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Exercise Set 10: Solutions  
Quantum Computation

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**Exercise 1** *Observables and Pauli operators*

(a) For  $Z$ , the computational basis vectors are already eigenvectors:

$$Z|0\rangle = |0\rangle, \quad Z|1\rangle = -|1\rangle.$$

Thus the eigenvalues are  $+1$  and  $-1$ , with normalized eigenvectors

$$+1 : |0\rangle, \quad -1 : |1\rangle.$$

For  $X$ , we solve  $X(a|0\rangle + b|1\rangle) = \lambda(a|0\rangle + b|1\rangle)$ . Since

$$X(a|0\rangle + b|1\rangle) = b|0\rangle + a|1\rangle,$$

we need  $b = \lambda a$  and  $a = \lambda b$ . Hence  $\lambda^2 = 1$ , so  $\lambda = \pm 1$ . For  $\lambda = +1$ , we have  $a = b$ , and for  $\lambda = -1$ , we have  $a = -b$ . Therefore

$$+1 : |+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}, \quad -1 : |-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}.$$

(b) Since the  $Z$ -eigenbasis is  $\{|0\rangle, |1\rangle\}$ , measuring  $Z$  gives

outcome	probability	post-measurement state
+1	$ \alpha ^2$	$ 0\rangle$
-1	$ \beta ^2$	$ 1\rangle$

provided the corresponding probability is non-zero. If, for example,  $\alpha = 0$ , then the outcome  $+1$  simply never occurs.

To measure  $X$ , first rewrite the state in the  $X$ -eigenbasis. We use

$$|0\rangle = \frac{|+\rangle + |-\rangle}{\sqrt{2}}, \quad |1\rangle = \frac{|+\rangle - |-\rangle}{\sqrt{2}}.$$

Therefore

$$|\psi\rangle = \alpha \frac{|+\rangle + |-\rangle}{\sqrt{2}} + \beta \frac{|+\rangle - |-\rangle}{\sqrt{2}} = \frac{\alpha + \beta}{\sqrt{2}} |+\rangle + \frac{\alpha - \beta}{\sqrt{2}} |-\rangle.$$

Thus measuring  $X$  gives

outcome	probability	post-measurement state
+1	$\frac{ \alpha + \beta ^2}{2}$	$ +\rangle$
-1	$\frac{ \alpha - \beta ^2}{2}$	$ -\rangle$

again ignoring outcomes whose probability is zero. The two probabilities add to one because

$$\frac{|\alpha + \beta|^2 + |\alpha - \beta|^2}{2} = |\alpha|^2 + |\beta|^2 = 1.$$

- (c) By the fact stated in the exercise, a measurement is guaranteed to leave the state unchanged exactly when the state is already an eigenvector of the measured observable. Thus a  $Z$ -measurement leaves the state unchanged precisely for

$$|\psi\rangle = e^{i\theta} |0\rangle \quad \text{or} \quad |\psi\rangle = e^{i\theta} |1\rangle,$$

where the phase  $e^{i\theta}$  is physically irrelevant. Equivalently, in  $\alpha |0\rangle + \beta |1\rangle$ , one of the two coefficients is zero.

Similarly, an  $X$ -measurement leaves the state unchanged precisely for

$$|\psi\rangle = e^{i\theta} |+\rangle \quad \text{or} \quad |\psi\rangle = e^{i\theta} |-\rangle.$$

Here is an example showing that a first measurement can change the statistics of a later one: Start with  $|0\rangle$ . If we measure  $Z$  immediately, the outcome is certainly  $+1$ . But if we first measure  $X$ , the state becomes  $|+\rangle$  or  $|-\rangle$ , each with probability  $1/2$ . A later  $Z$ -measurement then gives  $+1$  and  $-1$  with probabilities  $1/2$  and  $1/2$ .

The point is that measuring in one basis can destroy certainty in another, incompatible basis.

