Equilibria of collisionless systems

3rd part

Outlines

Models defined from DFs

- Polytropic models
- The isothermal sphere

Anisotropic distribution function in spherical systems

- Motivation
- General concepts
- Example of an anisotropic DF
- Models with constant anisotropy

The Jeans Equations

- Motivations
- The Jeans Equations and conservation laws
- The Jeans Equations in Spherical coordinates
- The Jeans Equations in Cylindrical coordinates

Distribution function for spherical systems

- Method (1)
  - from $\rho(r)$ $\phi(r)$ $\longrightarrow$ get $f(\mathcal{E}) \equiv f\left(\frac{1}{2}v^2 + \phi(r)\right)$
- Method (2)
  - assume $f(\mathcal{E})$ $\longrightarrow$ get $\rho(r)$

Spherical systems definded by DFs

Models defined from DFs: Polytropes

Polythropes and Plummer models

$$f(\mathcal{E}) = \begin{cases} F\,\mathcal{E}^{\,n-3/2} & (\mathcal{E} > 0) \\ 0 & (\mathcal{E} \leq 0) \end{cases}$$

F, a constant

$f = 0$ if $\mathcal{E} > 0$

$f = 0$

Corresponding density

$$\nu(r) = 4\pi \int_0^{\sqrt{2\Psi}} dv\, v^2 f\left(\Psi - \tfrac{1}{2}v^2\right)$$

$\times N \cdot m$

$\nu(r) \to \rho(r)$

$$\rho(r) = 4\pi F \int_0^{\sqrt{2\Psi}} dv\, v^2 \left(\Psi(r) - \tfrac{1}{2}v^2\right)^{n-3/2}$$

Which leads to :

$$\boxed{\rho(r) = C_n \, \psi(r)^n} \qquad (\text{for } \psi > 0)$$

relation between ρ and ϕ

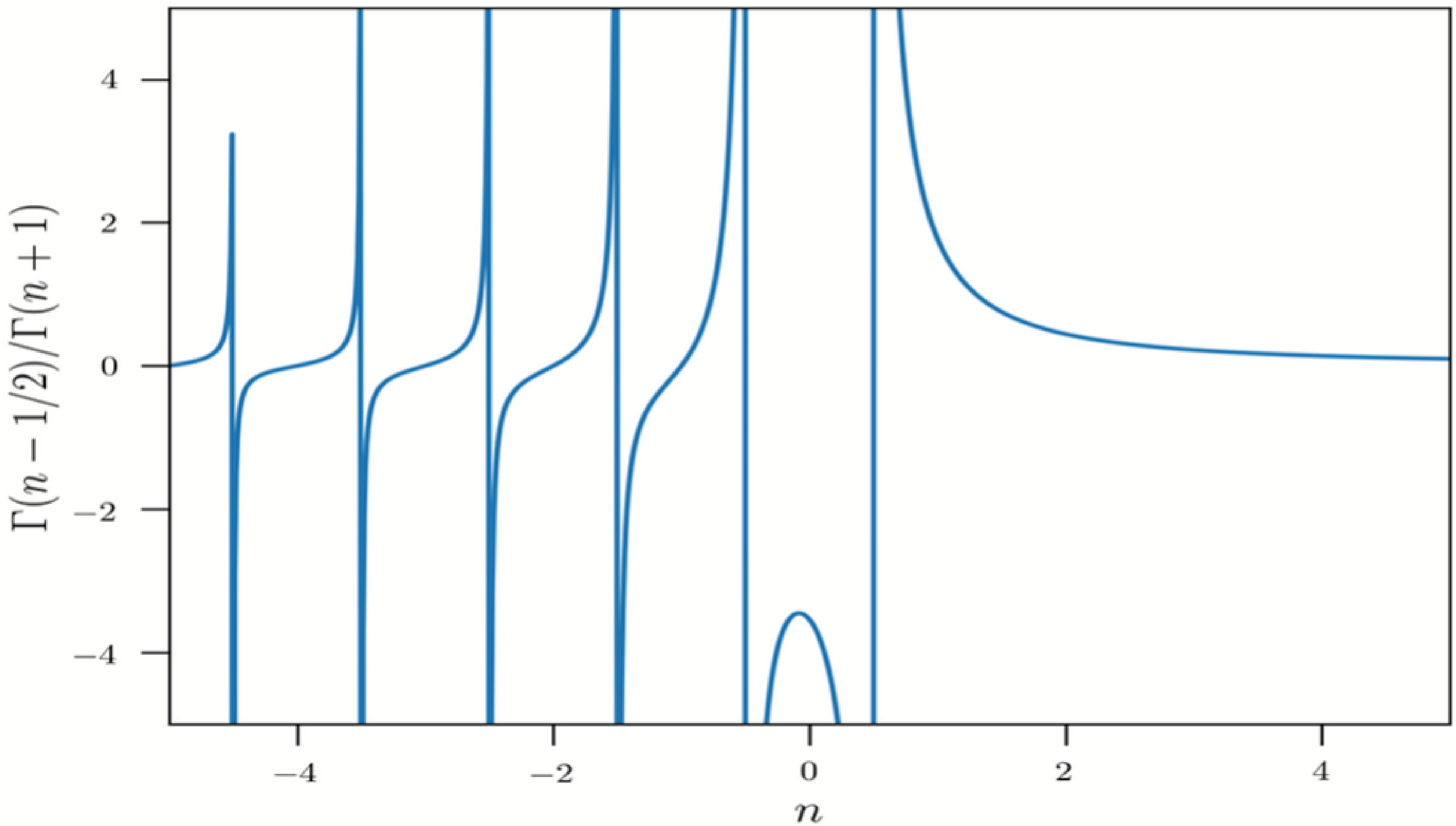
$$C_n = \frac{(2\pi)^{3/2} \left(n - \frac{3}{2}\right)! \, F}{n!} = \frac{(2\pi)^{3/2} \, \Gamma\left(n - \frac{1}{2}\right) F}{\Gamma(n+1)}$$

$$n! = \Gamma(n+1) = \int_0^\infty dt \; t^n e^{-t}$$

$$c_n \sim \frac{(n-\frac{3}{2})!}{n!} = \frac{\Gamma(n-\frac{1}{2})}{\Gamma(n+1)}$$

$n = \frac{1}{2}$

$n > \frac{1}{2}$, $c_n > 0$, $f > 0$



Demonstration

$$\rho(r) = 4\pi F \int_0^{\sqrt{2\psi}} dv \; v^2 \left(\psi(r) - \tfrac{1}{2} v^2\right)^{n-3/2}$$

smart substitution : introduce the variable $\theta(v)$ such that

$$v^2 = 2\psi \cos^2\theta \quad , \quad \theta = \arccos\left(\frac{v}{\sqrt{2\psi}}\right)$$

$$2v\,dv = -4\psi \cos\theta \sin\theta \, d\theta$$

$$\Rightarrow \quad dv = -\frac{2\psi \cos\theta \sin\theta \, d\theta}{\sqrt{2\psi}\cos\theta} = -\sqrt{2\psi}\, \sin\theta \, d\theta$$

$$\begin{cases} v = 0 \rightarrow \theta = \pi/2 \\ v = \sqrt{2\psi} \rightarrow \theta = 0 \end{cases}$$

$$\psi - \tfrac{1}{2} v^2 = \psi - \psi\cos^2\theta = \psi \sin^2\theta$$

$$\rho(r) = 4\pi F \int_0^{\pi/2} \left(\sqrt{2\psi}\, \sin\theta \, d\theta\right) \cdot \left(2\psi \cos^2\theta\right) \cdot \left(\psi \sin^2\theta\right)^{n-3/2}$$

$$= 4\pi F \int_0^{\pi/2} 2 \cdot 2^{\frac{1}{2}} \psi^{\frac{1}{2}} \psi \, \psi^{n-\frac{3}{2}} \cdot \cos^2\theta \, \sin^{2n-2}\theta \, d\theta$$

$$= 8\pi F \sqrt{2}\, \psi^{n} \int_0^{\frac{\pi}{2}} \underbrace{\cos^2\theta}_{1-\sin^2} \sin\theta^{2n-2}\, d\theta$$

So, we get

$$\rho(r) = C_n\, \psi(r)^{n} \qquad (\text{for } \psi > 0)$$

relation between ρ and ϕ

$$C_n = \frac{(2\pi)^{3/2} \left(n - \frac{3}{2}\right)!\, F}{n!} = \frac{(2\pi)^{3/2}\, \Gamma\left(n - \frac{1}{2}\right) F}{\Gamma(n+1)}$$

Corresponding "Pressure"

$$\boxed{P(\rho) = -\int_0^\rho d\rho' \, \rho' \, \frac{\partial \phi}{\partial \rho}(\rho')}$$

$$\rho = C_n \psi^n$$

$$\psi = \frac{1}{C_n^{1/n}} \rho^{\frac{1}{n}} \qquad \frac{\partial \psi}{\partial \rho} = \frac{1}{C_n^{1/n}} \frac{1}{n} \rho^{\frac{1}{n}-1}$$

$$\frac{\partial \phi}{\partial \rho} = -\frac{1}{C_n^{1/n}} \frac{1}{n} \rho^{\frac{1}{n}-1}$$

$$P(\rho) = \frac{1}{C_n^{1/n}} \frac{1}{n} \int_0^\rho d\rho' \, \rho'^{\frac{1}{n}} = \frac{1}{C_n^{1/n}} \frac{1}{n+1} \rho^{\frac{1}{n}+1}$$

$$\boxed{P(\rho) = K \rho^\gamma}$$

$\equiv$ Polytropic EoS

$$\begin{cases} \gamma = \dfrac{1}{n} + 1 & n = \dfrac{1}{\gamma - 1} \\ K = \dfrac{1}{C_n^{1/n}} \dfrac{1}{n+1} & C_n = \left(\dfrac{\gamma - 1}{K\gamma}\right)^{\frac{1}{\gamma - 1}} \end{cases}$$

Conclusion

The density of a stellar system described by and ergodic DF

$$f(\epsilon) \sim \epsilon^{n-3/2}$$

is the same as a polytropic gas sphere in hydrostatic equilibrium, with:

$$P(\rho) \sim \rho^{\gamma}$$

This is why these DFs are called polytropes.

Note: from $\rho(r) = C_n \, \psi(r)^n$

if $\rho = \text{cte}$ $\Rightarrow$ $n = 0$

But from $C_n = \dfrac{(2\pi)^{3/2} \, \Gamma(n - \frac{1}{2}) \, F}{\Gamma(n+1)}$ $\Rightarrow$ $C_n < 0$ $\quad \rho < 0$

No finite ergodic stellar system is homogeneous.

Self-gravity !

$$\vec{\nabla}^2(\Phi) = 4\pi G\rho$$

The Poisson equation for spherical systems (with ψ)

$$\boxed{\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\psi}{dr}\right) = -4\pi G\rho(r)}$$

$\rho = C_n \psi^n$

$\rho^{\frac{n-1}{n}} = C_n^{\frac{n-1}{n}} \psi^{n-1}$

With $\rho = C_n \psi^n$ $\qquad \dfrac{d\rho}{dr} = C_n n \psi^{n-1}\dfrac{d\psi}{dr} = C_n n \left(\dfrac{1}{C_n}\rho\right)^{\frac{n-1}{n}}\dfrac{d\psi}{dr}$

thus $\qquad \dfrac{d\psi}{dr} = \dfrac{1}{C_n^{1/n} n}\rho^{\frac{1-n}{n}}\dfrac{d\rho}{dr}$

$$\boxed{\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{1}{n C_n^{1/n}}\rho^{\frac{1-n}{n}}\right) + 4\pi G\rho = 0}$$

or eliminating ρ, using $\rho(r) = C_n \psi(r)^n$

$$\boxed{\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\psi}{dr}\right) + 4\pi G C_n \psi^n = 0}$$

Solutions

$$\boxed{\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Psi}{dr}\right) + 4\pi G c_n \Psi^n = 0}$$

A. Power laws

$$\left\{\begin{array}{l}\rho(r) \sim r^{-\alpha} \\ \Psi(r) \sim r^{-\alpha/n}\end{array}\right. \qquad \rho \sim \Psi^n$$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Psi}{dr}\right) \sim r^{-\frac{\alpha}{n}-2}$$

Poisson

$$\underbrace{\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Psi}{dr}\right)}_{r^{-\frac{\alpha}{n}-2}} + \underbrace{4\pi G \rho(r)}_{r^{-\alpha}} = 0$$

$$-\frac{\alpha}{n}-2 \sim -\alpha$$

$$\Rightarrow \boxed{\alpha = \frac{2n}{n-1}}$$

As the potential may not decrease faster than the Kepler potential $\frac{1}{r}$

$\left(\Psi \sim r^{-\frac{\alpha}{n}}\right)$ $\qquad \underline{\frac{\alpha}{n} \leq 1} \quad \Rightarrow \quad \boxed{n \geq 3}$

