
Exercise Set 10
Quantum Computation

This exercise sheet is a little longer than usual. However, Exercise 2 and 3 are very close to each other. We recommend doing both; but if you feel very comfortable with one of them then you will see that the other flows in the same way.

Exercise 1 *Observables and Pauli operators*

Here is a first important fact in quantum mechanics.

Let A be an observable, i.e., a self-adjoint matrix acting on a finite-dimensional Hilbert space $\mathcal{H}$. If the current state of the system is $|\psi\rangle$ and A has an orthonormal eigenbasis $(|\psi_i\rangle)$ with eigenvalues (λ_i), then "measuring A" gives outcome λ_i with probability $|\langle\psi_i|\psi\rangle|^2$ when the eigenvalue is non-degenerate. If an eigenvalue is degenerate, the outcome state is the normalized projection of $|\psi\rangle$ onto the corresponding eigenspace. In particular, if $|\psi\rangle$ is already an eigenvector of A, then measuring A leaves it unchanged.

Let

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

These are observables acting on $\mathbb{C}^2$ and satisfying

$$X|x\rangle = |\overline{x}\rangle, \qquad Z|x\rangle = (-1)^x|x\rangle$$

for $|x\rangle \in \{|0\rangle, |1\rangle\}$, where $\overline{0} = 1$ and $\overline{1} = 0$.

(a) Determine the eigenvalues and normalized eigenvectors of Z. Do the same for X.

(b) Let $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, with $|\alpha|^2 + |\beta|^2 = 1$. Compute the probabilities and post-measurement states for a measurement of Z. Then rewrite $|\psi\rangle$ as a superposition of X-eigenstates and compute the probabilities and post-measurement states for a measurement of X.

(c) For which states $|\psi\rangle$ is a Z-measurement guaranteed to leave the state unchanged? For which states is an X-measurement guaranteed to leave the state unchanged? Give an example where making a first measurement changes the statistics of a later measurement.

(d) Compute both the commutator and the anti-commutator of X and Z,

$$[X, Z] = XZ - ZX, \qquad \{X, Z\} = XZ + ZX.$$

Use your answer to explain why X and Z have no non-zero simultaneous eigenvector.

Consider now $\mathcal{H} = (\mathbb{C}^2)^{\otimes 3}$. We write X_i and Z_j for the Pauli operators acting on the i-th or j-th qubit, respectively. Thus

$$X_i \left|x_1, x_2, x_3\right\rangle = \left|y_1, y_2, y_3\right\rangle, \qquad y_i = \overline{x_i}, \quad y_j = x_j \text{ for } j \neq i,$$

and

$$Z_j \left|x_1, x_2, x_3\right\rangle = (-1)^{x_j} \left|x_1, x_2, x_3\right\rangle.$$

For example, X_1 is shorthand for $X \otimes I \otimes I$.

(e) Compute the commutators $[Z_i, Z_j]$, $[X_i, X_j]$, and $[X_i, Z_j]$ for $i, j \in \{1, 2, 3\}$. More generally, when do two tensor products made only of I, X, and Z commute?

(f) Define the three parity-check observables

$$A_1 = Z_2 Z_3, \qquad A_2 = Z_1 Z_3, \qquad A_3 = Z_1 Z_2.$$

Show that A_1, A_2, A_3 commute with each other and that $A_3 = A_1 A_2$. What does this imply about the information contained in the third measurement?

(g) Let $s_1, s_2 \in \{+1, -1\}$. Show that the projector onto the simultaneous eigenspace with associated eigenvalue

$$s_1 \text{ for } A_1, \qquad s_2 \text{ for } A_2$$

is

$$P_{s_1,s_2} = \frac{1}{4}(I + s_1 A_1)(I + s_2 A_2).$$

Explain why this is a projector onto a two-dimensional subspace.