- (d) We compute

$$XZ = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

and

$$ZX = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Thus

$$ZX = -XZ.$$

So

$$[X, Z] = XZ - ZX = 2XZ = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}, \quad \{X, Z\} = XZ + ZX = 0.$$

This anti-commutation implies that  $X$  and  $Z$  cannot have a non-zero simultaneous eigenvector. Indeed, suppose  $|\phi\rangle \neq 0$  satisfied

$$X|\phi\rangle = \lambda|\phi\rangle, \quad Z|\phi\rangle = \mu|\phi\rangle,$$

with  $\lambda, \mu \in \{+1, -1\}$ . Then

$$XZ|\phi\rangle = \mu X|\phi\rangle = \mu\lambda|\phi\rangle,$$

whereas

$$ZX|\phi\rangle = \lambda Z|\phi\rangle = \lambda\mu|\phi\rangle.$$

But  $ZX = -XZ$ , so these two expressions must also differ by a minus sign. Hence

$$\lambda\mu|\phi\rangle = -\lambda\mu|\phi\rangle,$$

which forces  $|\phi\rangle = 0$ . This contradicts the assumption. Therefore there is no non-zero simultaneous eigenvector.

(e) Operators acting on different qubits commute. Therefore

$$[Z_i, Z_j] = 0 \quad \text{for all } i, j,$$

and

$$[X_i, X_j] = 0 \quad \text{for all } i, j.$$

For  $X_i$  and  $Z_j$ , there are two cases. If  $i \neq j$ , they act on different qubits, so

$$[X_i, Z_j] = 0.$$

If  $i = j$ , they act on the same qubit and anti-commute:

$$Z_i X_i = -X_i Z_i.$$

Thus

$$[X_i, Z_i] = X_i Z_i - Z_i X_i = 2X_i Z_i \neq 0.$$

More generally, consider two tensor products made only of  $I$ ,  $X$ , and  $Z$ . At each tensor factor, the only possible obstruction to commutation is an  $X$  meeting a  $Z$ . Every such place contributes a minus sign when we swap the two tensor products. Hence the two tensor products commute if and only if the number of positions where one operator has  $X$  and the other has  $Z$  is even. They anti-commute if that number is odd.

(f) The operators

$$A_1 = Z_2 Z_3, \quad A_2 = Z_1 Z_3, \quad A_3 = Z_1 Z_2$$

are all products of  $Z$  operators. Since  $Z$  operators commute with one another, we have

$$[A_i, A_j] = 0 \quad \text{for all } i, j.$$

Moreover,

$$A_1 A_2 = (Z_2 Z_3)(Z_1 Z_3) = Z_1 Z_2 Z_3^2 = Z_1 Z_2 = A_3,$$

because  $Z_3^2 = I$ .

If  $A_1$  is measured with outcome  $s_1$  and  $A_2$  is measured with outcome  $s_2$ , then  $A_3 = A_1 A_2$  has outcome

$$s_3 = s_1 s_2.$$

Thus the third measurement is redundant: it gives a consistency check, but no new independent information.

- (g) For a self-adjoint operator  $A$  with  $A^2 = I$ , the projector onto the eigenspace with eigenvalue  $s \in \{+1, -1\}$  is

$$\frac{I + sA}{2}.$$

Indeed, if  $A|v\rangle = s|v\rangle$ , then

$$\frac{I + sA}{2} |v\rangle = \frac{1 + s^2}{2} |v\rangle = |v\rangle,$$

while if  $A|v\rangle = -s|v\rangle$ , then

$$\frac{I + sA}{2} |v\rangle = \frac{1 - s^2}{2} |v\rangle = 0.$$

Since  $A_1$  and  $A_2$  commute, the projector onto the common eigenspace with

$$A_1 = s_1, \quad A_2 = s_2$$

is the product of the two individual projectors:

$$P_{s_1, s_2} = \frac{I + s_1 A_1}{2} \frac{I + s_2 A_2}{2} = \frac{1}{4} (I + s_1 A_1)(I + s_2 A_2).$$

This proves the formula.

To see the dimension, look at computational basis vectors  $|x_1 x_2 x_3\rangle$ . On such a vector,

$$A_1 |x_1 x_2 x_3\rangle = (-1)^{x_2 + x_3} |x_1 x_2 x_3\rangle,$$

and

$$A_2 |x_1 x_2 x_3\rangle = (-1)^{x_1 + x_3} |x_1 x_2 x_3\rangle.$$

The simultaneous eigenspaces are

$(s_1, s_2)$	basis vectors
(+, +)	$ 000\rangle,  111\rangle$
(+, -)	$ 100\rangle,  011\rangle$
(-, +)	$ 010\rangle,  101\rangle$
(-, -)	$ 001\rangle,  110\rangle$

Each simultaneous eigenspace is therefore two-dimensional.