B Models with finite potential and density

Define new variables $s = \frac{r}{b}$ $\quad \psi' = \frac{\psi}{\psi_0}$

where $\begin{cases} b = \left(\frac{4}{3}\pi G \psi_0^{n-1} c_n\right)^{1/2} \\ \psi_0 = \psi(0) \end{cases}$

The Poisson equation becomes $\boxed{\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d\psi}{dr}\right) + 4\pi G c_n \psi^n = 0}$

$$\boxed{\frac{1}{s^2}\frac{d}{ds}\left(s^2 \frac{d\psi'}{ds}\right) = -3\psi'^n}$$

+ boundary conditions

$\begin{cases} \bullet\ \psi'(0) = 1 & \text{normalisation} \\ \bullet\ \left.\frac{d\psi'}{dr}\right)_0 = 0 & \text{no force at the center (smooth)} \end{cases}$

Lane - Emden Equation

(In general, non trivial solutions)

Two analytical solutions $\quad n=1, \quad n=5$

$\boxed{n=1}$

$$\boxed{\frac{1}{s^2}\frac{d}{ds}\left(s^2\frac{d\psi'}{ds}\right) = -3\psi'}$$

linear Helmholtz Equation

$$\psi'(s) = \begin{cases} \dfrac{\sin(\sqrt{3}\,s)}{\sqrt{3}\,s} & s < \dfrac{\pi}{\sqrt{3}} \\[2ex] \dfrac{\pi}{\sqrt{3}\,s} - 1 & s \geqslant \dfrac{\pi}{\sqrt{3}} \end{cases}$$

COMPLEMENT

COMPLEMENT

Two analytical solutions $\quad n = 1, \quad n = 5$

$\boxed{n = 1}$

$$\boxed{\frac{1}{s^2}\frac{d}{ds}\left(s^2 \frac{d\psi'}{ds}\right) = -3\psi'}$$

linear Helmholtz Equation

$$\psi'(s) = \begin{cases} \dfrac{\sin(\sqrt{3}\, s)}{\sqrt{3}\, s} & s < \dfrac{\pi}{\sqrt{3}} \\[2ex] \dfrac{\pi}{\sqrt{3}\, s} - 1 & s \geq \dfrac{\pi}{\sqrt{3}} \end{cases}$$

UNPHYSICAL SOLUTION

$n = 1 < 3$

non physical solution

$\boxed{n = 5}$ $\boxed{\frac{1}{s^2} \frac{d}{ds}\left(s^2 \frac{d\psi'}{ds} \right) = -3\psi'^5}$

consider $\psi'(s) = \frac{1}{\sqrt{1+s^2}}$

The Poisson Equation becomes

$$\frac{1}{s^2} \frac{d}{ds}\left(s^2 \frac{d\psi'}{ds} \right) = -\frac{1}{s^2} \frac{d}{ds}\left(\frac{s^3}{(1+s^2)^{3/2}} \right) = -\frac{3}{(1+s^2)^{5/2}} = -3\psi'^5$$

$\rightarrow$ $\psi'(s)$ is a solution !

$$\boxed{n = 5}$$

$$\boxed{\frac{1}{s^2}\frac{d}{ds}\left(s^2\frac{d\psi'}{ds}\right) = -3\psi'^5}$$

consider $\psi'(s) = \dfrac{1}{\sqrt{1+s^2}}$

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$$\frac{1}{s^2}\frac{d}{ds}\left(s^2\frac{d\psi'}{ds}\right) = -\frac{1}{s^2}\frac{d}{ds}\left(\frac{s^3}{(1+s^2)^{3/2}}\right) = -\frac{3}{(1+s^2)^{5/2}} = -3\psi'^5$$

$\rightarrow$ $\psi'(s)$ is a solution !

and corresponds to the <u>Plummer model</u>

$$\boxed{\phi(r) = -\frac{GM}{\sqrt{r^2+a^2}}}$$

$$\boxed{\rho(r) = \frac{3M}{4\pi a^3}\left(1+\frac{r^2}{a^2}\right)^{-5/2}}$$

Then : what do we learn concerning the Plummer model ?

We have access to its DF :

$$f(\mathcal{E}) \begin{cases} \sim \mathcal{E}^{n-3/2} \sim \left(\dfrac{GM}{\sqrt{r^2+a^2}} - \dfrac{1}{2} v^2 \right)^{7/2} \\ = 0 \quad \text{if} \quad \dfrac{GM}{\sqrt{r^2+a^2}} - \dfrac{1}{2} v^2 < 0 \end{cases}$$

We have access to the kinematics structure :

① Velocity distribution function

$$P_r(v) = \frac{f\left(\frac{1}{2}v^2 + \phi(r)\right)}{\nu(r)} \sim \underbrace{\left(1+\frac{r^2}{a^2}\right)^{5/2}}_{\frac{1}{\rho}} \underbrace{\left(\frac{GM}{\sqrt{r^2+a^2}} - \frac{1}{2} v^2 \right)^{7/2}}_{\mathcal{E}^{7/2}}$$

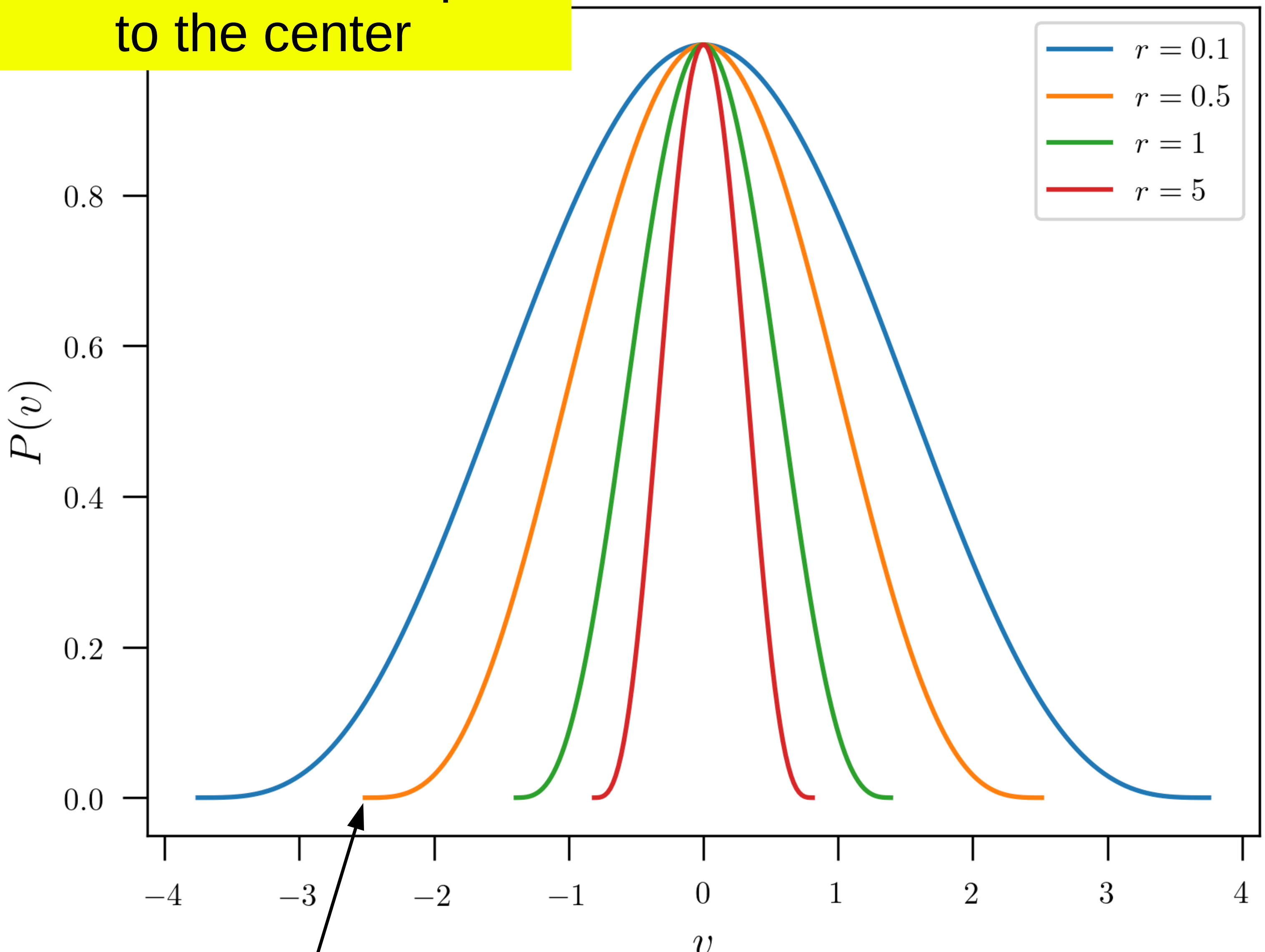
② Velocity dispersion

$$\sigma^2 = 4\pi \frac{1}{\nu(x)} \int_0^{v_{max} = \sqrt{2\psi}} v^4 f\left(\frac{1}{2}v^2 + \phi(\vec{x})\right) dv$$

$$= 4\pi \frac{1}{\nu(r)} \int_0^{v_{max}} v^4 \left(\frac{1}{2} v^2 - \frac{GM}{\sqrt{r^2+a^2}} \right)^{7/2} dv$$

The Plummer velocity distribution function

Normalized with respect to the center

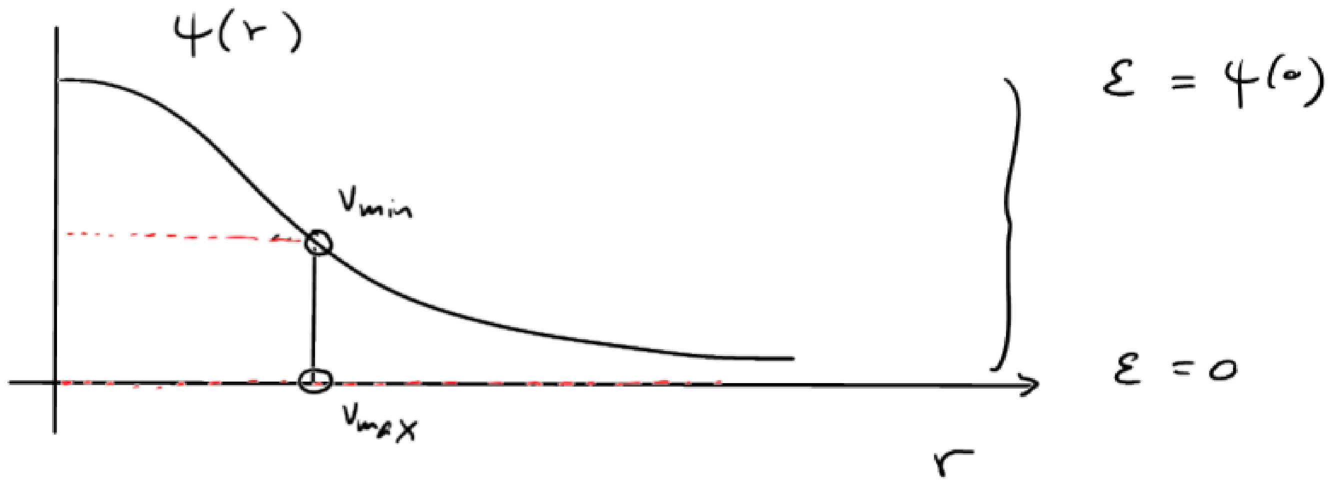


$$\forall r, \exists\, v_{\max} \text{ such that } \epsilon > 0 \Rightarrow f = 0$$

Interpretation

$$P_r(v) = \begin{cases} \left(\dfrac{GM}{\sqrt{r^2+a^2}} - \dfrac{1}{2} v^2 \right)^{7/2} & \mathcal{E} > 0 \\ 0 & \mathcal{E} \leq 0 \end{cases}$$

$$\mathcal{E} = \psi - \frac{1}{2} v^2$$



in r, the minimum velocity is $v_{min} = 0$

orbits with $r_{max} = r$, $v(r_{max}) = 0$

the maximum velocity is $v_{max} = \sqrt{2\psi(r)}$

orbits with $\mathcal{E} = 0$ $(r_{max} = \infty)$

Models defined from DFs: Isothermal spheres

Stellar system with the DF (Isothermal)

$$\boxed{f(\varepsilon) = \frac{\rho_1}{(2\pi\sigma^2)^{3/2}} e^{\frac{\varepsilon}{\sigma^2}}}$$

with $\varepsilon = \Psi - \frac{1}{2}v^2$

$$\rho(r) = 4\pi \int_0^\infty v^2 \frac{\rho_1}{(2\pi\sigma^2)^{3/2}} e^{\frac{\Psi - \frac{1}{2}v^2}{\sigma^2}} = \rho_1 e^{\frac{\Psi}{\sigma^2}}$$

$$\left(\int_0^\infty \frac{v^2 e^{\frac{\Psi - \frac{1}{2}v^2}{\sigma^2}}}{(2\pi\sigma^2)^{3/2}} dV = \frac{e^{\frac{\Psi}{\sigma^2}}}{4\pi} \right)$$

$$\boxed{\rho(r) = \rho_1 e^{\frac{\Psi}{\sigma^2}}}$$

$$\boxed{\rho(\Psi) = \rho_1 e^{\frac{\Psi}{\sigma^2}}}$$

"Pressure"

$$P(\rho) = -\int_0^{\rho} d\rho'\,\rho'\,\frac{\partial\phi}{\partial\rho'} \equiv \int_0^{\rho} d\rho'\,\rho'\,\frac{\partial\psi}{\partial\rho'}$$

