

Astrophysics IV: Stellar and galactic dynamics

Solutions

Problem 1:

The Jeans equations are obtained from the Boltzmann equations by computing moments of various orders.

- A) Direct integration on velocities (moment of order 0)
- B) Integration on the velocities after multiplying by one component of the velocity (moment of order 1)

Here are a few properties to keep in mind :

$$\begin{aligned}
 1) \ f \rightarrow 0 \text{ when } |v_i| \rightarrow \infty \quad & 2) \ m \int f \, d^3\mathbf{v} = \rho \quad & 3) \ m \int v_i f \, d^3\mathbf{v} = \rho \bar{v}_i \\
 4) \ \int v_i v_j f \, d^3\mathbf{v} = \rho \bar{v}_i \bar{v}_j \quad & 5) \ \bar{v}_i \bar{v}_j + \sigma_{ij}^2 = \overline{v_i v_j}
 \end{aligned}$$

where we set $m = 1$.

A - Moment 0: The equation is:

$$\frac{\partial \nu}{\partial t} + \sum_i \frac{\partial}{\partial x_i} (\nu \bar{v}_i) = 0, \quad (1)$$

which reads in vectorial notation:

$$\frac{\partial \nu}{\partial t} + \nabla \cdot (\nu \bar{\mathbf{v}}) = 0. \quad (2)$$

In spherical coordinates, the divergence of a vector reads:

$$\nabla \cdot \mathbf{F} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 F_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta F_\theta) + \frac{1}{r \sin \theta} \frac{\partial F_\phi}{\partial \phi}. \quad (3)$$

Consequently, the equation becomes :

$$\frac{\partial \nu}{\partial t} + \frac{\partial}{\partial r} (\nu \bar{v}_r) + \frac{2}{r} \nu \bar{v}_r + \frac{1}{r} \frac{\partial}{\partial \theta} (\nu \bar{v}_\theta) + \frac{\cot \theta}{r} \nu \bar{v}_\theta + \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} (\nu \bar{v}_\phi) = 0. \quad (4)$$

The systems with spherical symmetry have negligible meridional motions, hence $\bar{v}_\theta = 0$. Furthermore, a possible rotation of the system is done at an azimuthal symmetry, i.e. $\partial \bar{v}_\phi / \partial \phi = 0$. (In short, there can be no angular dependencies in a spherically symmetric system, hence $\partial / \partial \theta = 0$, $\partial / \partial \phi = 0$)

Thus, we get for the moment 0:

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \rho \bar{v}_r) = \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial r} (\rho \bar{v}_r) + \frac{2}{r} \rho \bar{v}_r = 0 \quad (5)$$

B - First moment In vectorial notation, the first moment reads:

$$\frac{\partial \bar{\mathbf{v}}}{\partial t} + (\bar{\mathbf{v}} \cdot \nabla) \bar{\mathbf{v}} = -\nabla \Phi - \frac{1}{\rho} \nabla \cdot (\rho \boldsymbol{\sigma}^2), \quad (6)$$

Transformation to spherical coordinates is risky (because of the divergence of the tensor), so it is better to start directly from the collisionless Boltzmann equation expressed in spherical coordinates.

$$\frac{\partial f}{\partial t} + v_r \frac{\partial f}{\partial r} + \frac{v_\theta}{r} \frac{\partial f}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial f}{\partial \phi} + \left(\frac{v_\theta^2 + v_\phi^2}{r} - \frac{\partial \Phi}{\partial r} \right) \frac{\partial f}{\partial v_r} \quad (7)$$

$$+ \frac{1}{r} (v_\phi^2 \cot \theta - v_r v_\theta) \frac{\partial f}{\partial v_\theta} - \frac{1}{r} [v_\phi (v_r + v_\theta \cot \theta)] \frac{\partial f}{\partial v_\phi} = 0. \quad (8)$$

We compute the radial Jeans equation by multiplying the collisionless Boltzmann equation by v_r and integrating over velocities:

$$\int v_r^2 \frac{\partial f}{\partial r} d^3 \mathbf{v} = \frac{\partial}{\partial r} \int f v_r^2 d^3 \mathbf{v} = \frac{\partial}{\partial r} (\rho \overline{v_r^2}) \quad (9)$$

$$\int \frac{v_r v_\theta}{r} \frac{\partial f}{\partial \theta} d^3 \mathbf{v} = \frac{1}{r} \frac{\partial}{\partial \theta} \int f v_r v_\theta d^3 \mathbf{v} = \frac{1}{r} \frac{\partial}{\partial \theta} (\rho \overline{v_r v_\theta}) = 0 \quad (10)$$

$$\int \frac{v_r v_\phi}{r \sin \theta} \frac{\partial f}{\partial \phi} d^3 \mathbf{v} = \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} \int f v_r v_\phi d^3 \mathbf{v} = \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} (\rho \overline{v_r v_\phi}) = 0, \quad (11)$$

where the null values in the last two equations come from the assumption of spherical symmetry,

