \wedge dx \wedge dy,$$

we compute that

$$
\begin{aligned}
d\omega &= d\left(\frac{x}{(x^2+y^2+z^2)^{3/2}}\right)\wedge dy\wedge dz + d\left(\frac{y}{(x^2+y^2+z^2)^{3/2}}\right)\wedge dz\wedge dx + \\
&\quad + d\left(\frac{z}{(x^2+y^2+z^2)^{3/2}}\right)\wedge dx\wedge dy \\
&= \frac{\partial}{\partial x}\left(\frac{x}{(x^2+y^2+z^2)^{3/2}}\right) dx\wedge dy\wedge dz + \frac{\partial}{\partial y}\left(\frac{y}{(x^2+y^2+z^2)^{3/2}}\right) dy\wedge dz\wedge dx + \\
&\quad + \frac{\partial}{\partial z}\left(\frac{x}{(x^2+y^2+z^2)^{3/2}}\right) dz\wedge dx\wedge dy \\
&= \frac{-2x^2+y^2+z^2}{(x^2+y^2+z^2)^{5/2}}\, dx\wedge dy\wedge dz + \frac{-2y^2+x^2+z^2}{(x^2+y^2+z^2)^{5/2}}\, dx\wedge dy\wedge dz + \\
&\quad + \frac{-2z^2+x^2+y^2}{(x^2+y^2+z^2)^{5/2}}\, dx\wedge dy\wedge dz \\
&= 0.
\end{aligned}
$$

Consider now the smooth function

$$f\colon N \to \mathbb{R},\ (x,y,z)\mapsto \sqrt{x^2+y^2+z^2}$$

and observe that $\eta = df$. Therefore,

$$d\eta = 0.$$

(b)(ii) Using the facts mentioned in (b)(i), we compute that

$$
\begin{aligned}
\omega\wedge\eta &= \frac{x^2}{(x^2+y^2+z^2)^2}\, dy\wedge dz\wedge dx + \frac{y^2}{(x^2+y^2+z^2)^2}\, dz\wedge dx\wedge dy + \\
&\quad + \frac{z^2}{(x^2+y^2+z^2)^2}\, dx\wedge dy\wedge dz \\
&= \frac{1}{x^2+y^2+z^2}\, dx\wedge dy\wedge dz.
\end{aligned}
$$

Moreover, since $d\eta = 0$ by (b)(i), we infer that

$$\eta\wedge d\eta = 0.$$

(b)(iii) Since

$$du\wedge du = dv\wedge dv = 0,\ du\wedge dv = -dv\wedge du$$

and

$$d\left(\sqrt{1-u^2-v^2}\right) = -\frac{u}{\sqrt{1-u^2-v^2}}\, du - \frac{v}{\sqrt{1-u^2-v^2}}\, dv,$$

we compute that

$$
\begin{aligned}
F^*\omega &= \frac{-u^2}{\sqrt{1-u^2-v^2}}\, dv\wedge du + \frac{-v^2}{\sqrt{1-u^2-v^2}}\, dv\wedge du + \sqrt{1-u^2-v^2}\, du\wedge dv \\
&= \frac{1}{\sqrt{1-u^2-v^2}}\, du\wedge dv.
\end{aligned}
$$

Moreover, since $d\eta = 0$ by (b)(i), we have

$$F^*(d\eta) = 0.$$

(b)(iv) Note that both $d(F^*\omega)$ and $F^*(d\omega)$ are 3-forms on the 2-dimensional manifold M. Hence they are both equal to 0; in particular, we have

$$d(F^*\omega) = F^*(d\omega) = 0.$$