Differential Geometry II - Smooth Manifolds
Winter Term 2024/2025
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Exam

Date: 16/01/2025
Duration: 180 minutes

About the exam: There are 6 exercises for a total of 100 points. Please write your solutions to the exercises with a pen, *not* with a pencil. You get maximum credit for solving any part of an exercise assuming the statements of previous parts of that exercise, even if you did not solve (all of) those previous parts.

Duration: The length of the exam is 180 minutes. If you did not leave the room before the final 20 minutes of the exam, then please remain seated and silent until the end of the exam, even if you finished your exam during these 20 minutes. Your solutions will be collected by the proctors at the end of the exam. You should remain seated and silent until the solutions of all students have been collected.

Papers: You have been provided with 16 blank A4 papers and you should write all your solutions there. The solution of each exercise should always begin in a new page and the number of the exercise to be solved should always be indicated clearly on the top left corner of that page. If necessary, ask for additional paper from the proctors. Write your name, surname and SCIPER clearly on the top left corner of each additional A4 paper. At the end of the exam, sign the last one of the additional papers, mention there clearly the number of additional papers that you used for your solutions, and put all the additional papers inside the given bulk of 16 A4 papers under the supervision of a proctor. Finally, we also provide colored scratch paper, and thus you are *neither allowed* to use your own scratch paper, *nor to hand in* colored paper.

Cheat sheet: You can use a cheat sheet, that is, two sides of an A4 paper written by yourself.

Personal items: Please place your CAMIPRO card on your table at a visible place. Your bag and your coat should be placed close to the walls of the room, *not* in the vicinity of your seat.

Results from the course: To solve the exercises of the exam, you can use all results seen in the lectures or in the exercise sessions (that is, all results contained in the lectures notes or in the exercise sheets), except if the given question asks exactly that result or a special case of it. If you are using such a result, then you should state explicitly what you are using and why its assumptions are satisfied.

Exercise 1 (2 + 3 + 2 + 3 + 2 + 6 points)**:**
Let M be a smooth n-manifold (without boundary). Let $p \in M$ and let $\big(U, \varphi = (x^i)\big)$ be a smooth coordinate chart for M such that $p \in U$.

(a) Define the *tangent space* T_pM to M at p and write down a basis for T_pM.

(b) Given $v \in T_pM$ and $f, g \in C^\infty(M)$, prove the following assertions:

  (i) If f is constant, then $vf = 0$.

  (ii) If $f(p) = g(p) = 0$, then $v(fg) = 0$.

(c) Define the *cotangent space* T_p^*M to M at p and write down a basis for T_p^*M.

(d) Define the notion of a *smooth vector bundle* of rank k over M. When is it said to be *(smoothly) trivial*?

(e) Write down a smooth local trivialization over U for both the tangent bundle TM and the cotangent bundle T^*M of M.

(f) Show that TM is a (smoothly) trivial vector bundle if and only if T^*M is a (smoothly) trivial vector bundle.

Exercise 2 (2 + 3 + 11 points)**:**

(a) Given a smooth manifold M (without boundary), define the notion of an *embedded submanifold* S of M. When is S said to be *properly embedded* in M?

(b) Formulate the *regular level set theorem* (in the setting of smooth manifolds without boundary) and explain what a *regular point* and a *regular value* of a smooth map are.

(c) Consider the smooth function

$$f\colon \mathbb{R}^2 \to \mathbb{R},\ (x, y) \mapsto x^3 + y^3 + 1.$$

  (i) Determine the regular values of f.

  (ii) For which $c \in \mathbb{R}$ is the level set $f^{-1}(c)$ a properly embedded submanifold of $\mathbb{R}^2$? Justify your answer.

  (iii) For each $c \in \mathbb{R}$ such that the corresponding level set $S = f^{-1}(c)$ is a properly embedded submanifold of $\mathbb{R}^2$, determine the tangent space $T_pS \cong d\iota_p(T_pS) \subset T_p\mathbb{R}^2 \cong \mathbb{R}^2$, where $p \in S$ and $\iota\colon S \hookrightarrow \mathbb{R}^2$ is the inclusion map.

Exercise 3 (6 + 6 points)**:**

(a) Compute the Lie bracket of each of the following pairs of smooth vector fields on $\mathbb{R}^3$:

(i) $X_1 = y\,\dfrac{\partial}{\partial z} - 2xy^2\dfrac{\partial}{\partial y}$ and $Y_1 = \dfrac{\partial}{\partial y}$.

(ii) $X_2 = -y\,\dfrac{\partial}{\partial x} + x\,\dfrac{\partial}{\partial y}$ and $Y_2 = -z\,\dfrac{\partial}{\partial y} + y\,\dfrac{\partial}{\partial z}$.