$$\int \frac{v_r v_\theta^2}{r} \frac{\partial f}{\partial v_r} d^3 \mathbf{v} = \frac{1}{r} \int dv_\phi \int v_\theta^2 dv_\theta \int v_r \frac{\partial f}{\partial v_r} dv_r = -\frac{1}{r} \int f v_\theta^2 d^3 \mathbf{v} = -\rho \frac{\overline{v_\theta^2}}{r}, \quad (12)$$

where the integral on v_r was integrated by parts. Similarly,

$$\int \frac{v_r v_\phi^2}{r} \frac{\partial f}{\partial v_r} d^3 \mathbf{v} = \frac{1}{r} \int dv_\theta \int v_\phi^2 dv_\phi \int v_r \frac{\partial f}{\partial v_r} dv_r = -\frac{1}{r} \int f v_\phi^2 d^3 \mathbf{v} = -\rho \frac{\overline{v_\phi^2}}{r} \quad (13)$$

$$\int \frac{\partial \Phi}{\partial r} v_r \frac{\partial f}{\partial v_r} d^3 \mathbf{v} = \frac{\partial \Phi}{\partial r} \int dv_\phi \int dv_\theta \int v_r \frac{\partial f}{\partial v_r} dv_r = -\frac{\partial \Phi}{\partial r} \int f d^3 \mathbf{v} = -\rho \frac{\partial \Phi}{\partial r}, \quad (14)$$

still with the same integration by parts,

$$\int v_r v_\phi^2 \frac{\cot \theta}{r} \frac{\partial f}{\partial v_\theta} d^3 \mathbf{v} = \frac{\cot \theta}{r} \int v_r dv_r \int v_\phi^2 dv_\phi \int \frac{\partial f}{\partial v_\theta} dv_\theta = 0, \quad (15)$$

after integration by parts of the integral on v_θ , we get:

$$\int \frac{v_r^2 v_\theta}{r} \frac{\partial f}{\partial v_\theta} d^3 \mathbf{v} = \frac{1}{r} \int v_r^2 dv_r \int dv_\phi \int v_\theta \frac{\partial f}{\partial v_\theta} dv_\theta = -\frac{1}{r} \int f v_r^2 d^3 \mathbf{v} = -\rho \frac{\overline{v_r^2}}{r}, \quad (16)$$

and similarly,

$$\int \frac{v_r^2 v_\phi}{r} \frac{\partial f}{\partial v_\phi} d^3\mathbf{v} = \frac{1}{r} \int v_r^2 dv_r \int dv_\theta \int v_\phi \frac{\partial f}{\partial v_\phi} dv_\phi = -\frac{1}{r} \int f v_r^2 d^3\mathbf{v} = -\frac{\overline{\rho v_r^2}}{r}. \quad (17)$$

Finally,

$$\begin{aligned} \int \frac{v_r v_\theta v_\phi \cot \theta}{r} \frac{\partial f}{\partial v_\phi} d^3\mathbf{v} &= \frac{\cot \theta}{r} \int v_r dv_r \int v_\theta dv_\theta \int v_\phi \frac{\partial f}{\partial v_\phi} dv_\phi \\ &= -\frac{\cot \theta}{r} \int v_r v_\theta f d^3\mathbf{v} = -\frac{\overline{\rho v_r v_\theta \cot \theta}}{r}, \end{aligned} \quad (18)$$

where we have again performed an integration by parts for the integral on v_ϕ . Since we're in a spherically symmetric case, we may choose any fixed θ , and we choose θ such that $\cot(\theta) = 0$.

Putting everything together finally results in the general Jeans equation for spherical symmetry:

$$\frac{\partial(\rho \overline{v_r})}{\partial t} + \frac{\partial(\rho \overline{v_r^2})}{\partial r} + \frac{\rho}{r} \left[2\overline{v_r^2} - (\overline{v_\theta^2} + \overline{v_\phi^2}) \right] = -\rho \frac{\partial \Phi}{\partial r}. \quad (19)$$

One can introduce the velocity dispersion : $\overline{v_i^2} = \sigma_i^2 + \overline{v_i}^2$

Isotropic systems: $\overline{v_\phi} = \overline{v_\theta} = \overline{v_r}$

For a stationary system with isotropic velocities, the Jeans equation reduces to :

$$\frac{d(\rho \sigma_r^2)}{dr} = -\rho \frac{d\Phi}{dr} \quad (20)$$

The potential Φ in the Jeans equation is always the gravitational potential representing the total mass of the system. ρ may be a mass density, a number density, or even a luminosity density.

