Final exam 2025
Quantum Computation

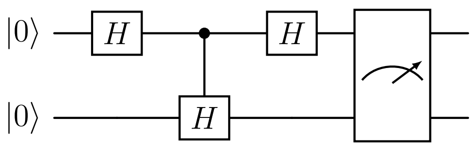
Please pay attention to the presentation of your answers! (4 points)

Exercise 1 *Quiz (16 points)*

For each question, 1 pt for the correct answer, 3 pts for the justification.

a) If a quantum state $\psi = |+,+,+,+\rangle$ is measured in the computational basis, then the probability that in the output state, there are as many qubits in state $|0\rangle$ as there are in state $|1\rangle$ is equal to $\frac{1}{2}$.

b) Let us consider the following circuit:

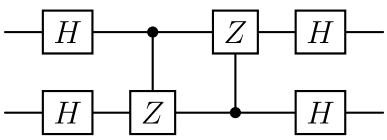


Then the probability that the output state of this circuit is equal to $|1,0\rangle$ is less than $\frac{1}{40}$.

Hint: $\sqrt{2} \simeq 1.41$

NB: The qubits are ordered from top to bottom in the circuit (as was always done in class).

c) The following gate:



is equivalent to a SWAP gate.

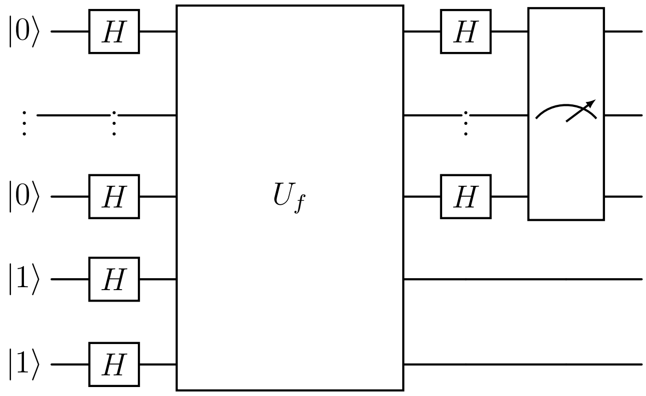
d) The 2-qubit gate with the following matrix representation: $U = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$ can be decomposed into a tensor product of 1-qubit gates.

Exercise 2 *Two Boolean functions (15 points)*

Let $n \geq 1$ and consider two Boolean functions $f_1, f_2 : \{0,1\}^n \to \{0,1\}$ satisfying one of the following three assumptions:

1. $f_1(x) = f_2(x),\ \forall x \in \{0,1\}^n$
2. $f_1(x) = \overline{f_2(x)},\ \forall x \in \{0,1\}^n$
3. $f_1(x) = f_2(x)$ for half of the values of $x \in \{0,1\}^n$ and $f_1(x) = \overline{f_2(x)}$ for the other half

Let also $f : \{0,1\}^n \to \{0,1\}^2$ be the function defined as $f(x) = (f_1(x), f_2(x))$ for $x \in \{0,1\}^n$ and consider the following circuit:



If the outcome of the final measurement is the state $|0,\ldots,0\rangle$, what can you deduce on the two functions f_1 and f_2? Explain your reasoning in detail.

Exercise 3 *Simon's algorithm (20 points)*

Consider the function $f : \{0,1\}^4 \to \{0,1\}^2$ defined as

$$f(x_1, x_2, x_3, x_4) = (x_1 \oplus x_2 \oplus x_3, x_3 \oplus x_4)$$

One applies Simon's algorithm in order to study this function f.

a) What are the possible output states of the algorithm?

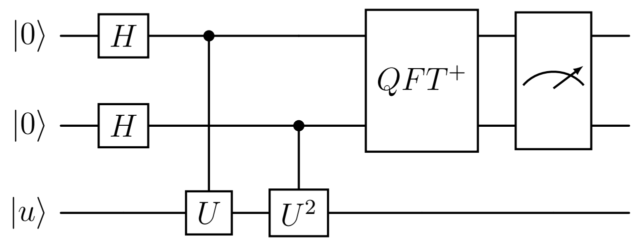
b) Assuming that a sufficiently large amount of measurements is performed, what can one then deduce on the subspace $H \subset \{0,1\}^4$ made of vectors h such that $f(x \oplus h) = f(x)$ for all $x \in \{0,1\}^4$?

Exercise 4 *Quantum phase estimation (25 points)*

Useful for this exercise:

The angle φ such that $\cos(\varphi) = \frac{1}{\sqrt{3}}$ is given approx. by $\varphi \simeq 57.4$ degrees ($\simeq 0.95$ radians).

Consider the following quantum circuit:



where U is the 1-qubit gate with the following matrix representation:

$$U = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & \sqrt{2} \\ -\sqrt{2} & 1 \end{pmatrix}$$

and $|u\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i\,|1\rangle)$.

a) Compute the successive states of the circuit, until the state before the final measurement of the first two qubits.

b) Write down the expression for the probabilities of the four possible output states.

c) *Without computing them explicitly*, can you tell which of these probabilities is the highest?

Exercise 5 *Quantum error correction (20 points)*

The following quantum state is sent through a channel:

$$|\psi\rangle = \alpha\,|00000\rangle + \beta\,|11111\rangle\,, \quad \text{where} \quad |\alpha|^2 + |\beta|^2 = 1$$

Assuming that up to 2 bit-flips occur during the transmission, describe *precisely* the error correction mechanism allowing to recover state $|\psi\rangle$.

In particular, please specify which operators to apply

a) first in order to detect the positions of the bit-flips (if any);

b) then in order to correct these bit-flips;

and **c)** apply your solution to the case where the received state is $|\psi'\rangle = \alpha\,|01001\rangle + \beta\,|10110\rangle$.

Hint: It may help to first compute how many error patterns may occur.