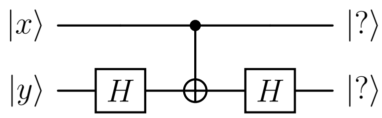
Final exam 2024
Quantum Computation

Please pay attention to the presentation of your answers! (2 points)

Exercise 1 *Analysis of a quantum circuit (14 points)*

Consider the following quantum circuit:



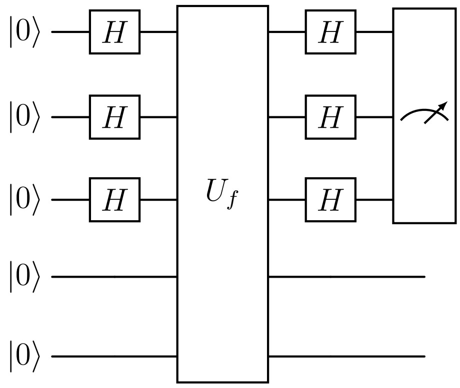
a) Compute the output of the circuit when the input is an element $|x\rangle \otimes |y\rangle$ of the computational basis in $(\mathbb{C}^2)^2$.

b) Assume now the input is in a product state $|\varphi\rangle \otimes |\psi\rangle$, where $|\varphi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$ with $|\alpha|^2 + |\beta|^2 = 1$ and $|\psi\rangle = \gamma\,|0\rangle + \delta\,|1\rangle$ with $|\gamma|^2 + |\delta|^2 = 1$. Under what condition(s) on $\alpha, \beta, \gamma, \delta$ is the output of the circuit also in a product state?

c) And if now the input is in an entangled state, is there a possibility that the output is in a product state? If yes, provide an example; if no, explain why this is impossible.

Exercise 2 *Period of a function (18 points)*

Let us consider the following circuit: where the 5-qubit gate U_f is the oracle corresponding



to the function $f : \{0,1\}^3 \to \{0,1\}^2$ defined as

$$f(x, y, z) = (x \oplus y, y \oplus z) \quad \text{for} \quad (x, y, z) \in \{0,1\}^3$$

a) Compute explicitly the 5-qubit output $|\psi\rangle$ of the circuit before the measurement.

As a reminder, the measurement is described by the collection of orthogonal projectors in the computational basis $\{P_{u,v,w},\ (u, v, w) \in \{0,1\}^3\}$, defined as

$$P_{u,v,w} = |u, v, w\rangle \langle u, v, w| \otimes I_2$$

and the output state $|\psi'\rangle = P_{u,v,w} |\psi\rangle \,/\, \| P_{u,v,w} |\psi\rangle \|$ occurs with probability

$$\text{prob}(u, v, w) = \langle\psi| P_{u,v,w} |\psi\rangle$$

b) Compute $\text{prob}(u, v, w)$ for every $(u, v, w) \in \{0,1\}^3$.

c) Deduce from there the period of the function f (that is, the value of $a \in \{0,1\}^3 \backslash \{(0,0,0)\}$ such that $f(x \oplus a) = f(x)$ for all $x \in \{0,1\}^3$).

Exercise 3 *Quantum search (18 points)*

Let $f : \{0,1\}^2 \to \{0,1\}$ be a binary function such that there exists a unique value of $(x_0, y_0) \in \{0,1\}^2$ with $f(x_0, y_0) = 1$. Let also U_f be the 3-qubit oracle gate whose action on the states in the computational basis is given by

$$U_f(|x, y, z\rangle) = |x, y, z \oplus f(x, y)\rangle$$

a) Build explicitly a circuit using gates X, H, C-NOT and U_f, taking as input $|0,0,0\rangle$ and whose output is given by $|x_0, y_0\rangle \otimes |\psi\rangle$, where $|\psi\rangle$ is some (irrelevant) state in $\mathbb{C}^2$.

b) Build explicitly the oracle gate U_f in the case where $(x_0, y_0) = (0, 0)$.

Note: Here, there is no restriction on the type of elementary gates which can be used.

c) Assume now that for the very same function f with $(x_0, y_0) = (0, 0)$, we run Deutsch-Josza's circuit with the corresponding gate U_f. After the measurement of the first two qubits of the output of the circuit, what are the probabilities of each output?

Exercise 4 *Quantum error correction (18 points)*

a) Two qubits in state $|\varphi\rangle = \sum_{x,y\in\{0,1\}} \alpha_{x,y}\, |x,y\rangle$ in $(\mathbb{C}^2)^2$, with $\sum_{x,y\in\{0,1\}} |\alpha_{x,y}|^2 = 1$, are sent through a channel and the received state is

$$|\psi\rangle = \sum_{x,y\in\{0,1\}} (-1)^y\, \alpha_{x,y}\, |\overline{x}, y\rangle$$

Propose a sequence of operations involving exclusively X and H operators allowing to recover the initial state $|\varphi\rangle$ from the received state $|\psi\rangle$, and draw the corresponding circuit.

b) A single qubit in state $|\varphi\rangle \in \mathbb{C}^2$, with $\| \, |\varphi\rangle \, \| = 1$, is sent through a channel and the received state is

$$|\psi\rangle = HXHXHXHX\, |\varphi\rangle$$

What is the *minimum* number of actions needed to recover state $|\varphi\rangle$ from state $|\psi\rangle$ (up to a global phase)? Justify your answer.

c) Consider the Shor code which, as a reminder, is defined as follows:

$$\begin{cases} |0\rangle_S = \dfrac{|000\rangle + |111\rangle}{\sqrt{2}} \otimes \dfrac{|000\rangle + |111\rangle}{\sqrt{2}} \otimes \dfrac{|000\rangle + |111\rangle}{\sqrt{2}} \\[2ex] |1\rangle_S = \dfrac{|000\rangle - |111\rangle}{\sqrt{2}} \otimes \dfrac{|000\rangle - |111\rangle}{\sqrt{2}} \otimes \dfrac{|000\rangle - |111\rangle}{\sqrt{2}} \end{cases}$$

With this code, is it possible to correct both a bit-flip occuring on a given qubit *and* a phase-flip occuring on another qubit?

If yes, explain the detection and correction procedure in the case where the bit-flip occurs on qubit 1 and the phase-flip occurs on qubit 4.

If no, explain why this is impossible (considering the same example as above).