Differential Geometry II - Smooth Manifolds
Winter Term 2023/2024
Lecturer: Dr. N. Tsakanikas
Assistant: L. E. Rösler

Exam

Date: 26/01/2024
Duration: 180 minutes

About the exam: There are 6 exercises for a total of 100 points. Please write your solutions to the exercises with a pen, *not* with a pencil. You get maximum credit for solving any part of an exercise assuming the statements of previous parts of that exercise, even if you did not solve (all of) those previous parts.

Duration: The length of the exam is 180 minutes. If you did not leave the room before the final 20 minutes of the exam, then please remain seated until the end of the exam, even if you finish your exam during these 20 minutes. Your solutions will be collected by the proctors at the end of the exam, during which period you should remain seated and silent.

Papers: You have been provided with 16 blank A4 papers and you should write all your solutions there. The solution of each exercise should always begin in a new page and the number of the exercise to be solved should be indicated clearly on the top left corner of that page. If necessary, ask for additional paper from the proctors. Write your name, surname and SCIPER clearly on the top left corner of each additional paper. At the end of the exam, sign the last one of the additional papers, mention there clearly the number of additional papers that you used for your solutions, and put all the additional papers inside the given bulk of 16 A4 papers under the supervision of a proctor. Finally, we also provide colored scratch paper, and thus you are *neither allowed* to use your own scratch paper, *nor to hand in* colored paper.

Cheat sheet: You can use a cheat sheet, that is, two sides of an A4 paper written by yourself.

Personal items: Please place your CAMIPRO card on your table at a visible place. Your bag and your coat should be placed close to the walls of the room, *not* in the vicinity of your seat.

Results from the course: To solve the exercises of the exam, you can use all results seen during the lectures or in the exercise sessions (that is, all results in the lectures notes or on the exercise sheets), except if the given question asks exactly that result or a special case of it. If you are using such a result, then you should state explicitly what you are using and why its assumptions are satisfied.

Exercise 1 (5 + 7 + 5 points)**:**

(a) Define the notion of a *smooth n-manifold* (without boundary). In particular, explain all the terms appearing in the definition.

(Note: you do not have to define the terms *topological space*, *basis* of a topological space, and the space $\mathbb{R}^n$ with its standard topology.)

(b) Let M be a smooth n-manifold (without boundary), let $p \in M$ and let (U, φ) be a smooth coordinate chart for M such that $p \in U$.

(i) What is the tangent space T_pM to M at p? Write down a basis for T_pM and explain how any of those basis elements acts on a function $f \in C^\infty(M)$.

(ii) What is the cotangent space T_p^*M to M at p? Write down a basis for its k-th exterior power $\Lambda^k\left(T_p^*M\right)$, where $1 \leq k \leq n$?

(c) Define the notion of a *smooth map* between smooth manifolds (without boundary). What is its *rank* (at a point) and when does one say that it has *constant rank*? Define also the notions of *smooth immersions* and *smooth submersions.* Finally, formulate the *rank theorem* (in the setting of smooth manifolds).

Exercise 2 (1 + 9 points)**:**

(a) Let M be a smooth n-manifold and let (U, φ) be a smooth coordinate chart for M. Write down a smooth coordinate chart for TM with domain $\pi^{-1}(U)$, where $\pi \colon TM \to M$ is the tangent bundle of M.

(b) Let U be an open subset of $\mathbb{R}^n$ and let $f \colon U \to \mathbb{R}^k$ be a smooth function. Define the *graph* $\Gamma(f)$ of f as a subset of $\mathbb{R}^n \times \mathbb{R}^k$ and give it the subspace topology. Show that it can be endowed with the structure of a smooth n-manifold (without boundary) and determine its tangent bundle $T\big(\Gamma(f)\big)$ up to diffeomorphism.

Exercise 3 (8 + 8 points)**:**

(a) Let M be a non-empty, compact, smooth manifold. Show that there exists no smooth submersion $F \colon M \to \mathbb{R}^k$ for any $k \in \mathbb{Z}_{\geq 1}$.