Derivating $\rho(\psi) = \rho_1\, e^{\frac{\psi}{\sigma^2}}$ with respect to ρ

$$\frac{\partial\rho}{\partial\rho} = 1 = \rho_1\, e^{\frac{\psi}{\sigma^2}}\,\frac{1}{\sigma^2}\,\frac{\partial\psi}{\partial\rho} = \frac{1}{\sigma^2}\,\rho\,\frac{\partial\psi}{\partial\rho}$$

$$\Rightarrow \quad \rho\,\frac{\partial\psi}{\partial\rho} = \sigma^2 \quad \text{and} \quad \boxed{P(\rho) = \sigma^2\rho}$$

Isothermal EoS

$$\sigma^2 \equiv \frac{k_B T}{m}$$

The structure of an isothermal self-gravitating sphere of gas with an EoS

$$\boxed{P(\rho) = \frac{k_B T}{m} \rho}$$

is identical to the one of a collisionless self-gravitating system with a DF

$$\boxed{f(\varepsilon) = \frac{\rho_1}{(2\pi\sigma^2)^{3/2}} e^{\frac{\varepsilon}{\sigma^2}}} \qquad \text{if} \quad \sigma^2 \equiv \frac{k_B T}{m}$$

wich leads to $\quad P(\rho) = \sigma^2 \rho$

Velocity distribution function

- collisionless isothermal sphere

$$P_r(v) = \frac{f(\varepsilon)}{\rho(\varepsilon)} \sim \frac{e^{\frac{1}{\sigma^2}\left(-\frac{1}{2}v^2+\psi(r)\right)}}{e^{\frac{1}{\sigma^2}\psi}} \sim e^{-\frac{v^2}{2\sigma^2}}$$

similar

- Gas sphere : (elastic collisions between particles)

$\Rightarrow$ Maxwell - Boltzman distribution $P_r(v) \sim e^{-\frac{mv^2}{2k_BT}} \equiv e^{-\frac{v^2}{2\sigma^2}}$

Note The correspondance between gaseous polythrope and stellar collisionless systems is not always as close a for the isothermal sphere

- gaseous polytrope : σ is allways Maxwellian and isothrope
- stellar system : σ given by f is no necessarily Maxwellian and may be anisothrope (if not ergodic)

Velocity dispersion

$$\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = \frac{1}{\nu} \int d^3v \; v^2 \frac{\rho_1}{(2\pi\sigma^2)^{3/2}} e^{\frac{\Psi - \frac{1}{2}v^2}{\sigma^2}}$$

spherical coord in vel. space

$$= \frac{\frac{4}{3}\pi \int_0^\infty v^4 \, e^{\frac{\Psi-\frac{1}{2}v^2}{\sigma^2}} dv}{4\pi \int_0^\infty v^2 \, e^{\frac{\Psi-\frac{1}{2}v^2}{\sigma^2}} dv} = \frac{2\sigma^2}{3} \frac{\int_0^\infty dx\, x^4 e^{-x^2}}{\int_0^\infty dx\, x^2 e^{-x^2}} = \boxed{\sigma^2}$$

$$-x^2 = \frac{\Psi - \frac{1}{2}v^2}{\sigma^2}$$

σ^2 is indep. of r

What is the corresponding density / potential

$\rho(r)$, $\phi(r)$ of the system ?

Self-gravity !

$$\vec{\nabla}^2(\Phi) = 4\pi G\rho$$

The Poisson Equation

$$\boxed{\rho(r) = \rho_1 \, e^{\frac{\Psi}{\sigma^2}}}$$

$$\frac{1}{r} \frac{d}{dr}\left(r^2 \frac{d\Psi}{dr} \right) = -4\pi G \rho(r)$$

$$\ln \rho = \ln \rho_1 + \frac{\Psi}{\sigma^2}$$

$$\frac{d \ln \rho}{dr} = \frac{1}{\sigma^2} \frac{d\Psi}{dr}$$

yields

$$\boxed{\frac{d}{dr}\left(r^2 \frac{d \ln \rho}{dr} \right) = -\frac{4\pi G}{\sigma^2} r^2 \rho(r)}$$

Solutions of the Poisson equation

$$\frac{d}{dr}\left(r^2 \frac{d\ln\rho}{dr}\right) = -\frac{4\pi G}{\sigma^2} r^2 \rho(r)$$

A. Power law

$$\rho \sim r^{-b}$$

Poisson $\Rightarrow$ $-b = -\frac{4\pi G}{\sigma^2} r^{2-b}$

$$b = 2$$

$$\boxed{\rho(r) = \frac{\sigma^2}{2\pi G r^2}}$$

Singular isothermal sphere

Notes

1. The specific energy (σ^2) is constant every where
2. The velocity dispersion is isotropic

Maximal equilibrium ?

But ρ and ϕ diverges at $r=0$!
$M(r)$ diverges at $r=\infty$

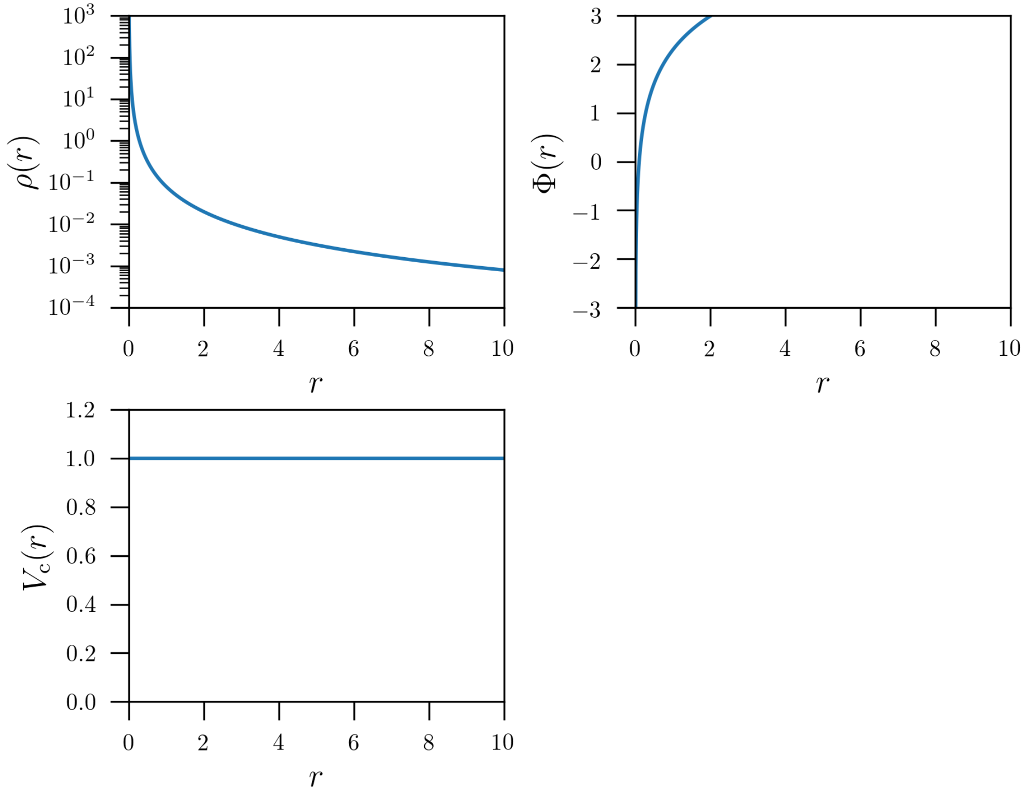
Isothermal sphere

$$\rho(r) = \rho_0 \frac{a^2}{r^2}$$

$$\Phi(r) = 4\pi G \rho_0 a^2 \ln\left(\frac{r}{a}\right)$$

$$M(r) = 4\pi \rho_0 a^2 \, r$$

$$V_c^2(r) = 4\pi G \rho_0 a^2$$



- often used for gravitational lens models
- But !
  - diverge towards the centre !
  - Infinite mass !

B Models with finite potential and density

$$\tilde{\rho} = \frac{\rho}{\rho_0} \qquad \tilde{r} = \frac{r}{r_0} \qquad r_0 = \sqrt{\frac{9\sigma^2}{4\pi G \rho_0}} \quad \text{(King radius)}$$

The Poisson equation becomes

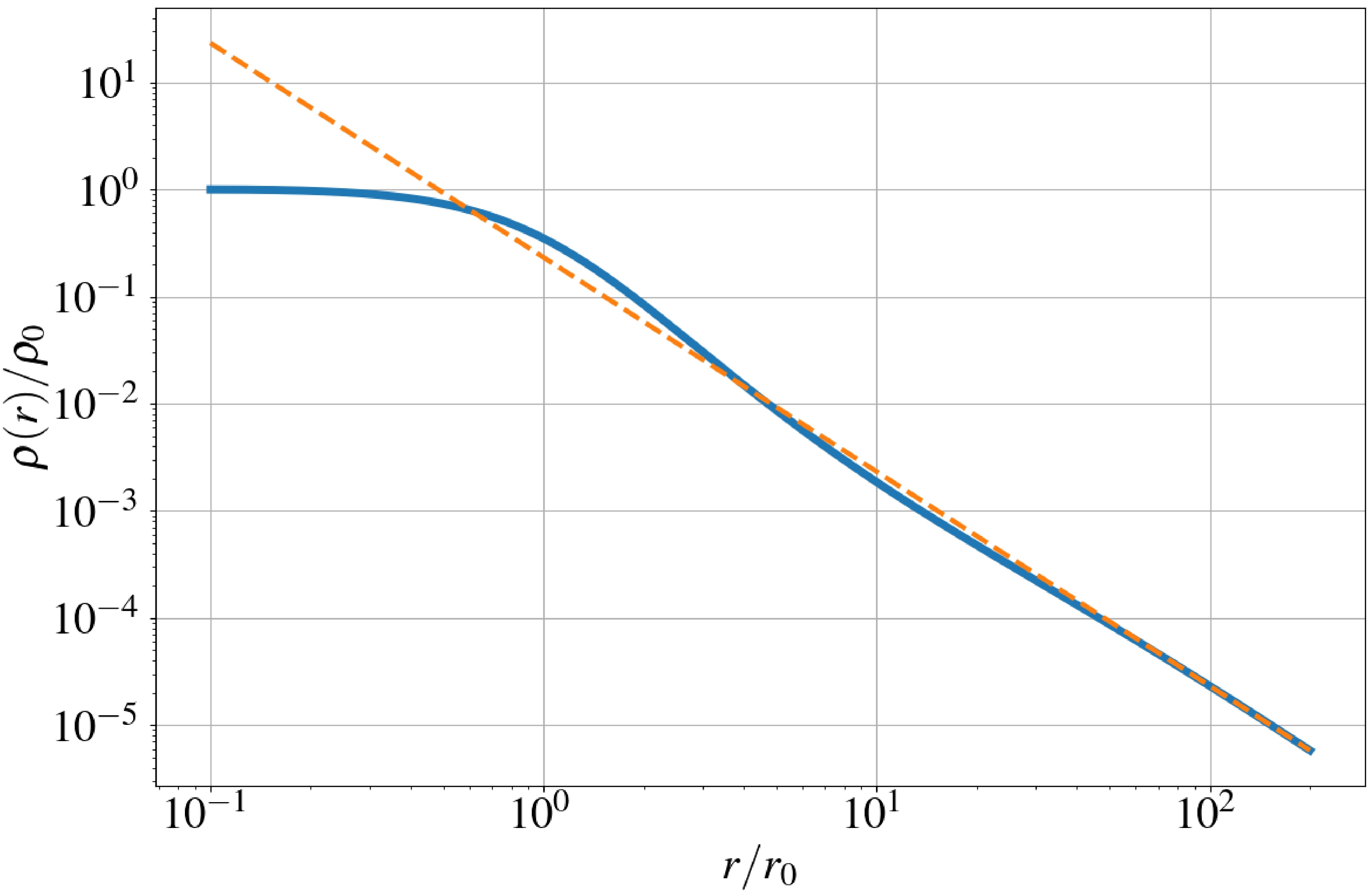
$$\boxed{\frac{d}{d\tilde{r}}\left(\tilde{r}^2 \frac{d\ln\tilde{\rho}}{d\tilde{r}}\right) = -9\,\tilde{r}\,\tilde{\rho}}$$