Problem 2:

Plummer:

$$\rho = \frac{3M}{4\pi a^3} \left[1 + \left(\frac{r}{a} \right)^2 \right]^{-5/2} \quad (21)$$

$$\Phi = -\frac{GM}{\sqrt{r^2 + a^2}} \quad (22)$$

$$\frac{d\Phi}{dr} = GM r (r^2 + a^2)^{-3/2}. \quad (23)$$

Introducing these expressions into the last equation of Problem 2, we get:

$$\frac{d(\rho \sigma_r^2)}{dr} = -\frac{3M}{4\pi a^3} \left[1 + \left(\frac{r}{a} \right)^2 \right]^{-5/2} \cdot GM r (r^2 + a^2)^{-3/2} \quad (24)$$

$$= -\frac{3GM^2 a^2}{4\pi} \frac{r}{(a^2 + r^2)^{5/2} (a^2 + r^2)^{3/2}} = -\frac{3GM^2 a^2}{4\pi} \frac{r}{(a^2 + r^2)^4} \quad (25)$$

By integration, taking into account that $\rho\sigma_r^2$ must tend to zero when M tends to zero, one obtains

$$\rho\sigma_r^2 = \frac{GM^2a^2}{8\pi(r^2 + a^2)^3}$$

Finally,

$$\sigma_r^2 = \frac{GM}{6\sqrt{r^2 + a^2}}$$

Problem 3:

From collisionless Boltzmann equation in cylindrical coordinates in term of velocities write:

$$\frac{\partial f}{\partial t} + v_R \frac{\partial f}{\partial R} + \frac{v_\phi}{R} \frac{\partial f}{\partial \phi} + v_z \frac{\partial f}{\partial z} + \left[\frac{v_\phi^2}{R} - \frac{\partial \Phi}{\partial R} \right] \frac{\partial f}{\partial v_R} - \frac{v_R v_\phi}{R} \frac{\partial f}{\partial v_\phi} - \frac{\partial \Phi}{\partial z} \frac{\partial f}{\partial v_z} = 0 \quad (26)$$

Assuming a steady state ($\frac{\partial f}{\partial t} = 0$) and an azimuthal symmetry ($\frac{\partial \Phi}{\partial \phi} = 0, \frac{\partial f}{\partial \phi} = 0$), we get:

$$v_R \frac{\partial f}{\partial R} + v_z \frac{\partial f}{\partial z} + \left[\frac{v_\phi^2}{R} - \frac{\partial \Phi}{\partial R} \right] \frac{\partial f}{\partial v_R} - \frac{v_R v_\phi}{R} \frac{\partial f}{\partial v_\phi} - \frac{\partial \Phi}{\partial z} \frac{\partial f}{\partial v_z} = 0 \quad (27)$$

The first moment in v_R writes:

$$\int dv_R dv_\phi dv_z v_R \left[v_R \frac{\partial f}{\partial R} + v_z \frac{\partial f}{\partial z} + \left[\frac{v_\phi^2}{R} - \frac{\partial \Phi}{\partial R} \right] \frac{\partial f}{\partial v_R} - \frac{v_R v_\phi}{R} \frac{\partial f}{\partial v_\phi} - \frac{\partial \Phi}{\partial z} \frac{\partial f}{\partial v_z} \right] = 0 \quad (28)$$

Using the rules given in Problem 1, we can write:

$$\int dv_R dv_\phi dv_z v_R v_R \frac{\partial f}{\partial R} = \frac{\partial}{\partial R} (\overline{\nu v_R^2}) \quad (29)$$

$$\int dv_R dv_\phi dv_z v_R v_z \frac{\partial f}{\partial z} = \frac{\partial}{\partial z} (\overline{\nu v_R v_z}) \quad (30)$$

$$\int dv_R dv_\phi dv_z v_R \frac{v_\phi^2}{R} \frac{\partial f}{\partial v_R} = -\frac{\nu}{R} \overline{v_\phi^2} \quad (31)$$

$$- \int dv_R dv_\phi dv_z v_R \frac{\partial \Phi}{\partial R} \frac{\partial f}{\partial v_R} = \nu \frac{\partial \Phi}{\partial R} \quad (32)$$

$$- \int dv_R dv_\phi dv_z v_R \frac{v_R v_\phi}{R} \frac{\partial f}{\partial v_\phi} = \frac{\nu}{R} \overline{v_R^2} \quad (33)$$

$$- \int dv_R dv_\phi dv_z v_R \frac{\partial \Phi}{\partial z} \frac{\partial f}{\partial v_z} = 0 \quad (34)$$

Putting all together, we get:

$$\frac{\partial}{\partial R} (\overline{\nu v_R^2}) + \frac{\partial}{\partial z} (\overline{\nu v_R v_z}) + \nu \left(\frac{\overline{v_R^2} - \overline{v_\phi^2}}{R} + \frac{\partial \Phi}{\partial R} \right) = 0 \quad (35)$$

The the first two other moment are obtained by successively multiplying by v_ϕ and v_z and integrating over the velocities. Using the same mathematical tricks, we obtain:

$$\frac{1}{R} \frac{\partial}{\partial R} (R \nu \overline{v_R v_z}) + \frac{\partial}{\partial z} (\nu \overline{v_z^2}) + \nu \frac{\partial \Phi}{\partial z} = 0, \quad (36)$$

and:

$$\frac{1}{R^2} \frac{\partial}{\partial R} (R^2 \nu \overline{v_R v_\phi}) + \frac{\partial}{\partial z} (\nu \overline{v_z v_\phi}) = 0. \quad (37)$$