Final exam 2025 - Solutions
Quantum Computation

Please pay attention to the presentation of your answers! (4 points)

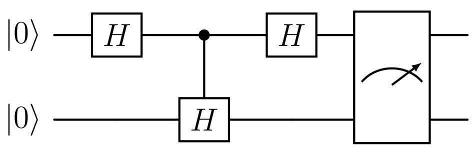
Exercise 1 *Quiz (16 points)*

For each question, 1 pt for the correct answer, 3 pts for the justification.

a) If a quantum state $\psi = |+,+,+,+\rangle$ is measured in the computational basis, then the probability that in the output state, there are as many qubits in state $|0\rangle$ as there are in state $|1\rangle$ is equal to $\frac{1}{2}$.

Solution: False.
When measured, any output state $|x\rangle$ for $x \in \{0,1\}^4$ has the probability $\frac{1}{16}$. Number of states with equal number $|0\rangle$ and $|1\rangle$ is $\binom{4}{2} = 6$. Hence, the probability is $\frac{6}{16}$

b) Let us consider the following circuit:



Then the probability that the output state of this circuit is equal to $|1,0\rangle$ is less than $\frac{1}{40}$.

Hint: $\sqrt{2} \simeq 1.41$

NB: The qubits are ordered from top to bottom in the circuit (as was always done in class).

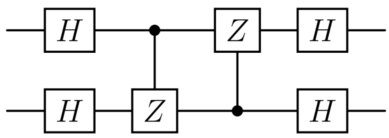
Solution: True.

Output state is given by

$$\begin{aligned}
|\psi\rangle &= (H \otimes I) CH (H \otimes I) |00\rangle \\
&= (H \otimes I) CH \frac{1}{\sqrt{2}} (|00\rangle + |10\rangle) \\
&= (H \otimes I) \frac{1}{\sqrt{2}} (|00\rangle + \frac{1}{\sqrt{2}} (|10\rangle + |11\rangle)) \\
&= \frac{1}{2} (|00\rangle + |10\rangle + \frac{1}{\sqrt{2}} (|00\rangle - |10\rangle + |01\rangle - |11\rangle)) \\
&= \frac{1}{2\sqrt{2}} ((\sqrt{2}+1) |00\rangle + (\sqrt{2}-1) |10\rangle + |01\rangle - |11\rangle)
\end{aligned}$$

$\Pr\{|10\rangle\} = \frac{(\sqrt{2}-1)^2}{8} < \frac{1}{40}$.

c) The following gate:



is equivalent to a SWAP gate.

Solution: False.
Note that $(ZC)(CZ)$ is identity. This is because, ZC and CZ are the same and have matrix representation given by $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$.

d) The 2-qubit gate with the following matrix representation: $U = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$ can be decomposed into a tensor product of 1-qubit gates.

Solution: True.

$$U = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = X \otimes X.$$

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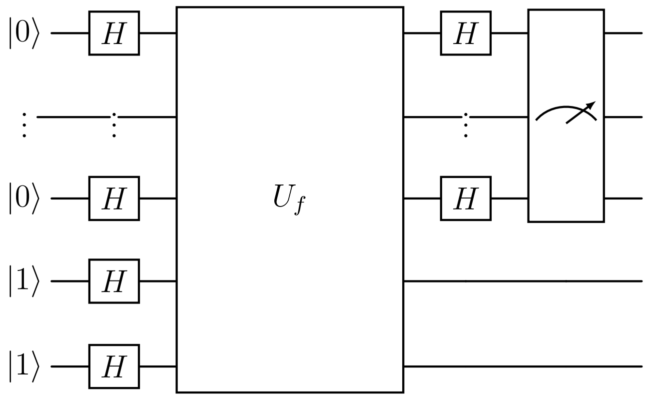
Exercise 2 *Two Boolean functions (15 points)*

Let $n \geq 1$ and consider two Boolean functions $f_1, f_2 : \{0,1\}^n \to \{0,1\}$ satisfying one of the following three assumptions:

