9 A pivoting algorithm for the Frobenius normal form

Let $A \in K^{n\times n}$ be a matrix. We wish to compute the *Frobenius normal form* of A and a transformation matrix $T \in K^{n\times n}$ such that

$$T^{-1} \cdot A \cdot T = \begin{pmatrix} C_{f_1} & & & \\ & C_{f_2} & & \\ & & \ddots & \\ & & & C_{f_k} \end{pmatrix} \tag{18}$$

where each C_{f_i} is a companion matrix with $f_{i+1} \mid f_i$ for each $i \in \{1, \ldots, k-1\}$. Here, the companion matrix C_f of the polynomial $f(x) = a_0 + a_1 x + \cdots + a_{n-1}x^{n-1} + x^n$ is the matrix

$$C_f = \begin{pmatrix} 0 & & & & -a_0 \\ 1 & 0 & & & \vdots \\ & \ddots & \ddots & & \vdots \\ & & 1 & 0 & -a_{d-2} \\ & & & 1 & -a_{d-1} \end{pmatrix} \in K^{d\times d} \tag{19}$$

Recall [4], that the characteristic polynomial of C_f satisfies

$$\det(C_f - xI) = f(x).$$

The algorithm presented here can be found in [1]. It follows the principle of using similarity transformations as in [3]. Recall [4], that the characteristic polynomial of C_f satisfies

$$\det(C_f - xI) = f(x).$$

Let us recall some *elementary row and column operation* and their respective inverse if applied from right and left. For $i \neq i$, let

$$P(i, j, \alpha) = I + \alpha E_{ij} \in K^{n\times n},$$

where E_{ij} denotes the matrix with a single 1 in position i, j and zeros elsewhere. Multiplication of a matrix A from the right by $P(i, j, \alpha)$ performs the column operation

$$A \cdot P(i, j, \alpha) = A(I + \alpha E_{ij}) = A + \alpha A E_{ij}.$$

The product $A \cdot P(i, j, \alpha)$ is obtained from A by the operation

$$C_j \leftarrow C_j + \alpha C_i,$$

where C_k denotes the k-th column of A. The inverse of $P(i, j, \alpha)$ is $P(i, j, \alpha)^{-1} = I - \alpha E_{ij} = P(i, j, -\alpha)$. The matrix $P(i, j, \alpha)^{-1} \cdot A \cdot P(i, j, \alpha)$ is similar to A. Multiplication from the *left* with $P(i, j, \alpha)^{-1}$ corresponds to the following elementary row operation

$$R_i \leftarrow R_i - \alpha R_j,$$

where R_k is the k-th row of A.

We start by computing the minimal polynomial $m_A(x) = a_0 + a_1 x + \cdots, a_{d-1}x^{d-1} + x^d \in K[x]$ as well as a vector $v \in K^n$ with minimal $p_v(x) = m_A(x)$ as in [2, Thm. 2.20]. The vectors

$$v, A\,v, \ldots, A^{d-1}\,v$$

are linearly independent and

$$A^d\,v = -a_0 v - a_1 A\,v - \cdots - a_{d-1}A^{d-1}\,v.$$

Let $J = \{j_1, \dots, j_{n-d}\} \subseteq \{1, \dots, n\}$ be an index set such that

$$\mathscr{B} = \{v, A\,v, \dots, A^{d-1}\,v, e_{j_1}, \dots, e_{j_{n-d}}\} \tag{20}$$

is a basis of K^n and let $T \in K^{n\times n}$ be the matrix with columns being the elements of $\mathscr{B}$ in this order. With respect to the basis $\mathscr{B}$, the matrix A has the form

$$A_{\mathscr{B}} = \begin{pmatrix} C_{m_A} & A_1 \\ 0 & A_2 \end{pmatrix} \in K^{n\times n}, \tag{21}$$

where

$$\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \in K^{nx(n-d)}$$

The idea is next to perform *pivoting* operations via successive *similarity transformations* such that the matrix in the upper right part (initially A_1) becomes all zero. Observe that one has