Exercise 2 *Syndrome measurements for the three-qubit repetition code*

We now use the observables from the previous exercise for the three-qubit repetition code. Let

$$A_1 = Z_2 Z_3, \qquad A_2 = Z_1 Z_3, \qquad A_3 = Z_1 Z_2.$$

The encoded state is

$$\left|\psi_0\right\rangle = \alpha \left|000\right\rangle + \beta \left|111\right\rangle, \qquad |\alpha|^2 + |\beta|^2 = 1.$$

Assume that at most one bit flip occurs, so that the new state is one of

$$\left|\psi_0\right\rangle, \qquad X_1 \left|\psi_0\right\rangle, \qquad X_2 \left|\psi_0\right\rangle, \qquad X_3 \left|\psi_0\right\rangle.$$

(a) Fill in the following syndrome table. The entries s_1, s_2, s_3 are the measurement outcomes of A_1, A_2, A_3, respectively. The correction should be a unitary operation, not a measurement.

actual error	resulting state	s_1	s_2	s_3	correction
none	$\alpha \left|000\right\rangle + \beta \left|111\right\rangle$				
X_1	$\alpha \left|100\right\rangle + \beta \left|011\right\rangle$				
X_2	$\alpha \left|010\right\rangle + \beta \left|101\right\rangle$				
X_3	$\alpha \left|001\right\rangle + \beta \left|110\right\rangle$				

(b) Show that each of the four possible resulting states above is a simultaneous eigenvector of A_1 and A_2, for all choices of α and β. Why do the measurements of A_1 and A_2 reveal the position of the bit flip but not the encoded quantum information $\alpha|0\rangle + \beta|1\rangle$?

(c) Does the order of measuring A_1 and A_2 matter? Would measuring A_3 as well help to identify the error? Explain your answer using the relation $A_3 = A_1 A_2$.

(d) After the syndrome has been measured, one applies the appropriate correction X_i. Is it a problem that this correction does not necessarily commute with the syndrome observables? What could go wrong if, instead of applying X_i as a unitary gate, one measured the observable X_i?

Exercise 3 *The phase-flip repetition code and the role of the Hadamard gate.*
Remark: This exercise is similar to the previous one. It is good to do it to test your understanding. If you feel comfortable, skip directly to the next exercise.

The three-qubit bit-flip repetition code encodes

$$|0\rangle \longmapsto |000\rangle\,, \qquad |1\rangle \longmapsto |111\rangle\,.$$

In this exercise we construct the analogous code for correcting one phase-flip error. Recall that the Hadamard gate is

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \qquad H|0\rangle = |+\rangle\,, \qquad H|1\rangle = |-\rangle\,,$$

where

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}, \qquad |-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}.$$

The phase-flip repetition code is the two-dimensional subspace spanned by

$$|\overline{0}\rangle = |{+}{+}{+}\rangle\,, \qquad |\overline{1}\rangle = |{-}{-}{-}\rangle\,.$$

Thus a general encoded state is

$$|\overline{\psi}\rangle = \alpha|{+}{+}{+}\rangle + \beta|{-}{-}{-}\rangle\,, \qquad |\alpha|^2 + |\beta|^2 = 1.$$

A phase-flip error on qubit i is the Pauli operator Z_i. Notice that $Z|+\rangle = |-\rangle$ and $Z|-\rangle = |+\rangle$, so a physical phase flip acts like a bit flip when the qubit is written in the $\{|+\rangle, |-\rangle\}$ basis.

(a) Show that this code is obtained from the bit-flip repetition code by applying $H^{\otimes 3}$ to all three physical qubits. That is, verify that

$$H^{\otimes 3}|000\rangle = |{+}{+}{+}\rangle\,, \qquad H^{\otimes 3}|111\rangle = |{-}{-}{-}\rangle\,.$$

(b) Use the identities $HXH = Z$ and $HZH = X$ to explain why the parity checks Z_1Z_2 and Z_2Z_3 of the bit-flip code are transformed into the checks

$$S_1 = X_1X_2, \qquad S_2 = X_2X_3$$

for the phase-flip code.

(c) Verify directly that every code state $|\overline{\psi}\rangle$ is stabilized by S_1 and S_2, meaning that

$$S_1 |\overline{\psi}\rangle = |\overline{\psi}\rangle , \qquad S_2 |\overline{\psi}\rangle = |\overline{\psi}\rangle .$$

Why can S_1 and S_2 be measured in either order?