Requires numerical integration

+ boundary conditions

$$\begin{cases} \bullet \ \tilde{\rho}(0) = 1 & \text{normalisation} \\ \bullet \ \left.\dfrac{d\tilde{\rho}}{d\tilde{r}}\right)_0 = 0 & \text{smooth} \end{cases}$$

Numerical solution of the non-singular isothermal sphere



Anisotropic DFs in spherical systems

Spherical systems with anisothropic velocities

Ergodic DF : $f(\mathcal{E}) \Rightarrow \sigma_{ij} = \begin{pmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & \sigma \end{pmatrix}$

If we know $\nu(r)$: Eddington's formula

$$\boxed{\begin{aligned} f(\mathcal{E}) &= \frac{1}{\sqrt{8}\,\pi^2} \frac{d}{d\mathcal{E}} \left[\int_0^{\mathcal{E}} \frac{d\Psi}{\sqrt{\mathcal{E}-\Psi}} \frac{d\nu}{d\Psi} \right] \\ &\text{or} \\ f(\mathcal{E}) &= \frac{1}{\sqrt{8}\,\pi^2} \left[\int_0^{\mathcal{E}} \frac{d\Psi}{\sqrt{\mathcal{E}-\Psi}} \frac{d^2\nu}{d\Psi^2} + \frac{1}{\sqrt{\mathcal{E}}} \left(\frac{d\nu}{d\Psi} \right)_{\Psi=0} \right] \end{aligned}}$$

<u>Note</u> : $f(\mathcal{E}) > 0$ only if $\int_0^{\mathcal{E}} \frac{d\Psi}{\sqrt{\mathcal{E}-\Psi}} \frac{d\nu}{d\Psi}$ is an increasing function of $\mathcal{E}$

⚠ For a given $\nu(r)$: no guarantee that $f(\mathcal{E}) > 0$ ⚠

By relaxing the assumption that $f = f(\mathcal{E})$ (isothropic in v)
Ex: $f = f(\mathcal{E}, L = |\vec{L}|)$, we can ensure $f > 0$

Idea:

1. Build a model based on circular orbits only.
By giving the appropriate weight to orbits at every radius, we can obtain a model with the desired $\nu(r)$

2. Add it to an ergodic DF that generates $\nu(r)$

We can ensure that the sum of both DFs is positive.

DF of a model based only on circular orbits

$$\boxed{f_c(\mathcal{E}, L) = \delta(L - L_c(\mathcal{E}))\, F(\mathcal{E})}$$

$L_c(\mathcal{E})$ = angular momentum of a circular orbit of energy $\mathcal{E}$

For a given $\mathcal{E}$, selects only orbits with the angular momentum corresponding to a circular orbit.

$F(\mathcal{E})$ is a weighting function such that

$$\boxed{\nu(r) = \int d^3v\, F(\mathcal{E})\, \delta(L - L_c(\mathcal{E}))}$$

Idea: If $f_i(\varepsilon)$ is an ergodic DF

we can define new DFs: (Note: we ensure $\nu(r) = \int f_\alpha \, d^3v$)

$$\boxed{f_\alpha(\varepsilon, L) = \alpha \, f_i(\varepsilon) + (1-\alpha) \, f_c(\varepsilon, L)}$$

$\alpha = 0$: circular orbits $\sigma_\theta = \sigma_\varphi \neq 0$, $\sigma_r = 0$

$\alpha = 1$: ergodic (isothrope) $\sigma_\theta = \sigma_\varphi = \sigma_r$

$\alpha > 1$: more elongated orbits "radial" $\sigma_\theta = \sigma_\varphi < \sigma_r$

eccentricity of orbits increases

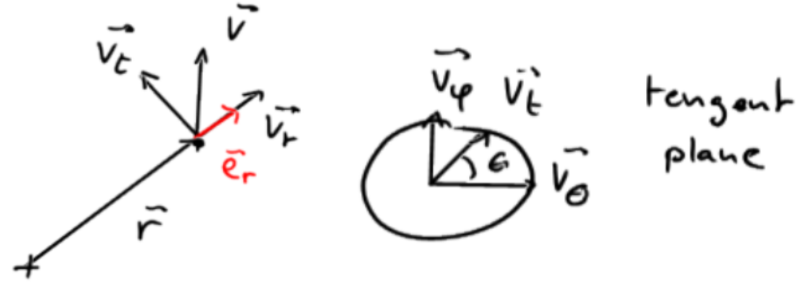
⚠ as long as $f_\alpha > 0$

If $f_i(\varepsilon) < 0$ we can then ensure $f_\alpha(\varepsilon, L) > 0$ as

1) $f_c(\varepsilon, L) > 0$

2) $(1-\alpha) > 0$ $\alpha \in [0, 1]$

i.e giving more weight to circular orbits



<u>Definition</u>: anisotropy parameter

$$\beta := 1 - \frac{\sigma_\theta^2 + \sigma_\phi^2}{2\sigma_r^2} = 1 - \frac{\sigma_t^2}{2\sigma_r^2}$$

$\beta = -\infty$

- Circular orbits

  $\sigma_\theta = \sigma_\phi \neq 0, \sigma_r = 0$

$\beta = 0$

- Isotrope ergodic

  $\sigma_\theta = \sigma_\phi = \sigma_r = \frac{1}{\sqrt{2}}\sigma_t$

$\beta = 1$

- Radial orbits

  $\sigma_\theta = \sigma_\phi = 0, \sigma_r \neq 0$

- tangentially biased orbits

  $\sigma_\theta = \sigma_\phi > \sigma_r$

- radially biased orbits

  $\sigma_\theta = \sigma_\phi < \sigma_r$

Models defined from an anisotropic DFs

Models with constant anisotropy

EXERCICE

$$\boxed{f(\mathcal{E}, L) = f_1(\mathcal{E})\, L^{\gamma} = f_1(\mathcal{E})\, L^{-2\beta}} \qquad f_1(\mathcal{E}) > 0$$

Can we find an expression for $f_1(\mathcal{E})$, for a given $\Psi(r)$ and $\rho(r)$?

From $\quad \nu(r) = \int d^3\vec{v}\, f_1(\mathcal{E})\, L^{-2\beta}$

$$\boxed{\frac{2^{\beta-\frac{1}{2}}}{2\pi\, \Gamma_\beta}\, r^{2\beta}\, \nu(\Psi) = \int_0^{\Psi} d\mathcal{E}\, \frac{f_1(\mathcal{E})}{(\Psi-\mathcal{E})^{\beta-\frac{1}{2}}}}$$

Differentiating with respect to Ψ, we get an Abel integral

$$\frac{2^{\beta-\frac{1}{2}}}{2\pi I_\beta} \frac{d}{d\Psi}\left(r^{2\beta}\nu\right) = \left(\frac{1}{2}-\beta\right)\int_0^{\Psi} d\mathcal{E} \, \frac{f_1(\mathcal{E})}{(\Psi-\mathcal{E})^{\beta+1/2}}$$

which can be inverted ("Eddington" formula)

$$\boxed{f_1(\mathcal{E}) = \frac{\sin(\pi\cdot(\beta+\frac{1}{2}))}{\pi} \frac{2\pi I_\beta}{2^{\beta-1/2}} \left(\frac{1}{2}-\beta\right) \times \left[\int_0^{\mathcal{E}} \frac{d\Psi}{(\mathcal{E}-\Psi)^{\frac{1}{2}-\beta}} \frac{d^2}{d\Psi^2}\left(r(\Psi)^{2\beta}\nu(\Psi)\right) + \frac{1}{\mathcal{E}^{\frac{1}{2}-\beta}}\left(\frac{d}{d\Psi}(r^{2\beta}\nu)\right)_{\Psi=0}\right]}$$

COMPLEMENT

<u>Density</u> : $\nu(r) = \int d^3\vec{v}\ f_-(\mathcal{E})\ L^{-2\beta}$

integration using polar coord. in velocity space :

$$\begin{cases} v_r = v\cos\eta \\ v_\theta = v\sin\eta\cos\varphi \\ v_\varphi = v\sin\eta\sin\varphi \end{cases} \qquad \begin{aligned} L &= r\sqrt{v_\theta^2+v_\varphi^2} = r\,v\sin\eta \\ d^3\vec{v} &= dv_r\,dv_\theta\,dv_\varphi\ v^2\sin\eta \end{aligned}$$

$$\nu(r) = \int d^3\vec{v}\ f_-(\mathcal{E})\ L^{-2\beta}$$

$$= 2\pi\int_0^\pi d\eta\,\sin\eta\int_0^\infty dv\,v^2\ f_-\left(\psi(r)-\tfrac{1}{2}v^2\right)L^{-2\beta}$$

$$= \frac{2\pi}{r^{2\beta}}\underbrace{\int_0^\pi d\eta\,\sin^{1-2\beta}\eta}_{\sqrt{\pi}\frac{(-\beta)!}{(\frac{1}{2}-\beta)!}\ :=\ I_\beta}\int_0^\infty dv\ v^{2-2\beta} f_-\left(\psi(r)-\tfrac{1}{2}v^2\right)$$

$(\,!\ \beta < 1\,)$

COMPLEMENT

And integrating through the energy $\varepsilon = \psi - \frac{1}{2} v^2$

$$\begin{cases} v = \sqrt{2(\psi - \varepsilon)} \qquad dv = \dfrac{-1}{\sqrt{2(\psi-\varepsilon)}}\, d\varepsilon \\ \frac{1}{2} v^2 + \phi = \phi_0 - \varepsilon \end{cases}$$

+ $\nu(r)$ is a monotonic function of ψ

$$\boxed{\frac{2^{\beta - 1/2}}{2\pi I_\beta}\, r^{2\beta}\, \nu(\psi) = \int_0^\psi d\varepsilon \, \frac{\rho_1(\varepsilon)}{(\psi - \varepsilon)^{\beta - 1/2}}}$$

#

$$I_\beta \equiv \int_0^\pi \mathrm{d}\eta \, \sin^{1-2\beta}\eta = \sqrt{\pi}\frac{(-\beta)!}{(\frac{1}{2}-\beta)!} \quad (\beta < 1).$$

Derivation of the "Eddington" formula

$$\frac{2^{\beta-\frac{1}{2}}}{2\pi I_\beta} \frac{d}{d\psi}\left(r^{2\beta}\nu\right) = \left(\frac{1}{2}-\beta\right)\int_0^\psi d\mathcal{E} \frac{f_1(\mathcal{E})}{(\psi-\mathcal{E})^{\beta+1/2}}$$

with $\alpha = \beta + \frac{1}{2}$ $\quad \beta = \alpha - \frac{1}{2}$ $\quad 1-\alpha = \frac{1}{2} - \beta$

$$\frac{d}{d\psi}\left(r^{2\beta}\nu\right) = \frac{2\pi I_\beta}{2^{\beta-1/2}}\left(\frac{1}{2}-\beta\right)\int_0^\psi d\mathcal{E} \frac{f_1(\mathcal{E})}{(\psi-\mathcal{E})^{\alpha}}$$

$$f_1(\mathcal{E}) = \frac{\sin(\pi\alpha)}{\pi} \frac{2\pi I_\beta}{2^{\beta-1/2}}\left(\frac{1}{2}-\beta\right)$$

$$\times\left[\int_0^{\mathcal{E}} \frac{d\psi}{(\mathcal{E}-\psi)^{1-\alpha}} \frac{d^2}{d\psi^2}\left(r(\psi)^{2\beta}\nu(\psi)\right) + \frac{1}{\mathcal{E}^{1-\alpha}}\left(\frac{d}{d\psi}\left(r^{2\beta}\nu\right)\right)_{\psi=0}\right]$$

COMPLEMENT

$$f_1(\varepsilon) = \frac{\sin\left(\pi \cdot \left(\beta + \frac{1}{2}\right)\right)}{\pi} \frac{2\pi I_\beta}{2^{\beta - 1/2}} \left(\frac{1}{2} - \beta\right)$$

$$\times \left[\int_0^{\varepsilon} \frac{d\psi}{(\varepsilon - \psi)^{\frac{1}{2} - \beta}} \frac{d^2}{d\psi^2}\left(r^{2\beta}(\psi)\, \nu(\psi) \right) + \frac{1}{\varepsilon^{\frac{1}{2} - \beta}} \left(\frac{d}{d\psi}\left(r^{2\beta} \nu\right)\right)_{\psi = 0} \right]$$

#

COMPLEMENT

COMPLEMENT

Abel integral

$$f(x) = \int_0^x dt \, \frac{g(t)}{(x-t)^\alpha} \qquad \boxed{0 < \alpha < 1}$$

$$g(t) = \frac{\sin(\pi\alpha)}{\pi} \left[\int_0^t \frac{dx}{(t-x)^{1-\alpha}} \frac{df}{dx} + \frac{f(0)}{t^{1-\alpha}} \right]$$

in term of ψ, ε

$x \to \psi$
$t \to \varepsilon$

$$f(\psi) = \int_0^\psi d\varepsilon \, \frac{g(\varepsilon)}{(\psi-\varepsilon)^\alpha}$$

$$g(\varepsilon) = \frac{\sin(\pi\alpha)}{\pi} \left[\int_0^\varepsilon \frac{d\psi}{(\varepsilon-\psi)^{1-\alpha}} \frac{df(\psi)}{d\psi} + \frac{f(0)}{\varepsilon^{1-\alpha}} \right]$$

Equilibria of collisionless systems

Jeans Equations

The Jeans Equations

- From observations, we usually obtain velocity moments:

  Examples : mean velocity $\bar{v}_i$

  velocity dispersions $\overline{v_i v_j} \equiv \sigma_{ij}$

- Computing moments from a DF is "easy":

$$\bar{v}_i = \frac{1}{\nu(\vec{x})} \int v_i \, f(\vec{x}, \vec{v}) \, d^3\vec{v}$$

- Obtaining a DF compatible with an observed $\nu(\vec{x})$ ($\rho(\vec{x})$) is less easy and solutions are often not unique.