**Exercise 2** *Syndrome measurements for the three-qubit repetition code*

(a) The observables are

$$A_1 = Z_2 Z_3, \quad A_2 = Z_1 Z_3, \quad A_3 = Z_1 Z_2.$$

They compare the parities of pairs of qubits. For example,  $A_1 = Z_2 Z_3$  gives  $+1$  if qubits 2 and 3 are equal in the computational basis, and  $-1$  if they are different.

The completed table is

actual error	resulting state	$s_1$	$s_2$	$s_3$	correction
none	$\alpha  000\rangle + \beta  111\rangle$	$+1$	$+1$	$+1$	$I$
$X_1$	$\alpha  100\rangle + \beta  011\rangle$	$+1$	$-1$	$-1$	$X_1$
$X_2$	$\alpha  010\rangle + \beta  101\rangle$	$-1$	$+1$	$-1$	$X_2$
$X_3$	$\alpha  001\rangle + \beta  110\rangle$	$-1$	$-1$	$+1$	$X_3$

As an example, consider the error  $X_1$ . The state is

$$X_1 |\psi_0\rangle = \alpha |100\rangle + \beta |011\rangle.$$

For both  $|100\rangle$  and  $|011\rangle$ , qubits 2 and 3 are equal, so  $A_1$  gives  $+1$ . But qubits 1 and 3 are different, and qubits 1 and 2 are different, so  $A_2$  and  $A_3$  give  $-1$ .

(b) Each possible resulting state is a simultaneous eigenvector of  $A_1$  and  $A_2$ . More precisely,

state	$A_1$ -eigenvalue	$A_2$ -eigenvalue
$\alpha  000\rangle + \beta  111\rangle$	$+1$	$+1$
$\alpha  100\rangle + \beta  011\rangle$	$+1$	$-1$
$\alpha  010\rangle + \beta  101\rangle$	$-1$	$+1$
$\alpha  001\rangle + \beta  110\rangle$	$-1$	$-1$

The key point is that, in each row, the two basis vectors appearing in the superposition have the same syndrome. For instance,

$$A_1 |100\rangle = + |100\rangle, \quad A_1 |011\rangle = + |011\rangle,$$

and

$$A_2 |100\rangle = - |100\rangle, \quad A_2 |011\rangle = - |011\rangle.$$

Therefore

$$A_1(\alpha |100\rangle + \beta |011\rangle) = +(\alpha |100\rangle + \beta |011\rangle),$$

and

$$A_2(\alpha |100\rangle + \beta |011\rangle) = -(\alpha |100\rangle + \beta |011\rangle).$$

The same reasoning applies to the other rows.

Thus the pair  $(s_1, s_2)$  identifies which qubit was flipped. However, the syndrome does not depend on the values of  $\alpha$  and  $\beta$ . The measurement distinguishes four two-dimensional subspaces, not the individual states inside those subspaces. Inside one such subspace, the state may still be any superposition with coefficients  $\alpha$  and  $\beta$ .

This is exactly what we want: the syndrome tells us about the error, but not about the encoded quantum information.

(c) The order of measuring  $A_1$  and  $A_2$  does not matter because

$$[A_1, A_2] = 0.$$

Commuting observables have a common eigenbasis, and measuring one does not disturb a measurement of the other.

Measuring  $A_3$  does not help to identify the error in the ideal setting because

$$A_3 = A_1 A_2.$$

Consequently, if  $A_1$  gives  $s_1$  and  $A_2$  gives  $s_2$ , then  $A_3$  must give

$$s_3 = s_1 s_2.$$

Thus  $A_3$  is redundant. It may be useful as a consistency check in a more realistic situation with measurement errors, but it contains no additional independent information in this ideal exercise.

(d) After the syndrome has been measured, we know which correction should be applied. For example, if the syndrome is  $(+1, -1)$  for  $(A_1, A_2)$ , the error was  $X_1$ , so we apply the unitary gate  $X_1$ :

$$X_1(\alpha |100\rangle + \beta |011\rangle) = \alpha |000\rangle + \beta |111\rangle.$$

This restores the encoded state.

It is not a problem that the correction  $X_i$  may fail to commute with some syndrome observables. The syndrome has already been extracted. After correction, we want the state to return to the code space, where the syndrome is again  $(+1, +1, +1)$ . We are not trying to preserve the old syndrome; the old syndrome described the error before correction.

By contrast, measuring the observable  $X_i$  is not the same operation as applying the unitary gate  $X_i$ . A unitary correction deterministically flips the  $i$ -th bit. A measurement of  $X_i$  projects onto the  $+1$  or  $-1$  eigenspace of  $X_i$  and has a random outcome.