1. $f_1(x) = f_2(x)$, $\forall x \in \{0,1\}^n$
2. $f_1(x) = \overline{f_2(x)}$, $\forall x \in \{0,1\}^n$
3. $f_1(x) = f_2(x)$ for half of the values of $x \in \{0,1\}^n$ and $f_1(x) = \overline{f_2(x)}$ for the other half

Let also $f : \{0,1\}^n \to \{0,1\}^2$ be the function defined as $f(x) = (f_1(x), f_2(x))$ for $x \in \{0,1\}^n$ and consider the following circuit:



If the outcome of the final measurement is the state $|0,\dots,0\rangle$, what can you deduce on the two functions f_1 and f_2? Explain your reasoning in detail.

Solution: The output of the circuit before measurement is given by

$$\begin{aligned}
|\psi\rangle &= (H^{\otimes n} \otimes I^{\otimes 2}) U_f H^{\otimes (n+2)} (|0\rangle^{\otimes n} \otimes |11\rangle) \\
&= (H^{\otimes n} \otimes I^{\otimes 2}) U_f \frac{1}{2^{n/2}} \sum_{x \in \{0,1\}^n} |x\rangle \otimes |-,-\rangle \\
&= (H^{\otimes n} \otimes I^{\otimes 2}) \frac{1}{2^{n/2+1}} \sum_{x \in \{0,1\}^n} |x\rangle \otimes \left| f_1(x) - \overline{f_1(x)}, f_2(x) - \overline{f_2(x)} \right\rangle \\
&= (H^{\otimes n} \otimes I^{\otimes 2}) \frac{1}{2^{n/2}} \sum_{x \in \{0,1\}^n} (-1)^{f_1(x) \oplus f_2(x)} |x\rangle \otimes |-,-\rangle \\
&= \frac{1}{2^n} \sum_{x \in \{0,1\}^n} (-1)^{f_1(x) \oplus f_2(x)} \sum_{z \in \{0,1\}^n} (-1)^{x.z} |z\rangle \otimes |-,-\rangle \\
&= \frac{1}{2^n} \sum_{z \in \{0,1\}^n} \sum_{x \in \{0,1\}^n} (-1)^{f_1(x) \oplus f_2(x) \oplus x.z} |z\rangle \otimes |-,-\rangle
\end{aligned}$$

Therefore, $\Pr\{|0\rangle^{\otimes n}\} = \frac{1}{2^{2n}}\left|\sum_{x\in\{0,1\}^n}(-1)^{f_1(x)\oplus f_2(x)}\right|^2$. So, for cases 1 and 2, this probability is 1 and for case 3 this probability is 0. Hence, given that the measurement is $|0\rangle^{\otimes n}$, we can conclude that either case 1 or case 2 is true.

Exercise 3 *Simon's algorithm (20 points)*

Consider the function $f : \{0,1\}^4 \to \{0,1\}^2$ defined as

$$f(x_1, x_2, x_3, x_4) = (x_1 \oplus x_2 \oplus x_3, x_3 \oplus x_4)$$

One applies Simon's algorithm in order to study this function f.

a) What are the possible output states of the algorithm?

Solution: Observe that, for $x \in \mathbb{F}_2^4$, we can write the function f as a linear code:

$$f(x) = x \cdot \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Let $g_1 = (1,1,1,0), g_2 = (0,0,1,1)$. Let $H = \{h \in \mathbb{F}_2^4 \mid h \cdot g_1 = 0, h \cdot g_1 = 0\}$. Then, we can see that, for any $h \in H$ we have $f(x+h) = f(x)$. Thus, recollecting the properties of Simon's algorithm, it will output one of the vectors in $H^\perp = \text{span}\{g_1, g_2\} = (0,0,0,0), (0,0,1,1), (1,1,0,1), (1,1,1,0)$ with equal probability.

b) Assuming that a sufficiently large amount of measurements is performed, what can one then deduce on the subspace $H \subset \{0,1\}^4$ made of vectors h such that $f(x \oplus h) = f(x)$ for all $x \in \{0,1\}^4$?