Our goal

Find a method that let infer moments from stellar systems, without recovering the DF.

Idea

Compute moments of the collisionless Boltzman equation.

In carthesian coordinates

$$\frac{\partial f}{\partial t} + \vec{v}\,\frac{\partial f}{\partial \vec{x}} - \vec{\nabla}\phi\,\frac{\partial f}{\partial \vec{v}} = 0$$

$$\boxed{\frac{\partial}{\partial t} f + \sum_i v_i \frac{\partial f}{\partial x_i} - \sum_i \frac{\partial \phi}{\partial x_i}\frac{\partial f}{\partial v_i} = 0}$$

Zeroth moment

$$\frac{\partial}{\partial t} f + \sum_i v_i \frac{\partial f}{\partial x_i} - \sum_i \frac{\partial \phi}{\partial x_i} \frac{\partial f}{\partial v_i} = 0$$

integrate over velocities

$$\int \frac{\partial}{\partial t} f \, d^3 + \sum_i \int d^3 v_i \frac{\partial f}{\partial x_i} - \sum_i \frac{\partial \phi}{\partial x_i} \int d^3 v \frac{\partial f}{\partial v_i} = 0$$

$$\frac{\partial}{\partial t} \underbrace{\int f \, d^3 v}_{\nu(\vec{x})} + \sum_i \frac{\partial}{\partial x_i} \underbrace{\int d^3 v \, v_i f}_{\nu \bar{v}_i} - \sum_i \frac{\partial \phi}{\partial x_i} \underbrace{\oint_S d^2 s \, f}_{=0} = 0$$

v_i does not dep. on x_i (canonical coords)

div. theorem ⊕ + $f(x, v, t) = 0$ for $v \to \infty$ $= 0$

We get

$$\boxed{\frac{\partial}{\partial t} \nu(\vec{x}) + \sum_i \frac{\partial}{\partial x_i} (\nu \bar{v}_i) = 0}$$

continuity equation for $\nu(x)$

$$\boxed{\frac{\partial}{\partial t} \nu + \vec{\nabla} \cdot (\nu \vec{\bar{v}}) = 0}$$

$$\left(\frac{\partial}{\partial t} \rho + \vec{\nabla} \cdot (\rho \vec{v}) \right) \qquad \nu = \rho, \quad \vec{\bar{v}} = \vec{v}$$

⊕ div. theorem $\int d^3 x \, \vec{\nabla} \vec{F} = \int d^2 \vec{s} \cdot \vec{F}$

for $\vec{F} = f \vec{e}_j$ $\quad \int d^3 x \frac{\partial f}{\partial x_j} = \int d^2 s_j \, f$

First moment

$$\boxed{\frac{\partial}{\partial t} f + \sum_i v_i \frac{\partial f}{\partial x_i} - \sum_i \frac{\partial \phi}{\partial x_i}\frac{\partial f}{\partial v_i} = 0}$$

multiply by v_j and integrate over velocities

$$\frac{\partial}{\partial t}\underbrace{\int d^3v\, v_j\, f}_{\nu \overline{v_j}} + \underbrace{\int d^3v \sum_i v_i v_j \frac{\partial f}{\partial x_i}}_{2} - \sum_i \frac{\partial \phi}{\partial x_i} \underbrace{\int d^3v\, v_j \frac{\partial f}{\partial v_i}}_{3} = 0$$

$3 = \delta_{ij}\nu$

$$2 \quad \int d^3v \sum_i v_i v_j \frac{\partial f}{\partial x_i} = \sum_i \frac{\partial}{\partial x_i}\int d^3v\, v_i v_j f = \sum_i \frac{\partial}{\partial x_i}\left(\overline{v_i v_j}\,\nu\right)$$

$$\underbrace{\int d^3v \frac{\partial}{\partial v_i}(v_j f)}_{\oint d^2s\, v_j f = 0} = \underbrace{\int d^3v\, v_i \frac{\partial}{\partial v_i} f}_{3} + \underbrace{\int d^3v\, f \underbrace{\frac{\partial v_j}{\partial v_i}}_{\delta_{ij}}}_{\delta_{ij}\nu}$$

$$\boxed{\frac{\partial}{\partial t}\left(\overline{v_j}\,\nu\right) + \sum_i \frac{\partial}{\partial x_i}\left(\overline{v_i v_j}\,\nu\right) + \nu\frac{\partial \phi}{\partial x_j} = 0}$$

Using the continuity equation multiplied by $\bar{v}_j$

$$\bar{v}_j \left(\frac{\partial}{\partial t}\nu + \sum_i \frac{\partial}{\partial x_i}\left(\nu \bar{v}_i\right) \right) = 0$$

and substracting it from the previous result

$$\underbrace{\frac{\partial}{\partial t}\left(\bar{v}_j \nu\right) - \bar{v}_j \frac{\partial}{\partial t}\nu}_{\nu \frac{\partial}{\partial t}\bar{v}_j} + \underbrace{\sum_i \frac{\partial}{\partial x_i}\left(\overline{v_i v_j}\,\nu\right) - \bar{v}_j \sum_i \frac{\partial}{\partial x_i}\left(\nu \bar{v}_i\right)}_{1} + \nu \frac{\partial \phi}{\partial x_j} = 0$$

with $\sigma_{ij}^2 = \overline{v_i v_j} - \bar{v}_i \bar{v}_j$

$$1 = \sum_i \frac{\partial}{\partial x_i}\left(\sigma_{ij}^2 \nu\right) + \underbrace{\sum_i \frac{\partial}{\partial x_i}\left(\bar{v}_i \bar{v}_j \nu\right)} - \bar{v}_j \sum_i \frac{\partial}{\partial x_i}\left(\nu \bar{v}_i\right)$$

$$\nu \sum_i \bar{v}_i \frac{\partial}{\partial x_i}\left(\bar{v}_j\right) + \underbrace{\sum_i \bar{v}_j \frac{\partial}{\partial x_i}\left(\bar{v}_i \nu\right) - \bar{v}_j \sum_i \frac{\partial}{\partial x_i}\left(\nu \bar{v}_i\right)}_{=0}$$

$$\boxed{\nu \frac{\partial}{\partial t}(\bar{v}_j) + \nu \sum_i \bar{v}_i \frac{\partial}{\partial x_i} \bar{v}_j = - \sum_i \frac{\partial}{\partial x_i}(\sigma_{ij}^2 \nu) - \nu \frac{\partial \phi}{\partial x_j}}$$

Jeans 1919

Interpretation

Euler equation in hydrodynamics

Lagrangian form
$$\frac{d}{dt}\vec{v} = -\frac{\vec{\nabla} P}{\rho} - \vec{\nabla}\phi$$

Eulerian form
$$\text{(*)} \quad \frac{\partial}{\partial t}\vec{v} + \vec{v}\cdot\vec{\nabla}\vec{v} = -\frac{\vec{\nabla} P}{\rho} - \vec{\nabla}\phi$$

$$\rho \frac{\partial}{\partial t}\vec{v} + \rho\, \vec{v}\cdot\vec{\nabla}\vec{v} = -\vec{\nabla} P - \rho\, \vec{\nabla}\cdot\phi$$

"j" component only
$$\boxed{\rho \frac{\partial}{\partial t} v_j + \rho \sum_i v_i \frac{\partial v_j}{\partial x_i} = -\frac{\partial P}{\partial x_j} - \rho \frac{\partial \phi}{\partial x_j}}$$

$$\text{(*)} \quad \frac{dV_i}{dt} = \frac{\partial V_i}{\partial t} + \sum \frac{\partial V_i}{\partial x} \dot{x}$$

Both equations are similar

if $\boxed{\rho = \nu}$ $\boxed{V_i = \bar{V}_i}$ $\boxed{\dfrac{\partial P}{\partial x_j} = \sum_i \dfrac{\partial}{\partial x_i}\left(\sigma_{ij}^2 \nu\right)}$

$$\begin{pmatrix} P & & \\ & P & \\ & & P \end{pmatrix} \equiv \begin{pmatrix} \sigma_{xx}^2 & \sigma_{xy}^2 & \sigma_{xz}^2 \\ \sigma_{yx}^2 & \sigma_{yy}^2 & \sigma_{yz}^2 \\ \sigma_{zx}^2 & \sigma_{zy}^2 & \sigma_{zz}^2 \end{pmatrix} \nu$$

anisotropic stress tensor

(symmetric)

diagonal in an appropriate rest frame $\begin{pmatrix} \sigma_1^2 & & \\ & \sigma_2^2 & \\ & & \sigma_3^2 \end{pmatrix} \nu$

Thus $\boxed{\dfrac{\partial P}{\partial x_j} = \dfrac{\partial}{\partial x_i}\left(\sigma_{ij}^2 \nu\right)}$

Note: it is possible to show that for an ergodic system,

$$P = \int_0^\rho d\rho' \rho' \frac{\partial \phi}{\partial \rho'}$$

leads to

$$P = \sigma^2 \nu$$

Comments

$f(\vec{x},\vec{v})$ is unknown

2 known quantities : $\rho(\vec{x})$, $\phi(\vec{x})$

6 unknown quantities : $\bar{v}_x \; \bar{v}_y \; \bar{v}_z$, σ_{xx} , σ_{yy} , σ_{zz} (assuming it is diagonal)

4 equations : zeroth moment (1) + first moment (3)

$$\boxed{\text{The Jeans equations are not closed !}}$$

- if we multiply the CB by $v_i v_j$ → new terms $\overline{v_i v_j v_k}$
  → not a solution
- we need to do some assumptions (closure conditions)

<u>example</u> : σ_{ij} (3) → σ (1) ok if f is ergodic

Equilibria of collisionless systems

"Static" Jeans Equations for spherical systems

The Jeans equations for spherical systems

Canonical momenta

$$
\begin{cases}
p_r = \dot{r} = v_r \\
p_\theta = r^2 \dot{\theta} = r v_\theta \\
p_\phi = r^2 \sin^2(\theta \dot{\phi}) = r \sin(\theta)\, v_\phi
\end{cases}
$$

The static Collisionless Boltzmann Equation, for spherical systems

$$
\cancel{\frac{\partial f}{\partial t}} + p_r \frac{\partial f}{\partial r} + \frac{p_\theta}{r^2}\frac{\partial f}{\partial \theta} + \frac{p_\phi}{r^2 \cancel{\sin^2}(\theta)}\frac{\partial f}{\partial \phi} - \left(\frac{\partial \Phi}{\partial r} - \frac{p_\theta^2}{r^3} - \frac{p_\phi^2}{r^3 \sin^2(\theta)}\right)\frac{\partial f}{\partial p_r} - \left(\cancel{\frac{\partial \Phi}{\partial \theta}} - \frac{p_\phi^2 \cos(\theta)}{r^2 \sin^3(\theta)}\right)\frac{\partial f}{\partial p_\theta} - \cancel{\frac{\partial \Phi}{\partial \phi}}\frac{\partial f}{\partial p_\phi} = 0
$$

f can depend on θ as $p_\phi = r\sin(\theta)\, v_\phi$

Zeroth order moment of the Jeans Equation

EXERCICE

$$
\frac{\partial}{\partial r}\left(\sin(\theta)\nu\overline{v_r}\right) = \frac{\partial}{\partial \theta}\left(\sin(\theta)\nu\overline{v_\theta}\right)
$$