$$A_{\mathscr{B}} = T^{-1}AT.$$

In more detail, the matrix $A_{\mathscr{B}}$ is as follows

$$A_{\mathscr{B}} = \begin{pmatrix} 0 & & & & -a_0 & b_{1,1} & \cdots\cdots & b_{1,n-d} \\ 1 & 0 & & & \vdots & b_{2,1} & \cdots\cdots & b_{2,n-d} \\ & \ddots & \ddots & & \vdots & & \cdots\cdots & \\ & & 1 & 0 & -a_{d-2} & & \cdots\cdots & \\ & & & 1 & -a_{d-1} & b_{d,1} & \cdots\cdots & b_{d,n-d} \\ 0 & \cdots & \cdots & \cdots & 0 & b_{d+1,1} & \cdots\cdots & b_{d+1,n-d} \\ & \cdots & \cdots & \cdots & & & \cdots\cdots & \\ 0 & \cdots & \cdots & \cdots & 0 & b_{n,1} & \cdots\cdots & b_{n,n-d} \end{pmatrix} \in K^{n\times n}, \tag{22}$$

We next perform elementary column operations and the corresponding *inverse* row operations to zero out all rows of the upper right matrix, except possibly the first row to obtain

$$A' = \begin{pmatrix} 0 & & & & -a_0 & b'_{1,1} & \cdots\cdots & b'_{1,n-d} \\ 1 & 0 & & & \vdots & 0 & \cdots\cdots & 0 \\ & \ddots & \ddots & & \vdots & & \cdots\cdots & \\ & & 1 & 0 & -a_{d-2} & & \cdots\cdots & \\ & & & 1 & -a_{d-1} & 0 & \cdots\cdots & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & b_{d+1,1} & \cdots\cdots & b_{d+1,n-d} \\ & \cdots & \cdots & \cdots & & & \cdots\cdots & \\ 0 & \cdots & \cdots & \cdots & 0 & b_{n,1} & \cdots\cdots & b_{n,n-d} \end{pmatrix} \in K^{n\times n}, \tag{23}$$

This is done in a bottom-up fashion. We begin by subtracting in (22) $b_{d,1}$ times column $d-1$ from column $d+1$, which eliminates $b_{d,1}$ in (22). The inverse operation from the right corresponds to the addition of $b_{d,1}$ times row $d+1$ to row $d-1$. This does not change row d. We continue with subtracting $b_{d,2}$ times column $d-1$ from column $d+2$, which eliminates $b_{d,2}$ and we add $b_{d,2}$ times row $d+2$ to row $d-1$, leaving row d again invariant. These similarity transformations will bring the matrix $A_{\mathscr{B}}$ into the following form

$$\begin{pmatrix} 0 & & & & -a_0 & b'_{1,1} & \cdots\cdots & b'_{1,n-d} \\ 1 & 0 & & & \vdots & b'_{2,1} & \cdots\cdots & b'_{2,n-d} \\ & \ddots & \ddots & & \vdots & & \cdots\cdots & \\ & & 1 & 0 & -a_{d-2} & b'_{d-1,1} & \cdots\cdots & b'_{d-1,n-d} \\ & & & 1 & -a_{d-1} & 0 & \cdots\cdots & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & b_{d+1,1} & \cdots\cdots & b_{d+1,n-d} \\ & \cdots & \cdots & \cdots & & & \cdots\cdots & \\ 0 & \cdots & \cdots & \cdots & 0 & b_{n,1} & \cdots\cdots & b_{n,n-d} \end{pmatrix} \in K^{n\times n}, \tag{24}$$

Next, we are subtracting in (24) $b'_{d-1,1}$ times column $d-2$ from column $d+1$, which eliminates $b'_{d-2,1}$ in (24). And we add $b'_{d-1,1}$ times row $d+1$ to row $d-2$, leaving row $d-1$ invariant. Continuing in this way, we finally arrive at the form (23).

Lemma 9.1. *After these steps, also the first row of the upper right matrix is zero. In other words, all b'_{1j} in* (23) *are zero.*

Proof. Suppose that $b'_{1,j} \neq 0$. We now show that $e_{d+j}, A'e_{d+j}, \ldots, A'^d e_{d+j}$ are linearly independent which is a contradiction to the fact that the minimal polynomial of A is of degree d. These vectors look like this

$$\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ e_j \end{pmatrix}, \begin{pmatrix} b'_{1,j} \\ 0 \\ \vdots \\ 0 \\ * \end{pmatrix}, \begin{pmatrix} * \\ b'_{1,j} \\ \vdots \\ 0 \\ * \end{pmatrix}, \cdots \begin{pmatrix} * \\ * \\ \vdots \\ b'_{1,j} \\ * \end{pmatrix}.$$

They are linearly independent. □

After this procedure, we have transformed the matrix into a shape

$$\begin{pmatrix} C_{m_A} & \\ 0 & A' \end{pmatrix} \in K^{n\times n}.$$

The minimal polynomial of A' divides $m_A(x)$. By induction, we have therefore proved the following theorem.