(b) For each $\lambda \in \mathbb{R}$, consider the set

$$M_\lambda := \left\{(x, y) \in \mathbb{R}^2 \mid y^2 = x(x - \lambda)\right\}.$$

For which values of λ is M_λ a properly embedded submanifold of $\mathbb{R}^2$?

Exercise 4 (3 + 6 + 5 + 4 points)**:**
Consider the open submanifold

$$M := \{(x, y) \in \mathbb{R}^2 \mid x > 0,\ y > 0\} \subseteq \mathbb{R}^2,$$

the map

$$F\colon M \to M,\ (x, y) \mapsto \left(xy, \frac{y}{x}\right),$$

and the smooth vector fields

$$X := x\,\frac{\partial}{\partial x} + y\,\frac{\partial}{\partial y} \quad \text{and} \quad Y := y\,\frac{\partial}{\partial x}$$

on M.

(a) Show that F is a diffeomorphism, compute its Jacobian matrix $DF(x, y)$ at an arbitrary point $(x, y) \in M$, and determine its inverse F^{-1}.

(b) Compute the pushforwards F_*X and F_*Y of X and Y, respectively.

(c) Compute the Lie brackets $[X, Y]$ and $[F_*X, F_*Y]$.

(d) Find the maximal integral curve of Y starting at the point $(1, 1) \in M$ and describe its image geometrically.

Exercise 5 (6 + 4 + 4 + 4 + 3 points)**:**
Consider the vector field

$$X = -x\,\frac{\partial}{\partial x} + y\,\frac{\partial}{\partial y} \in \mathfrak{X}(\mathbb{R}^2)$$

and the covector field

$$\omega = 2xy^3\,dx + 3x^2y^2\,dy \in \mathfrak{X}^*(\mathbb{R}^2) = \Omega^1(\mathbb{R}^2).$$

(a) Compute the flow of X. Is it a complete vector field?

(b) Show that ω is closed and examine whether it is exact.

(c) Compute the line integral $\displaystyle\int_\gamma \omega$, where γ is the arc of the parabola $y = x^2$ from $(0, 0)$ to $(1, 1)$.

(d) Compute the smooth functions $\omega(X)\colon \mathbb{R}^2 \to \mathbb{R}$ and $X\big(\omega(X)\big)\colon \mathbb{R}^2 \to \mathbb{R}$.

(e) Denote by f the function $\omega(X)$ from (d). Determine the locus of points $(x, y) \in \mathbb{R}^2$ where $\operatorname{grad}(f)(x, y) = (0, 0)$ and show that if $c \in \mathbb{R} \setminus \{0\}$, then the level set $f^{-1}(c)$ is a properly embedded submanifold of $\mathbb{R}^2$.

Exercise 6 (6 + 12 points)**:**

(a) Let M be a smooth n-manifold. Let $\omega \in \Omega^k(M)$ and $\eta \in \Omega^\ell(M)$. Assume that ω is closed and that η is exact. Show that $\omega \wedge \eta$ is closed and exact.

(b) Consider the smooth manifolds

$$M = \big\{(u,v) \in \mathbb{R}^2 \mid u^2 + v^2 < 1\big\} \qquad \text{and} \qquad N = \mathbb{R}^3 \setminus \{0\},$$

the smooth map

$$F \colon M \to N, \; (u,v) \mapsto \Big(u, v, \sqrt{1 - u^2 - v^2}\Big),$$

and the differential forms

$$\omega = \frac{x\,dy \wedge dz + y\,dz \wedge dx + z\,dx \wedge dy}{(x^2+y^2+z^2)^{3/2}} \in \Omega^2(N)$$

and

$$\eta = \frac{x\,dx + y\,dy + z\,dz}{(x^2+y^2+z^2)^{1/2}} \in \Omega^1(N).$$

(i) Compute $d\omega$ and $d\eta$.

(ii) Compute $\omega \wedge \eta$ and $\eta \wedge d\eta$.

(iii) Compute $F^*\omega$ and $F^*(d\eta)$.

(iv) Verify that $d(F^*\omega) = F^*(d\omega)$.