Differential Geometry II - Smooth Manifolds
Winter Term 2025/2026
Lecturer: Dr. N. Tsakanikas
Assistant: L. E. Rösler

Exam

Date: 24/01/2026
Duration: 180 minutes

About the exam: There are 6 exercises for a total of 100 points. Please write your solutions to the exercises with a pen, *not* with a pencil. You get maximum credit for solving any part of an exercise assuming the statements of previous parts of that exercise, even if you did not solve (all of) those previous parts.

Duration: The length of the exam is 180 minutes. If you did not leave the room before the final 20 minutes of the exam, then please remain seated and silent until the end of the exam, even if you finished your exam during these 20 minutes. Your solutions will be collected by the proctors at the end of the exam. You should remain seated and silent until the solutions of all the students have been collected.

Papers: You have been provided with 16 blank A4 papers and you should write all your solutions there. The solution of each exercise should always begin in a new page and the number of the exercise to be solved should always be indicated clearly on the top left corner of that page. If necessary, ask for additional paper from the proctors. You should write your name, surname and SCIPER clearly on the top left corner of each additional A4 paper. At the end of the exam, sign the last one of the additional papers, mention there clearly the number of additional papers that you used for your solutions, and put all the additional papers inside the given bulk of 16 A4 papers under the supervision of a proctor. Finally, we also provide colored scratch paper, and thus you are *neither allowed* to use your own scratch paper, *nor to hand in* colored paper.

Cheat sheet: You can use a cheat sheet, that is, two sides of an A4 paper written by yourself.

Personal items: Place your CAMIPRO card on your table at a visible place, so that the proctors can easily check your identity during the exam. Your bag and your coat should be placed close to the walls of the room, *not* in the vicinity of your seat.

Results from the course: To solve the exercises of the exam, you can use all results seen in the lectures or in the exercise sessions (that is, all results contained in the lectures notes or in the exercise sheets), except if the given question asks exactly that result or a special case of it. If you are using such a result, then you should state explicitly what you are using and why its assumptions are satisfied.

Exercise 1 (5 + 10 + 5 + 2 points): Let n be a positive integer.

(a) Define the notion of a *smooth n-manifold* (without boundary). In particular, explain all the terms appearing in the definition.

(Note: you do not have to define the terms *topological space*, *basis* of a topological space, and the space $\mathbb{R}^n$ with its standard topology.)

(b) Let M be a smooth n-manifold (without boundary). Let $p \in M$ and let $\big(U, \varphi = (x^i)\big)$ be a smooth coordinate chart for M such that $p \in U$.

(i) Define the *tangent space* T_pM to M at p, write down a basis for T_pM and explain how these basis elements are defined.

(ii) Define the *cotangent space* T_p^*M to M at p, write down a basis for T_p^*M and explain how these basis elements are defined.

(iii) Define the notion of a *smooth vector bundle* of rank k over M.

(iv) Describe the projection map and the fibers of the tangent bundle TM of M. Subsequently, write down a smooth local trivialization for TM over U and construct a smooth local frame for TM over U.

(c) Define the notion of a *smooth map* between smooth manifolds (without boundary) and explain how one defines the *differential* of a smooth map at every point of its domain. Define also the notions of *smooth immersions*, *smooth submersions* and *smooth embeddings*.

(d) Define the notion of a *regular value* of a smooth map and formulate the *regular level set theorem* (in the setting of smooth manifolds without boundary).

Exercise 2 (4 + 5 + 5 + 3 points):
Consider the smooth function

$$f\colon \mathbb{R}^2 \to \mathbb{R},\ (x, y) \mapsto y^2 - x^3 - x^2.$$

(a) Determine the regular values of f.

(b) Deduce that the level set $S := f^{-1}(4)$ is a properly embedded submanifold of $\mathbb{R}^2$ and describe its tangent space $T_pS \subset T_p\mathbb{R}^2 \simeq \mathbb{R}^2$ at the point $p \in \{(-2, 0), (0, 2)\}$.

(c) Determine the function $g := df(V)\colon \mathbb{R}^2 \to \mathbb{R}$, where

$$V = y\,\frac{\partial}{\partial x} + x\,\frac{\partial}{\partial y} \in \mathfrak{X}(\mathbb{R}^2),$$

and examine whether the level set $g^{-1}(0)$ is an embedded submanifold of $\mathbb{R}^2$.

(d) Compute the line integrals

$$\int_{\widetilde{\alpha}} df \quad \text{and} \quad \int_{\widetilde{\alpha}} dg,$$

where $\widetilde{\alpha}$ denotes the restriction of the smooth plane curve α from *Exercise 3*(a) below to the closed interval $[-1, 1] \subset \mathbb{R}$, i.e., $\widetilde{\alpha} = \alpha|_{[-1,1]}$.