(iii) $X_3 = -y\,\dfrac{\partial}{\partial x} + x\,\dfrac{\partial}{\partial y}$ and $Y_3 = y\,\dfrac{\partial}{\partial x} + x\,\dfrac{\partial}{\partial y}$.

(b) Compute the flow of $X_2 = X_3$ and examine whether it is a *complete* smooth vector field on $\mathbb{R}^3$.

Exercise 4 (5 + 8 + 3 points)**:**

(a) Let M be a smooth n-manifold (without boundary) and let $\omega \in \Omega^1(M)$.

(i) Let $\big(U, (x^i)\big)$ be a smooth coordinate chart for M, and write $\omega|_U = \sum_{i=1}^n \omega_i\, dx^i$ in this chart. Find an expression for the *exterior derivative* $d\omega \in \Omega^2(M)$ of ω in this chart (that is, an expression of $d\omega$ in terms of the natural basis induced in each fiber of $\Lambda^2(T^*M)$ by the given chart).

(ii) Deduce that ω is closed if and only if for every point $p \in M$ there exists a smooth coordinate chart $\big(U, (x^i)\big)$ such that $p \in U$ and

$$\frac{\partial \omega_j}{\partial x^i} = \frac{\partial \omega_i}{\partial x^j} \quad \text{for all } 1 \le i, j \le n,$$

where $\omega|_U = \sum_{i=1}^n \omega_i\, dx^i$ in this chart.

(b) Consider the smooth 1-forms

$$\omega = y\cos(xy)\, dx + x\cos(xy)\, dy \in \Omega^1(\mathbb{R}^2)$$

and

$$\eta = x\cos(xy)\, dx + y\cos(xy)\, dy \in \Omega^1(\mathbb{R}^2).$$

(i) Show that ω is closed and exact.

(ii) Show that η is neither closed nor exact.

(iii) Compute $\omega \wedge \eta$.

(c) Evaluate the line integral

$$\int_\gamma \omega,$$

where γ is the straight line segment from $(0,0)$ to $(\sqrt{\pi}, \sqrt{\pi})$.

Exercise 5 (3 + 2 + 6 + 5 points)**:**
Consider the smooth map

$$F\colon \mathbb{R}^2 \to \mathbb{R}^2,\ (s,t) \mapsto (st, e^t)$$

and the smooth 1-form

$$\omega = x\,dy \in \Omega^1(\mathbb{R}^2).$$

Note that we denote the standard coordinates on the source by (s,t) and the standard coordinates on the target by (x,y).

(a) Compute $d\omega$, $F^*\omega$ and $F^*\omega \wedge F^*\omega$.

(b) Verify by direct computation that $d(F^*\omega) = F^*(d\omega)$.

(c) Consider the set

$$D = \big\{(s,t) \in \mathbb{R}^2 \mid s > 0,\ t > 0\big\}$$

and determine its image under F. Show that the restriction $G \coloneqq F|_D\colon D \to F(D)$ is a diffeomorphism, compute its Jacobian matrix $DG(s,t)$ at an arbitrary point $(s,t) \in D$, and determine the inverse G^{-1} of the diffeomorphism G.

(d) Compute the pushforward G_*V of the smooth vector field

$$V_{(s,t)} = st\,\frac{\partial}{\partial s}\bigg|_{(s,t)} + t^2 e^t\,\frac{\partial}{\partial t}\bigg|_{(s,t)} \in \mathfrak{X}(D).$$

Exercise 6 (3 + 1 + 2 + 4 + 6 + 6 points)**:**

(a) Define the *integral* $\int_M \omega$ of an n-form ω on a smooth n-manifold M, where $n \geq 1$.

(You do not have to recall the definition of the integral of a (continuous) n-form on $\mathbb{R}^n$ or $\mathbb{H}^n$, but you should write explicitly the conditions that M and ω must satisfy so that the integral $\int_M \omega$ makes sense.)

(b) Formulate *Stokes' theorem*.

(c) State and prove *Green's theorem*.

(d) Using Green's theorem, compute the area of the disc

$$D_r \coloneqq \big\{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq r^2\big\},$$

where $r > 0$.

(e) Let M be an oriented, smooth n-manifold with or without boundary, where $n \geq 1$. Prove the following assertion: If η is a positively oriented orientation form on M, then

$$\int_M \eta > 0.$$

(f) Let M be a compact, connected, oriented, smooth n-manifold without boundary (i.e., $\partial M = \emptyset$), where $n \geq 1$, and let $\omega \in \Omega^{n-1}(M)$. Show that there exists a point $p \in M$ such that $(d\omega)_p = 0 \in \Lambda^n(T_p^*M)$.