For instance, suppose the error was  $X_1$ , so the state is

$$|\phi\rangle = \alpha |100\rangle + \beta |011\rangle.$$

Applying  $X_1$  gives the desired state

$$X_1 |\phi\rangle = \alpha |000\rangle + \beta |111\rangle.$$

But measuring  $X_1$  uses the projectors

$$\frac{I + X_1}{2}, \quad \frac{I - X_1}{2}.$$

The two possible post-measurement states are proportional to

$$|\phi\rangle + X_1 |\phi\rangle = \alpha |100\rangle + \beta |011\rangle + \alpha |000\rangle + \beta |111\rangle,$$

or

$$|\phi\rangle - X_1 |\phi\rangle = \alpha |100\rangle + \beta |011\rangle - \alpha |000\rangle - \beta |111\rangle.$$

These are not the corrected code state. Instead, the measurement has created a state involving both the error subspace and the code subspace. Thus measuring  $X_i$  is not a valid substitute for applying  $X_i$  as a unitary correction.

### Exercise 3 *The Shor code*

The circuit is best understood in two stages. Label the qubits  $1, 2, \dots, 9$ , with qubit 1 initially in the state

$$|\psi\rangle = \alpha |0\rangle + \beta |1\rangle,$$

and all other qubits initialized as  $|0\rangle$ .

**First stage: create a three-qubit repetition code on qubits 1, 4, 7.**

The first two CNOT gates use qubit 1 as the control and qubits 4 and 7 as targets. Therefore

$$\alpha |0\rangle_1 |0\rangle_4 |0\rangle_7 + \beta |1\rangle_1 |0\rangle_4 |0\rangle_7$$

becomes

$$\alpha |0\rangle_1 |0\rangle_4 |0\rangle_7 + \beta |1\rangle_1 |1\rangle_4 |1\rangle_7.$$

In short, on the qubits 1, 4, 7 we have

$$\alpha |000\rangle_{1,4,7} + \beta |111\rangle_{1,4,7}.$$

Next, the circuit applies Hadamard gates to qubits 1, 4, 7. Since

$$H |0\rangle = |+\rangle, \quad H |1\rangle = |-\rangle,$$

this state becomes

$$\alpha |+++ \rangle_{1,4,7} + \beta |-- - \rangle_{1,4,7}.$$

This is the phase-flip repetition structure: the logical information has been repeated in the  $\{|+\rangle, |-\rangle\}$  basis across the three blocks.

**Second stage: encode each of the three qubits into a bit-flip repetition block.**

Now consider what happens to one block. Suppose the first qubit of a block is in the state  $|+\rangle$  and the other two qubits are  $|0\rangle$ . Applying two CNOT gates from the first qubit to the other two gives

$$|+\rangle |0\rangle |0\rangle = \frac{|0\rangle |0\rangle |0\rangle + |1\rangle |0\rangle |0\rangle}{\sqrt{2}} \mapsto \frac{|000\rangle + |111\rangle}{\sqrt{2}} = |+_B\rangle.$$

Similarly,

$$|-\rangle |0\rangle |0\rangle = \frac{|0\rangle |0\rangle |0\rangle - |1\rangle |0\rangle |0\rangle}{\sqrt{2}} \mapsto \frac{|000\rangle - |111\rangle}{\sqrt{2}} = |-_B\rangle.$$

The circuit applies this procedure to the three blocks (1, 2, 3), (4, 5, 6), and (7, 8, 9). Therefore

$$|+++ \rangle_{1,4,7} \mapsto |+_B\rangle_{123} |+_B\rangle_{456} |+_B\rangle_{789},$$

and

$$|-- - \rangle_{1,4,7} \mapsto |-_B\rangle_{123} |-_B\rangle_{456} |-_B\rangle_{789}.$$

By linearity, the final state is

$$\alpha |+_B\rangle_{123} |+_B\rangle_{456} |+_B\rangle_{789} + \beta |-_B\rangle_{123} |-_B\rangle_{456} |-_B\rangle_{789}.$$

Using the definitions

$$|0\rangle_{\mathcal{S}} = |+_B\rangle |+_B\rangle |+_B\rangle, \quad |1\rangle_{\mathcal{S}} = |-_B\rangle |-_B\rangle |-_B\rangle,$$

we get exactly

$$\alpha |0\rangle_{\mathcal{S}} + \beta |1\rangle_{\mathcal{S}}.$$

Thus the circuit realizes the Shor encoding.