The Jeans equations for spherical systems

Canonical momenta

$$\begin{cases} p_r = \dot{r} = v_r \\ p_\theta = r^2\dot{\theta} = r v_\theta \\ p_\phi = r^2 \sin^2(\theta\dot{\phi}) = r\sin(\theta)\, v_\phi \end{cases}$$

The static Collisionless Boltzmann Equation, for spherical systems

$$\cancel{\frac{\partial f}{\partial t}} + p_r \frac{\partial f}{\partial r} + \frac{p_\theta}{r^2}\frac{\partial f}{\partial \theta} + \frac{p_\phi}{r^2 \sin^{\cancel{2}}(\theta)}\frac{\partial f}{\partial \phi} - \left(\frac{\partial \Phi}{\partial r} - \frac{p_\theta^2}{r^3} - \frac{p_\phi^2}{r^3\sin^2(\theta)}\right)\frac{\partial f}{\partial p_r} - \left(\cancel{\frac{\partial \Phi}{\partial \theta}} - \frac{p_\phi^2\cos(\theta)}{r^2\sin^3(\theta)}\right)\frac{\partial f}{\partial p_\theta} - \cancel{\frac{\partial \Phi}{\partial \phi}}\frac{\partial f}{\partial p_\phi} = 0$$

f can depend on θ as $p_\phi = r\sin(\theta)\, v_\phi$

Zeroth order moment of the Jeans Equation

$$\frac{\partial}{\partial r}\left(\sin(\theta)\nu\overline{v_r}\right) = \frac{\partial}{\partial \theta}\left(\sin(\theta)\nu\overline{v_\theta}\right)$$

if $\quad f = f(H) \;\text{ or }\; f(H,L) \Rightarrow \overline{v_r} = \overline{v_z} = \overline{v_\theta} = 0$

$$\overline{v_r^2} = \sigma_r^2 \;\; \overline{v_\theta^2} = \sigma_\theta^2 \;\; \overline{v_\phi^2} = \sigma_\phi^2$$

The Jeans equations for spherical systems

Canonical momenta

$$\begin{cases} p_r = \dot{r} = v_r \\ p_\theta = r^2 \dot{\theta} = r v_\theta \\ p_\phi = r^2 \sin^2(\theta \dot{\phi}) = r \sin(\theta)\, v_\phi \end{cases}$$

The static Collisionless Boltzmann Equation, for spherical systems

$$\cancel{\frac{\partial f}{\partial t}} + p_r \frac{\partial f}{\partial r} + \frac{p_\theta}{r^2} \frac{\partial f}{\partial \theta} + \cancel{\frac{p_\phi}{r^2 \sin^2(\theta)}} \frac{\partial f}{\partial \phi} - \left(\frac{\partial \Phi}{\partial r} - \frac{p_\theta^2}{r^3} - \frac{p_\phi^2}{r^3 \sin^2(\theta)} \right) \frac{\partial f}{\partial p_r} - \left(\cancel{\frac{\partial \Phi}{\partial \theta}} - \frac{p_\phi^2 \cos(\theta)}{r^2 \sin^3(\theta)} \right) \frac{\partial f}{\partial p_\theta} - \cancel{\frac{\partial \Phi}{\partial \phi}} \frac{\partial f}{\partial p_\phi} = 0$$

f can depend on θ as $p_\phi = r \sin(\theta)\, v_\phi$

Zeroth order moment of the Jeans Equation

EXERCICE

$$0 = 0$$

if $f = f(H)$ or $f(H, L) \Rightarrow \overline{v_r} = \overline{v_z} = \overline{v_\theta} = 0$

$$\overline{v_r^2} = \sigma_r^2 \quad \overline{v_\theta^2} = \sigma_\theta^2 \quad \overline{v_\phi^2} = \sigma_\phi^2$$

First order moment of the Jeans Equation

EXERCICE

$$\frac{\partial}{\partial r}\left(\nu \sigma_r^2\right) + \nu \left(\frac{\partial \Phi}{\partial r} + \frac{2\sigma_r^2 - \sigma_\theta^2 - \sigma_\phi^2}{r} \right) = 0$$

or

$$\boxed{\frac{\partial}{\partial r}\left(\nu \sigma_r^2\right) + 2 \frac{\beta}{r} \nu \sigma_r^2 = -\nu \frac{\partial \Phi}{\partial r}}$$

where

$$\beta = 1 - \frac{\sigma_\theta^2 + \sigma_\phi^2}{2\sigma_r^2} = 1 - \frac{\sigma_t^2}{2\sigma_r^2}$$

Discussion

$$\boxed{\frac{\partial}{\partial r}\left(\nu \sigma_r^2\right) + \nu\left(\frac{\partial \phi}{\partial r} + \frac{2\sigma_r^2 - \sigma_\theta^2 - \sigma_\varphi^2}{r}\right) = 0}$$

Case $\boxed{\sigma_r = \sigma_\varphi = \sigma_\theta}$ $\Rightarrow \frac{1}{\nu}\frac{\partial}{\partial r}\left(\nu \sigma_r^2\right) = -\frac{\partial \phi}{\partial r}$

Ergodic $\equiv \frac{\vec{\nabla} p}{\rho} = \vec{F}_{grav}$

Discussion

$$\boxed{\frac{\partial}{\partial r}(\nu \sigma_r^2) + \nu\left(\frac{\partial \phi}{\partial r} + \frac{2\sigma_r^2 - \sigma_\theta^2 - \sigma_\varphi^2}{r}\right) = 0}$$

Case $\boxed{\sigma_r = 0}$ $\rightarrow$ $\sigma_t^2 = r\frac{\partial \phi}{\partial r}$

interpretation

only circular orbits

$$v_t^2 = r\frac{\partial \phi}{\partial r}$$

but from all possible planes

Demonstration

associated dispersion : in the tangential plane

$$V_\varphi = V_t \cos\eta \qquad \sigma_\varphi^2 = \frac{1}{2\pi}\int V_t^2 \cos^2\eta \, d\eta = \frac{1}{2} V_t^2$$

$$V_e = V_t \sin\eta \qquad \sigma_\epsilon^2 = \qquad = \frac{1}{2} V_t^2$$

thus $\sigma_t^2 := \sigma_\varphi^2 + \sigma_\epsilon^2 = V_t^2$

#

COMPLEMENT

Discussion

$$\boxed{\frac{\partial}{\partial r}\left(\nu\,\sigma_r^2\right) + \nu\left(\frac{\partial\phi}{\partial r} + \frac{2\sigma_r^2 - \sigma_\theta^2 - \sigma_\varphi^2}{r}\right) = 0}$$

Case

$$\boxed{\sigma_t = 0} \quad \Rightarrow \quad \frac{1}{\nu}\frac{\partial}{\partial r}\left(\nu\,\sigma_r^2\right) + \frac{2\sigma_r^2}{r} = -\frac{\partial\phi}{\partial r}$$

purely radial orbits

The Jeans equations for spherical systems

$$\frac{\partial}{\partial r}\left(\nu\sigma_r^2\right) + 2\frac{\beta}{r}\nu\sigma_r^2 = -\nu\frac{\partial\Phi}{\partial r}$$

$$r^{-2\beta}\frac{\partial}{\partial r}\left(\nu\sigma_r^2 r^{2\beta}\right) = -\nu\frac{\partial\Phi}{\partial r}$$

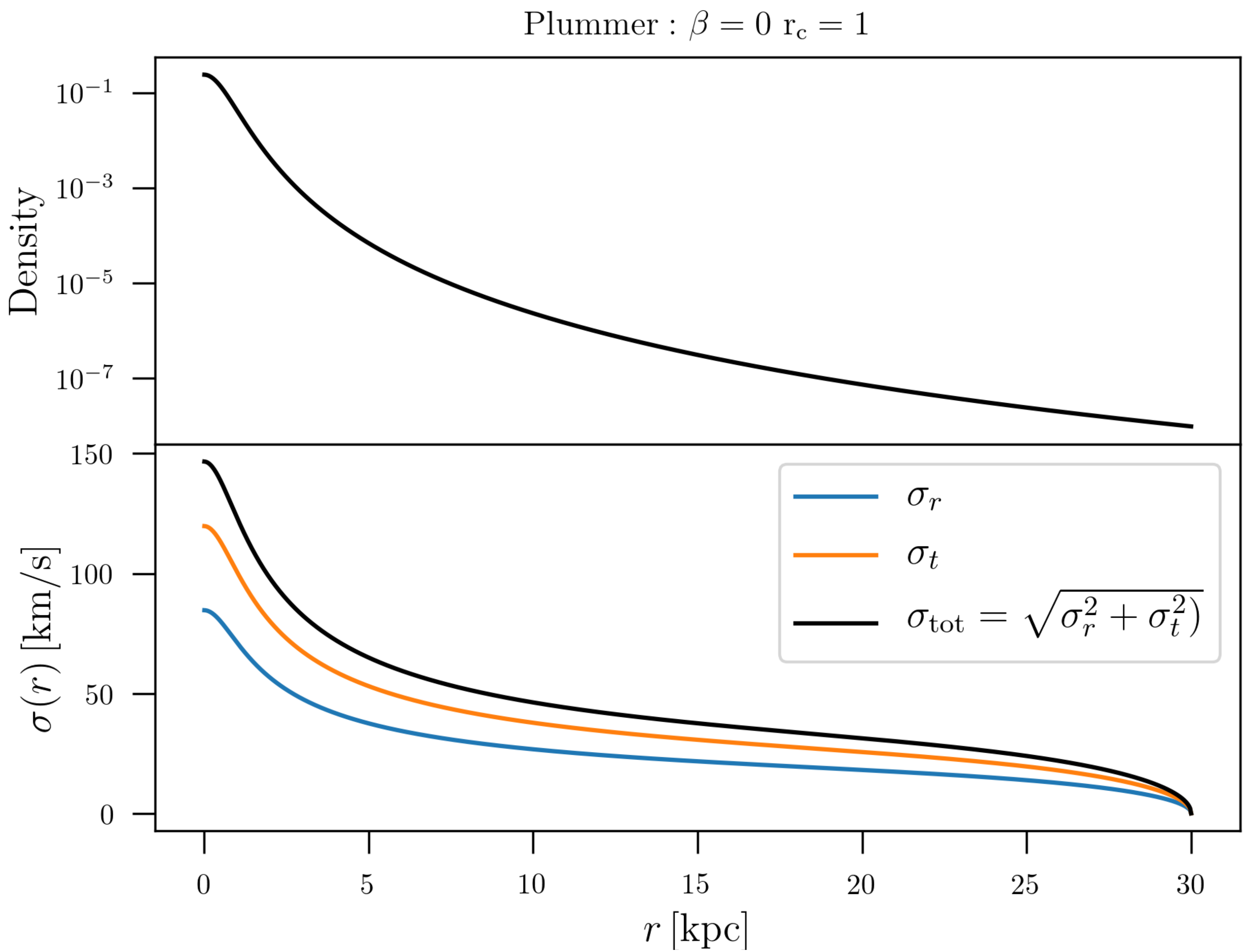
If the system has a constant anisotrpy parameter $\beta = cte$

$$\sigma_r^2(r) = \frac{1}{r^{2\beta}\nu(r)}\int_r^\infty \mathrm{d}r' r'^{2\beta}\nu(r')\frac{\partial\Phi}{\partial r'} = \frac{G}{r^{2\beta}\nu(r)}\int_r^\infty \mathrm{d}r' r'^{2\beta-2}\nu(r')M(r')$$