Theorem 9.2. *Each matrix $A \in K^{n\times n}$ is similar to a matrix of the form* (18).

A nonzero matrix $A \in K^{n\times n}$ is nilpotent if and only if $m_A(x) = x^i$ for some $i \in \mathbb{N}_+$. The elements $a_0, \ldots, a_{i-1}$ are zero in this case. By changing inversing order of the basis and applying the corresponding permutation transformation matrix from left and right, we can therefore record the following.

Corollary 9.3. *Each nilpotent matrix $A \in K^{n\times n}$ is similar to a matrix in Jordan normal form.*

Example

Let

$$A = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 2 & 2 & 0 & 0 & 2 & 1 \\ 0 & 2 & 1 & 2 & 2 & 1 \\ 2 & 2 & 0 & 2 & 2 & 2 \end{pmatrix} \in \mathbb{Z}_3^{6\times 6}.$$

The minimal polynomial of A is$m_A(x) = (x-1)^4$. Choose

$$v = e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

The four vectors v, Av, A^2v, A^3v are linearly independent. Completing them with e_5, e_6 yields the basis

$$\mathscr{B} = \{v, Av, A^2v, A^3v, e_5, e_6\}.$$

One has

$$A_{\mathscr{B}} = P^{-1}AP = \begin{pmatrix} 0 & 0 & 0 & 2 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

where P is the matrix of the basis change. First use column 3 to eliminate the two entries with 2 in row 4 and columns 5 and 6:

$$C_5 \leftarrow C_5 + C_3, \qquad C_6 \leftarrow C_6 + C_3.$$

The corresponding inverse row operations are

$$R_3 \leftarrow R_3 - R_5, \qquad R_3 \leftarrow R_3 - R_6.$$

This yields

$$\begin{pmatrix} 0 & 0 & 0 & 2 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 & 2 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Next use column 2 to eliminate the entries in row 3 and columns 5 and 6:

$$C_5 \leftarrow C_5 + C_2, \qquad C_6 \leftarrow C_6 + C_2.$$

The inverse row operations are

$$R_2 \leftarrow R_2 - R_5, \qquad R_2 \leftarrow R_2 - R_6.$$

This gives

$$\begin{pmatrix} 0 & 0 & 0 & 2 & 1 & 0 \\ 1 & 0 & 0 & 1 & 2 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Finally use column 1 to eliminate the entry in row 2 andcolumn 5:

$$C_5 \leftarrow C_5 + C_1.$$

The inverse row operation is

$$R_1 \leftarrow R_1 - R_5.$$

This gives the final matrix

$$\begin{pmatrix} 0 & 0 & 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The upper-left 4×4 block is

$$C = \begin{pmatrix} 0 & 0 & 0 & 2 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

This is the companion matrix corresponding to the polynomial $(x-1)^4$. Indeed,

$$(x-1)^4 = x^4 + 2x^3 + 0x^2 + 2x + 1$$

in $\mathbb{Z}_3[x]$.

Exercises

1. Show that the matrix

$$C_f = \begin{pmatrix} 0 & & & & -a_0 \\ 1 & 0 & & & \vdots \\ & \ddots & \ddots & & \vdots \\ & & 1 & 0 & -a_{d-2} \\ & & & 1 & -a_{d-1} \end{pmatrix} \text{ is similar to } \begin{pmatrix} -a_{d-1} & 1 & & & \\ -a_{d-2} & 0 & 1 & & \\ \vdots & & \ddots & \ddots & \\ -a_1 & & & 0 & 1 \\ -a_0 & & & & 0 & 1 \end{pmatrix}.$$

References

[1] Adolfo Ballester-Bolinches, Ramón Esteban Romero, and Vicent Pérez Calabuig. "A note on the rational canonical form of an endomorphism of a vector space of finite dimension". In: *Operators and Matrices* 12.3 (2018), pp. 823–836.

[2] Friedrich Eisenbrand. *Algèbre Linéaire Avancée 2*. Lecture Notes. 2026.

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[4] Philippe Michel. *Algèbre Linéaire Avancée*. Notes du cours 1-er Semestre. 2026.