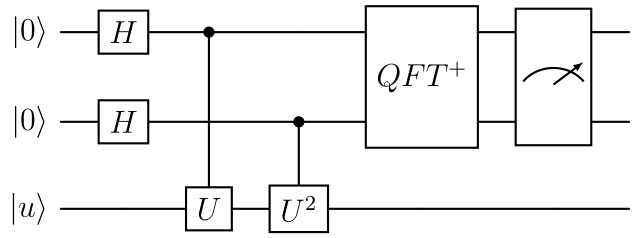
Solution: After a large number of measurements, we obtain the set $H^\perp$. Hence we can infer H by taking the orthogonal complement of $H^\perp$.
We get $H = \{(0,0,0,0), (0,1,1,1), (1,0,1,1), (1,1,0,0)\}$.

Exercise 4 *Quantum phase estimation (25 points)*

Useful for this exercise:

The angle φ such that $\cos(\varphi) = \frac{1}{\sqrt{3}}$ is given approx. by $\varphi \simeq 57.4$ degrees ($\simeq 0.95$ radians).

Consider the following quantum circuit:



where U is the 1-qubit gate with the following matrix representation:

$$U = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & \sqrt{2} \\ -\sqrt{2} & 1 \end{pmatrix}$$

and $|u\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i\,|1\rangle)$.

a) Compute the successive states of the circuit, until the state before the final measurement of the first two qubits.

Solution: First, observe that $|u\rangle$ is an eigen vector of U. We can obtain the corresponding eigen value λ.

$$\begin{aligned} U\,|u\rangle &= \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & \sqrt{2} \\ -\sqrt{2} & 1 \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} \\ &= \frac{1}{\sqrt{3}} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 + i\sqrt{2} \\ -\sqrt{2} + i \end{pmatrix} = \frac{1 + i\sqrt{2}}{\sqrt{3}} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{1 + i\sqrt{2}}{\sqrt{3}}\,|u\rangle \end{aligned}$$

Hence, $\lambda = \frac{1+i\sqrt{2}}{\sqrt{3}} = e^{i\varphi}$. The state before the measurement is given by

$$\begin{aligned} |\psi\rangle &= (QFT^\dagger \otimes I)(CU^2_{2,3})(CU_{1,3})(H^{\otimes 2} \otimes I)(|0,0\rangle \otimes |u\rangle) \\ &= (QFT^\dagger \otimes I)(CU^2_{2,3})(CU_{1,3})\frac{1}{2}\sum_{x_1,x_2} |x_1, x_2\rangle \otimes |u\rangle \\ &= (QFT^\dagger \otimes I)(CU^2_{2,3})\frac{1}{2}\sum_{x_1,x_2} |x_1, x_2\rangle \otimes U^{x_1}\,|u\rangle \\ &= (QFT^\dagger \otimes I)\frac{1}{2}\sum_{x_1,x_2} |x_1, x_2\rangle \otimes U^{x_1+2x_2}\,|u\rangle \\ &= (QFT^\dagger \otimes I)\frac{1}{2}\sum_{x_1,x_2} |x_1, x_2\rangle \otimes U^{x_1+2x_2}\,|u\rangle \\ &= (QFT^\dagger \otimes I)\frac{1}{2}\sum_{x} e^{i\varphi x}\,|x\rangle \otimes |u\rangle \\ &= \frac{1}{2}\sum_{x} e^{i\varphi x} QFT^\dagger\,|x\rangle \otimes |u\rangle \\ &= \frac{1}{2}\sum_{x} e^{i\varphi x} \frac{1}{2}\sum_{z} e^{-i2\pi\frac{xz}{4}}\,|z\rangle \otimes |u\rangle \\ &= \frac{1}{4}\sum_{z}\sum_{x} e^{\frac{i2\pi x}{4}(\frac{4\varphi}{2\pi} - z)}\,|z\rangle \otimes |u\rangle \end{aligned}$$

b) Write down the expression for the probabilities of the four possible output states.