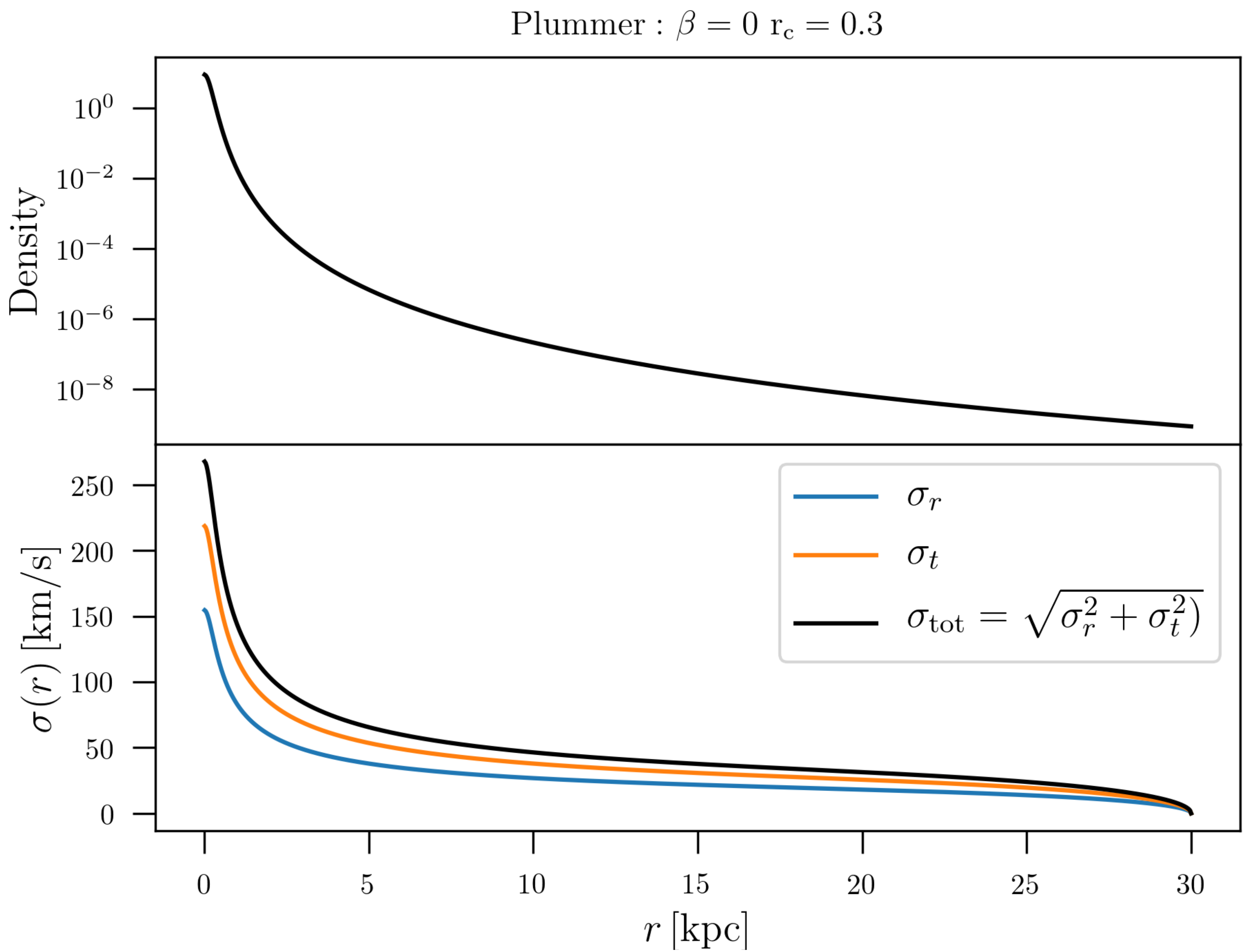
If the system is ergodic (isotropic in velocities) $\beta = 0$

$$\sigma_r^2(r) = \frac{1}{\nu(r)}\int_r^\infty \mathrm{d}r' \nu(r')\frac{\partial\Phi}{\partial r'} = \frac{G}{\nu(r)}\int_r^\infty \mathrm{d}r' \frac{1}{r'^2}\nu(r')M(r')$$