(d) Assume that at most one phase flip occurs, so that the received state is one of

$$|\overline{\psi}\rangle , \qquad Z_1 |\overline{\psi}\rangle , \qquad Z_2 |\overline{\psi}\rangle , \qquad Z_3 |\overline{\psi}\rangle .$$

Fill in the following syndrome table. The entries s_1 and s_2 are the measurement outcomes of S_1 and S_2, respectively. The correction should again be a unitary Pauli operation, not a measurement.

actual error	resulting state	s_1	s_2	correction
none	$\alpha \|+++\rangle + \beta \|---\rangle$			
Z_1	$\alpha \|-++\rangle + \beta \|+--\rangle$			
Z_2	$\alpha \|+-+\rangle + \beta \|-+-\rangle$			
Z_3	$\alpha \|++-\rangle + \beta \|--+\rangle$			

(e) Show that the four possible states in part (d) lie in four different simultaneous eigenspaces of S_1 and S_2. Explain why measuring S_1 and S_2 identifies the error position but does not reveal the coefficients α and β.

(f) Explain why this code protects against one phase flip but not against one bit flip. In particular, compute the action of X_1, X_2, and X_3 on $|\overline{\psi}\rangle$, and determine what syndrome would be obtained after such an error.

(g) Let

$$|+_B\rangle = \frac{|000\rangle + |111\rangle}{\sqrt{2}}, \qquad |-_B\rangle = \frac{|000\rangle - |111\rangle}{\sqrt{2}}.$$

The codewords of the Shor coe can be written as

$$|0\rangle_{\mathcal{S}} = |+_B\rangle |+_B\rangle |+_B\rangle , \qquad |1\rangle_{\mathcal{S}} = |-_B\rangle |-_B\rangle |-_B\rangle .$$

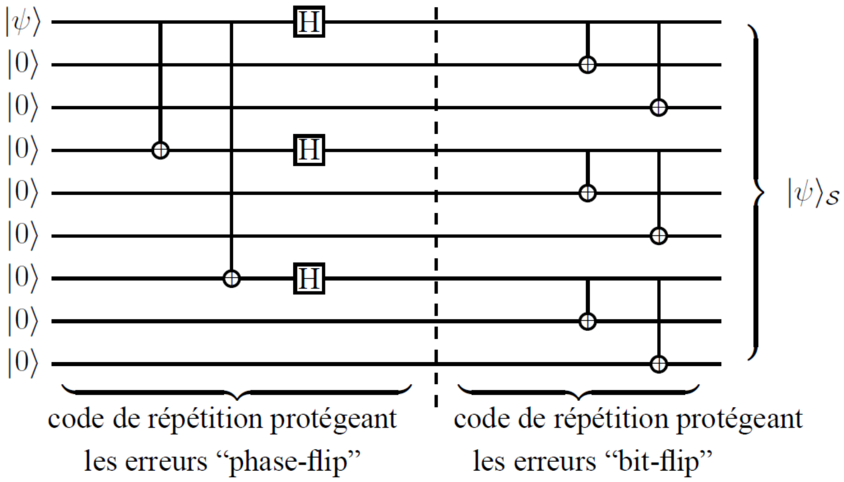
Explain how this displays the phase-flip repetition code at the level of three blocks. What happens to one block if a single physical Z error occurs inside that block?

Exercise 4 *The Shor code*

The Shor code uses 9 qubits one qubit and corrects one error (a bit flip or a phase flip). The code length is 9, the code dimension is 2 (viewed as a subspace of the Hilbert space). Codewords are of the form $\alpha|0\rangle_S + \beta|1\rangle_S$ with

$$\frac{|000\rangle + |111\rangle}{\sqrt{2}} \otimes \frac{|000\rangle + |111\rangle}{\sqrt{2}} \otimes \frac{|000\rangle + |111\rangle}{\sqrt{2}} \equiv |0\rangle_S$$
$$\frac{|000\rangle - |111\rangle}{\sqrt{2}} \otimes \frac{|000\rangle - |111\rangle}{\sqrt{2}} \otimes \frac{|000\rangle - |111\rangle}{\sqrt{2}} \equiv |1\rangle_S$$

Verify that the following circuit realizes the proposed (unitary) encoding (where $|\psi\rangle = \alpha |0\rangle + \beta |1\rangle$):


$|\psi\rangle$
$|0\rangle$
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$|\psi\rangle_S$
code de répétition protégeant
les erreurs “phase-flip”
code de répétition protégeant
les erreurs “bit-flip”