Solution 10 Part 1
CS-526 Learning Theory

1. Tensor decomposition

1.
$$X = \sum_{i,j=1}^{K} \lambda_i \, \lambda_j \, \vec{a}_i \big(\vec{c}_i \otimes_{\text{Kro}} \vec{b}_i\big)^{\mathrm{T}} \big(\vec{c}_j \otimes_{\text{Kro}} \vec{b}_j\big) \vec{a}_j^{\mathrm{T}}$$

From the definition of the Kronecker product, we have that $\big(\vec{c}_i \otimes_{\text{Kro}} \vec{b}_i\big)^{\mathrm{T}} \big(\vec{c}_j \otimes_{\text{Kro}} \vec{b}_j\big) = \big(\vec{c}_i^{\mathrm{T}} \vec{c}_j\big)\big(\vec{b}_i^{\mathrm{T}} \vec{b}_j\big)$. Using the orthogonality of $\vec{b}_i$'s, and the assumption that the vectors are unit norm, we find:

$$X = \sum_{i=1}^{K} \lambda_i^2 \, \vec{a}_i \vec{a}_i^{\mathrm{T}} \tag{1}$$

Since $\vec{a}_i$'s are orthogonal, (**??**) is the spectral decomposition of X, thus X has rank K.

From the tensor T, we can find the matrix X. Computing spectral decomposition of X, we find λ_i^2, and the vectors $\vec{a}_i$'s. Since, λ_i's are assumed to be positive, we can find λ_i's.

2. Form the mode-2 matrization of T, which can be expressed as:

$$T_{(2)} = \sum_{i=1}^{K} \lambda_i \, \vec{b}_i \big(\vec{a}_i \otimes_{\text{Kro}} \vec{c}_i\big)^{\mathrm{T}}$$

Then, compute the matrix $Y = T_{(2)} T_{(2)}^{\mathrm{T}}$. Following the same steps as in the previous part, the spectral decomposition of Y is:

$$Y = \sum_{i=1}^{K} \lambda_i^2 \, \vec{b}_i \vec{b}_i^{\mathrm{T}}$$

Therefore, $\vec{b}_i$'s can be recovered as the eigenvectors of Y.

3. For each $1 \le i \le K$, we consider the following linear transformation of T:

$$T(\vec{a}_i, \vec{b}_i, .) = \sum_{j=1}^{K} \lambda_j \, (\vec{a}_i^{\mathrm{T}} \vec{a}_j) \, (\vec{b}_i^{\mathrm{T}} \vec{b}_j) \vec{c}_j = \lambda_i \, \vec{c}_i$$

where in the last equality we used the orthogonality assumption of $\vec{a}_i$'s (or $\vec{b}_i$'s). Since, we know λ_i, we can find $\vec{c}_i$ from the above transformation.