Solution:

$$\Pr\{|0,0\rangle\} = \Pr\{|0\rangle\} = \frac{1}{16}\left|\sum_{x=0}^{3} e^{\frac{i2\pi x}{4}(\frac{4\varphi}{2\pi}-0)}\right|^2 \qquad \Pr\{|1,0\rangle\} = \Pr\{|1\rangle\} = \frac{1}{16}\left|\sum_{x=0}^{3} e^{\frac{i2\pi x}{4}(\frac{4\varphi}{2\pi}-1)}\right|^2$$

$$\Pr\{|0,1\rangle\} = \Pr\{|2\rangle\} = \frac{1}{16}\left|\sum_{x=0}^{3} e^{\frac{i2\pi x}{4}(\frac{4\varphi}{2\pi}-2)}\right|^2 \qquad \Pr\{|1,1\rangle\} = \Pr\{|3\rangle\} = \frac{1}{16}\left|\sum_{x=0}^{3} e^{\frac{i2\pi x}{4}(\frac{4\varphi}{2\pi}-3)}\right|^2$$

c) *Without computing them explicitly*, can you tell which of these probabilities is the highest?

Solution: The intuiition is that we have $\frac{4\varphi}{2\pi} \approx 0.6$, whose closest integer is 1. So, we expect that the state $|1,0\rangle$ to have the highest probability.

A closer look at the expressions for the probabilities in the previous answer also shows that for the state $|1,0\rangle$, the four complex numbers involved in the sum align better than for the other states, leading therefore to a higher probability.

Exercise 5 *Quantum error correction (20 points)*

The following quantum state is sent through a channel:

$$|\psi\rangle = \alpha\,|00000\rangle + \beta\,|11111\rangle\,, \quad \text{where} \quad |\alpha|^2 + |\beta|^2 = 1$$

Assuming that up to 2 bit-flips occur during the transmission, describe the error correction mechanism allowing to recover state $|\psi\rangle$. *Hint:* It may help to first compute how many error patterns may occur.

In particular, please specify which operators to apply:

a) first in order to detect the positions of the bit-flips (if any);

Solution: First, note that total number of possible patterns we need to distinguish is $\binom{5}{0} + \binom{5}{1} + \binom{5}{2} = 16$, corresponding to patterns with 0, 1, and 2 errors. Hence, we need at least 4 measurements. But also, we have 5 quantum bits, and the code "lives" in 1 dimension. Hence, we need exactly 4 measurements.

Let us use the stabilisers $Z_1Z_2, Z_2Z_3, Z_3Z_4, Z_4Z_5$, were we follow the notation used in the lectures. First note that these measurements are compatible (i.e., these stabilisers commute). Second, these are indeed stabilisers of $|\psi\rangle$ as a quick check shows, i.e., for any of them, call it S, $S\,|\psi\rangle = +\,|\psi\rangle$.

Next we need to show that the 16 error patterns that we want to distinguish all map to distinct syndromes so that we can uniquely map back from the syndrome to the error pattern. This can be done by either writing down a table of all the 16 cases and checking directly.

It can also be done conceptually by realising that this corresponds to using a parity-check matrix of the form

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

and that the code encoded by this parity-check matrix has minimum distance 5 since no strict subset of the columns is linearly dependent.

We conclude that the measurements obtained using the above 4 stabilizers allow us to uniquely identify the position of the errors. For example, for $|\psi'\rangle = \alpha |01001\rangle + \beta |10110\rangle$, the measurements would be $(-1, -1, 1, -1)$. The only error pattern that gives this syndrome and has at most 2 errors is $(0, 1, 0, 0, 1)$. (The other solution is $(1, 0, 1, 1, 0)$ but this solution has 3 errors.). Selecting the error pattern with ≤ 2 errors, we see that the error pattern is $(0, 1, 0, 0, 1)$.

b) then in order to correct these bit-flips.

Solution: Once we have the position of the errors, we can apply qubit flips for those particular positions. Hence, inorder to correct the errors, we have to apply one gate from the following set: $\{X_i : 1 \leq i \leq 5\} \cup \{X_i X_j : 1 \leq i < j \leq 5\}$. For example, when the error pattern is $(0, 1, 0, 0, 1)$, we should apply $X_2 X_5$.