Exercise 3 (5 + 10 points)**:**

(a) Examine whether the following plane curves are smooth immersions:

(i) $\alpha\colon \mathbb{R} \to \mathbb{R}^2,\ t \mapsto (t^3,\, t^2)$.

(ii) $\beta\colon \mathbb{R} \to \mathbb{R}^2,\ t \mapsto (t^3 - 4t,\, t^2 - 4)$.

If so, then examine subsequently whether they are smooth embeddings.

(b) Let $\pi\colon M \to N$ be a smooth submersion. Prove the following assertions:

(i) Every point of M is in the image of a smooth local section of π.

(ii) π is an open map.

Exercise 4 (5 + 5 + 4 + 6 points)**:**
Consider the following three smooth vector fields on $\mathbb{R}^2$:

$$X = x\,\frac{\partial}{\partial x} + y\,\frac{\partial}{\partial y}\,, \quad Y = -y\,\frac{\partial}{\partial x} + x\,\frac{\partial}{\partial y}\,, \quad Z = (x^2 + y^2)\,\frac{\partial}{\partial x} \in \mathfrak{X}(\mathbb{R}^2).$$

(a) Show that the map

$$\begin{aligned} \Phi\colon (0, +\infty) \times (-\pi, \pi) &\to \mathbb{R}^2 \\ (r, \theta) &\mapsto (r\cos\theta,\, r\sin\theta) \end{aligned}$$

is a diffeomorphism onto its image $U \coloneqq \Phi(V)$, where $V \coloneqq (0, +\infty) \times (-\pi, \pi)$, and compute the inverse Ψ of the restriction of Φ to the subset $(0, +\infty) \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ of V.

(b) Compute the coordinate representation of Z in polar coordinates on the right half-plane $M = \{(x, y) \in \mathbb{R}^2 \mid x > 0\}$, that is, determine the pushforward $\Psi_*\big(Z|_M\big)$.

[You may use (without proof) the formula $\dfrac{d}{du}(\arctan u) = \dfrac{1}{1 + u^2}$.]

(c) Compute the Lie brackets $[X, Z]$ and $[Y, Z]$.

(d) Compute the flow of

$$W = -x\,\frac{\partial}{\partial x} - y\,\frac{\partial}{\partial y} \in \mathfrak{X}(\mathbb{R}^2)$$

and examine whether it is a *complete* vector field on $\mathbb{R}^2$. Describe also geometrically the image of the maximal integral curve of W starting at the point $(-1, -1) \in \mathbb{R}^2$.

Exercise 5 (5 + 4 + 4 points)**:**
Consider the smooth map

$$F\colon \mathbb{R}^2 \to \mathbb{R}^3,\ (\theta,\varphi) \mapsto \big((\cos\varphi + 2)\cos\theta,\ (\cos\varphi + 2)\sin\theta,\ \sin\varphi\big),$$

the smooth 1-form

$$\eta = z^2\, dx \in \Omega^1(\mathbb{R}^3),$$

and the smooth 2-form

$$\omega = y\, dz \wedge dx \in \Omega^2(\mathbb{R}^3).$$

Note that we denote the standard coordinates on the source by (θ,φ) and the standard coordinates on the target by (x,y,z).

(a) Compute the Jacobian matrix $J_F(\theta,\varphi)$ of F and determine the rank of F at every point $(\theta,\varphi) \in \mathbb{R}^2$.

(b) Compute $d\eta$ and $F^*\eta$, and verify by direct computation that $d(F^*\eta) = F^*(d\eta)$.

(c) Compute $d\omega$ and $F^*\omega$, and verify by direct computation that $d(F^*\omega) = F^*(d\omega)$.

Exercise 6 (6 + 2 + 5 points)**:** Let n be a positive integer.

(a) Recall that the *length* of a smooth curve segment $\gamma\colon [a,b] \to \mathbb{R}^n$, where $n > 0$, is defined to be the value of the (ordinary) integral

$$L(\gamma) := \int_a^b \big|\gamma'(t)\big|\, dt.$$

Show that there is no smooth covector field ω on $\mathbb{R}^n$ with the property that

$$\int_\gamma \omega = L(\gamma)$$

for every smooth curve segment γ in $\mathbb{R}^n$.

(b) Show that the integral of an exact n-form over a compact, oriented, smooth n-manifold without boundary is zero.

(c) Examine whether the smooth covector field

$$\eta = -\frac{4xz}{(x^2+1)^2}\, dx + \frac{2y}{y^2+1}\, dy + \frac{2}{x^2+1}\, dz \in \Omega^1(\mathbb{R}^3)$$

is exact. If so, then find a potential function for η. In any case, compute the line integral along the straight line segment from $(0,0,0)$ to $(1,1,1)$.