**Exercise 4** *The phase-flip repetition code and the role of the Hadamard gate*

(a) Since  $H|0\rangle = |+\rangle$ , we have

$$H^{\otimes 3}|000\rangle = H|0\rangle \otimes H|0\rangle \otimes H|0\rangle = |+++ \rangle.$$

Similarly, since  $H|1\rangle = |-\rangle$ ,

$$H^{\otimes 3}|111\rangle = H|1\rangle \otimes H|1\rangle \otimes H|1\rangle = |-- \rangle.$$

Therefore applying  $H^{\otimes 3}$  to the bit-flip repetition code

$$\alpha |000\rangle + \beta |111\rangle$$

gives

$$\alpha |+++ \rangle + \beta |-- \rangle,$$

which is precisely the phase-flip repetition code.

(b) The bit-flip repetition code is checked by  $Z$ -type parity observables, for example

$$Z_1 Z_2, \quad Z_2 Z_3.$$

The phase-flip code is obtained by applying  $H^{\otimes 3}$  to the bit-flip code. Therefore its check operators are obtained by conjugating the old check operators by  $H^{\otimes 3}$ .

Using

$$HZH = X,$$

we get

$$H^{\otimes 3}(Z_1 Z_2)H^{\otimes 3} = X_1 X_2,$$

and

$$H^{\otimes 3}(Z_2 Z_3)H^{\otimes 3} = X_2 X_3.$$

Thus the checks for the phase-flip repetition code are

$$S_1 = X_1 X_2, \quad S_2 = X_2 X_3.$$

(c) Recall that

$$X|+\rangle = |+\rangle, \quad X|-\rangle = -|-\rangle.$$

Therefore

$$S_1 |+++ \rangle = X_1 X_2 |+++ \rangle = |+++ \rangle,$$

and

$$S_1 |---\rangle = X_1 X_2 |---\rangle = (-1)(-1) |---\rangle = |---\rangle.$$

So  $S_1$  fixes both basis codewords. By linearity,

$$S_1(\alpha |+++ \rangle + \beta |--- \rangle) = \alpha |+++ \rangle + \beta |--- \rangle.$$

The same argument applies to  $S_2 = X_2 X_3$ :

$$S_2 |+++ \rangle = |+++ \rangle, \quad S_2 |--- \rangle = |--- \rangle.$$

Thus every code state is stabilized by both  $S_1$  and  $S_2$ .

The two checks can be measured in either order because they commute:

$$S_1 S_2 = (X_1 X_2)(X_2 X_3) = X_1 X_3,$$

and

$$S_2 S_1 = (X_2 X_3)(X_1 X_2) = X_1 X_3.$$

Hence

$$[S_1, S_2] = 0.$$

- (d) In the  $\{|+\rangle, |-\rangle\}$  basis, the operator  $X$  has eigenvalue  $+1$  on  $|+\rangle$  and eigenvalue  $-1$  on  $|-\rangle$ . Hence  $S_1 = X_1 X_2$  compares the first two signs, and  $S_2 = X_2 X_3$  compares the last two signs.

The completed table is

actual error	resulting state	$s_1$	$s_2$	correction
none	$\alpha  +++ \rangle + \beta  --- \rangle$	$+1$	$+1$	$I$
$Z_1$	$\alpha   - + + \rangle + \beta   + - - \rangle$	$-1$	$+1$	$Z_1$
$Z_2$	$\alpha   + - + \rangle + \beta   - + - \rangle$	$-1$	$-1$	$Z_2$
$Z_3$	$\alpha   + + - \rangle + \beta   - - + \rangle$	$+1$	$-1$	$Z_3$

For example, after a  $Z_2$  error the state is

$$Z_2 |\bar{\psi}\rangle = \alpha |+-+\rangle + \beta |-+-\rangle.$$

In both terms, qubits 1 and 2 have opposite  $X$ -eigenvalues, so  $S_1 = -1$ . Also, qubits 2 and 3 have opposite  $X$ -eigenvalues, so  $S_2 = -1$ .

- (e) The syndrome pairs from part (d) are

state	$(s_1, s_2)$
$ \bar{\psi}\rangle$	$(+1, +1)$
$Z_1  \bar{\psi}\rangle$	$(-1, +1)$
$Z_2  \bar{\psi}\rangle$	$(-1, -1)$
$Z_3  \bar{\psi}\rangle$	$(+1, -1)$

These four pairs are all different. Therefore the four possible received states lie in four different simultaneous eigenspaces of  $S_1$  and  $S_2$ .

The measurements identify the error position because each possible error gives a different syndrome. They do not reveal the coefficients  $\alpha$  and  $\beta$  because, for a fixed error, both components of the superposition have the same syndrome.

For example, for the error  $Z_3$  we have

$$Z_3 |\bar{\psi}\rangle = \alpha |++-\rangle + \beta |--+\rangle.$$

Both  $|++-\rangle$  and  $|--+\rangle$  have syndrome  $(+1, -1)$ . Thus the measurement tells us that the third qubit was hit by a phase flip, but it does not distinguish the  $\alpha$  part from the  $\beta$  part. The encoded quantum information remains hidden inside the two-dimensional syndrome subspace.

- (f) The code protects against one phase flip because a phase flip  $Z_i$  acts as a bit flip in the  $\{|+\rangle, |-\rangle\}$  basis:

$$Z|+\rangle = |-\rangle, \quad Z|-\rangle = |+\rangle.$$

The checks  $S_1 = X_1X_2$  and  $S_2 = X_2X_3$  can therefore detect which of the three  $\{|+\rangle, |-\rangle\}$  entries was flipped.

Now consider instead a bit-flip error  $X_i$ . Since

$$X|+\rangle = |+\rangle, \quad X|-\rangle = -|-\rangle,$$

we get

$$X_1 |\bar{\psi}\rangle = \alpha |+++ \rangle - \beta |-- - \rangle,$$

$$X_2 |\bar{\psi}\rangle = \alpha |+++ \rangle - \beta |-- - \rangle,$$

and

$$X_3 |\bar{\psi}\rangle = \alpha |+++ \rangle - \beta |-- - \rangle.$$

Thus every single-qubit  $X_i$  has the same effect on the encoded state: it changes the relative phase between the two logical codewords.

Moreover, the resulting state

$$\alpha |+++ \rangle - \beta |-- - \rangle$$

is still stabilized by  $S_1$  and  $S_2$ . Therefore the syndrome is

$$(s_1, s_2) = (+1, +1),$$

which is the same syndrome as no error. The code cannot detect this error.

In logical terms, a physical  $X_i$  acts as a logical phase flip on the encoded qubit:

$$\alpha |\bar{0}\rangle + \beta |\bar{1}\rangle \mapsto \alpha |\bar{0}\rangle - \beta |\bar{1}\rangle.$$

So the phase-flip repetition code corrects one  $Z$  error, but it does not correct one  $X$  error.

(g) Define the block states

$$|+_{B}\rangle = \frac{|000\rangle + |111\rangle}{\sqrt{2}}, \quad |-_{B}\rangle = \frac{|000\rangle - |111\rangle}{\sqrt{2}}.$$

The Shor codewords are

$$|0\rangle_{\mathcal{S}} = |+_{B}\rangle |+_{B}\rangle |+_{B}\rangle, \quad |1\rangle_{\mathcal{S}} = |-_{B}\rangle |-_{B}\rangle |-_{B}\rangle.$$

This has exactly the same form as the phase-flip repetition code

$$|\bar{0}\rangle = |+++ \rangle, \quad |\bar{1}\rangle = |-- - \rangle,$$

except that the single-qubit states  $|+\rangle$  and  $|-\rangle$  have been replaced by the three-qubit block states  $|+_{B}\rangle$  and  $|-_{B}\rangle$ . Thus the Shor code is a phase-flip repetition code at the level of three blocks.

Now suppose a single physical  $Z$  error occurs inside one block. For any of the three qubits in that block,

$$Z_i |000\rangle = |000\rangle, \quad Z_i |111\rangle = -|111\rangle.$$

Therefore

$$Z_i |+_{B}\rangle = \frac{|000\rangle - |111\rangle}{\sqrt{2}} = |-_{B}\rangle,$$

and

$$Z_i |-_{B}\rangle = \frac{|000\rangle + |111\rangle}{\sqrt{2}} = |+_{B}\rangle.$$

So a single physical phase flip inside a block flips that block from  $|+_{B}\rangle$  to  $|-_{B}\rangle$ , and conversely. At the block level, it looks like a bit flip of one of the three block qubits. The outer phase-flip repetition structure of the Shor code can therefore identify which block was affected.