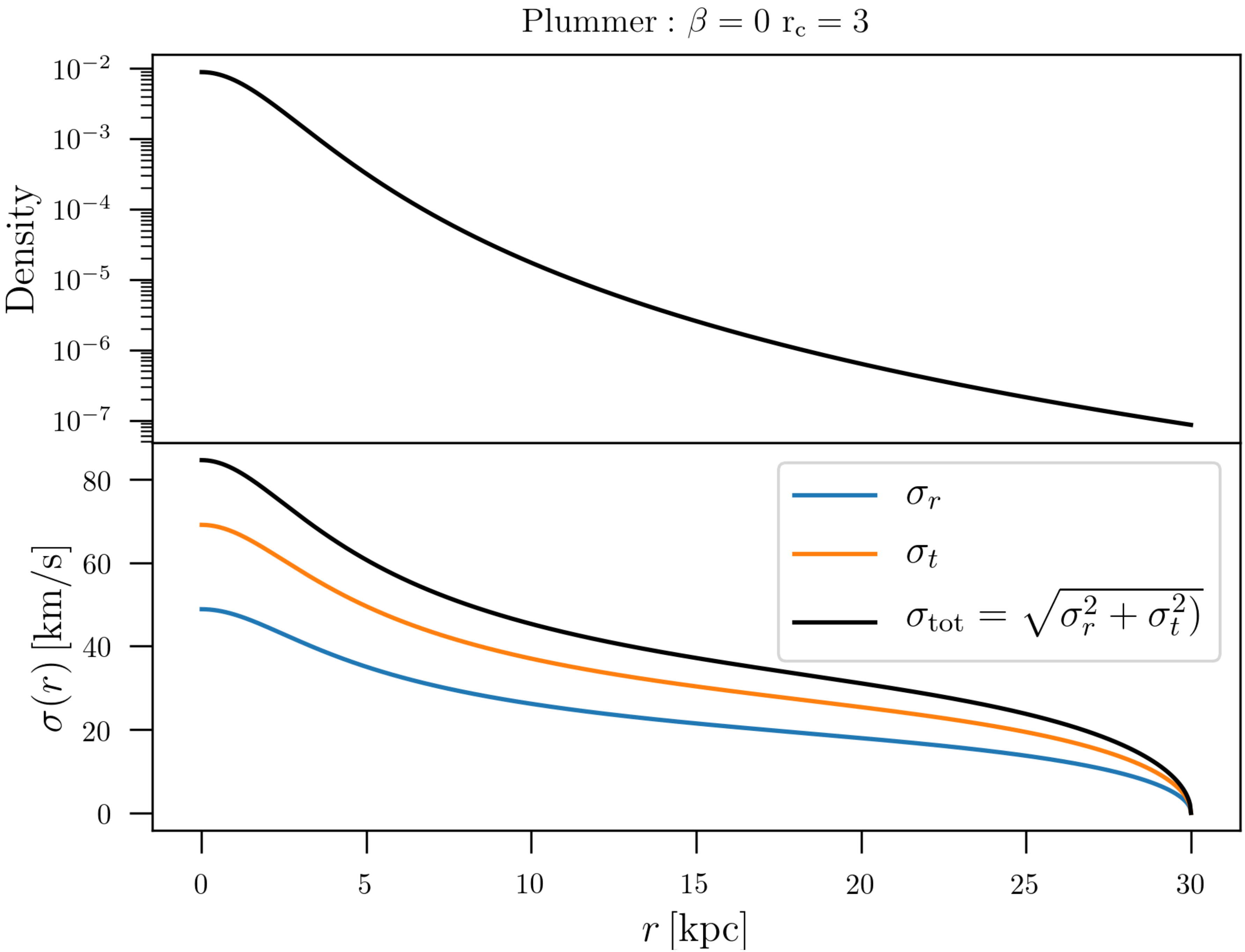
Play with the core radius R_c



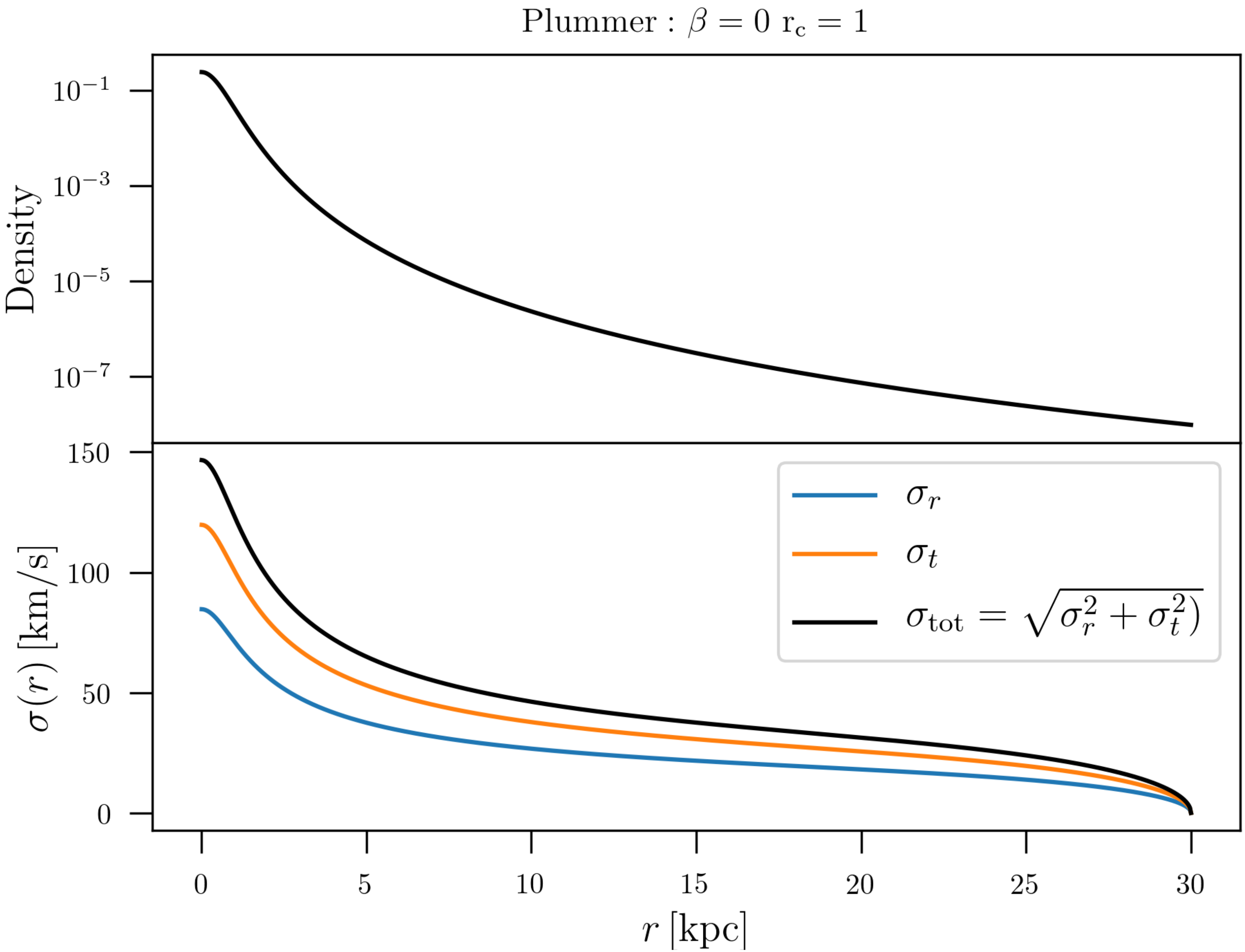
Play with the core radius R_c



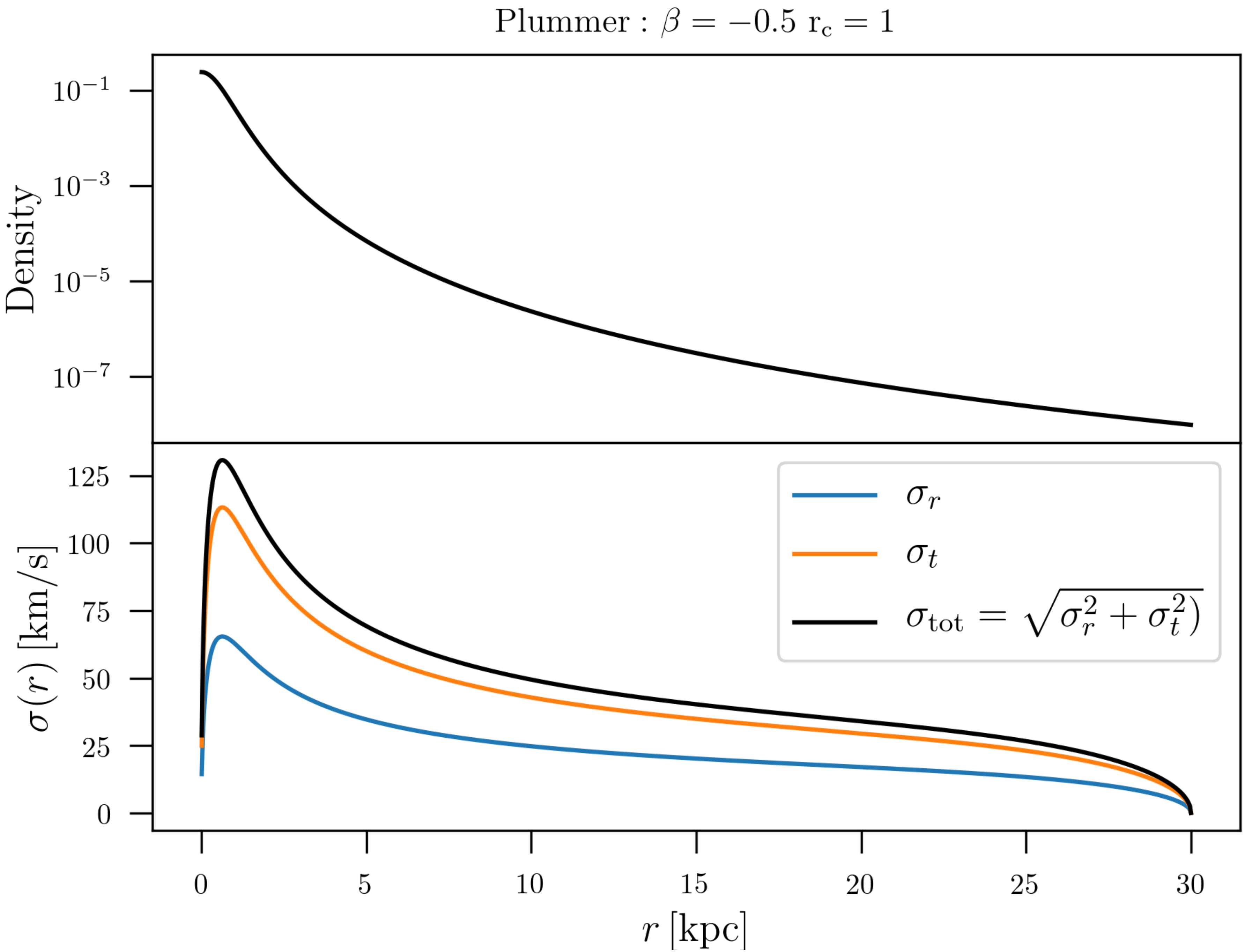
Play with the core radius R_c



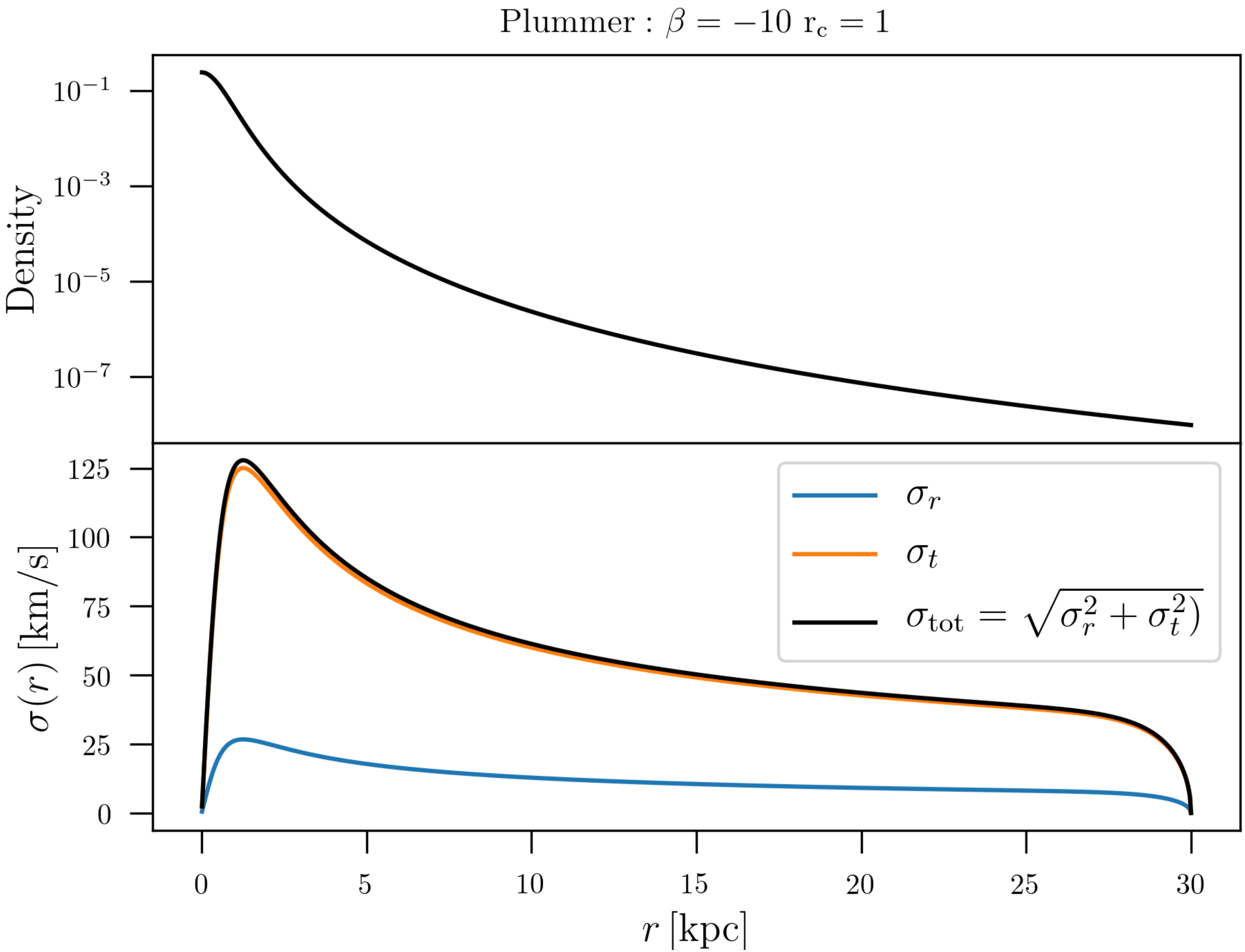
Play with the core radius R_c



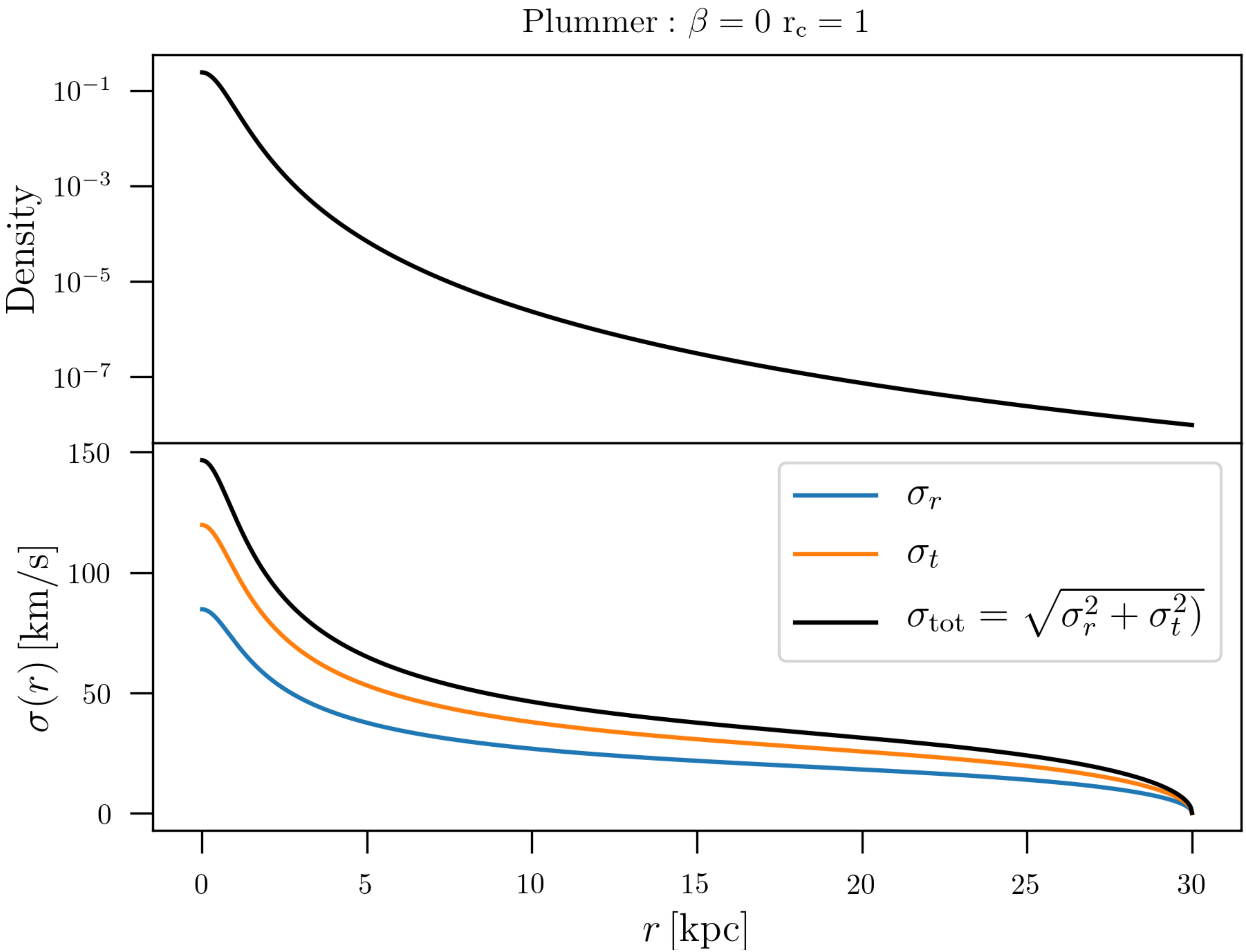
Play with the anisotropy parameter



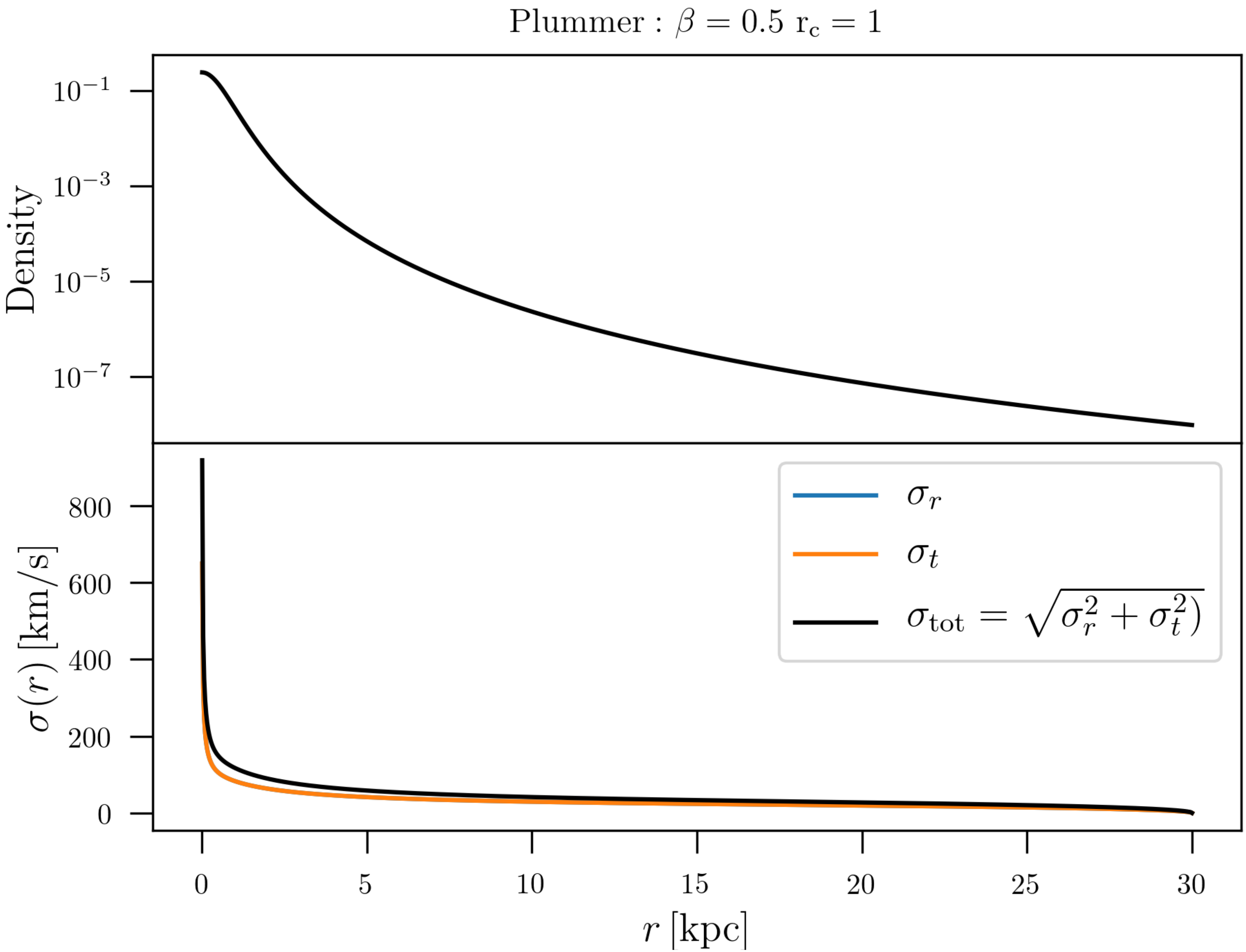
Play with the anisotropy parameter



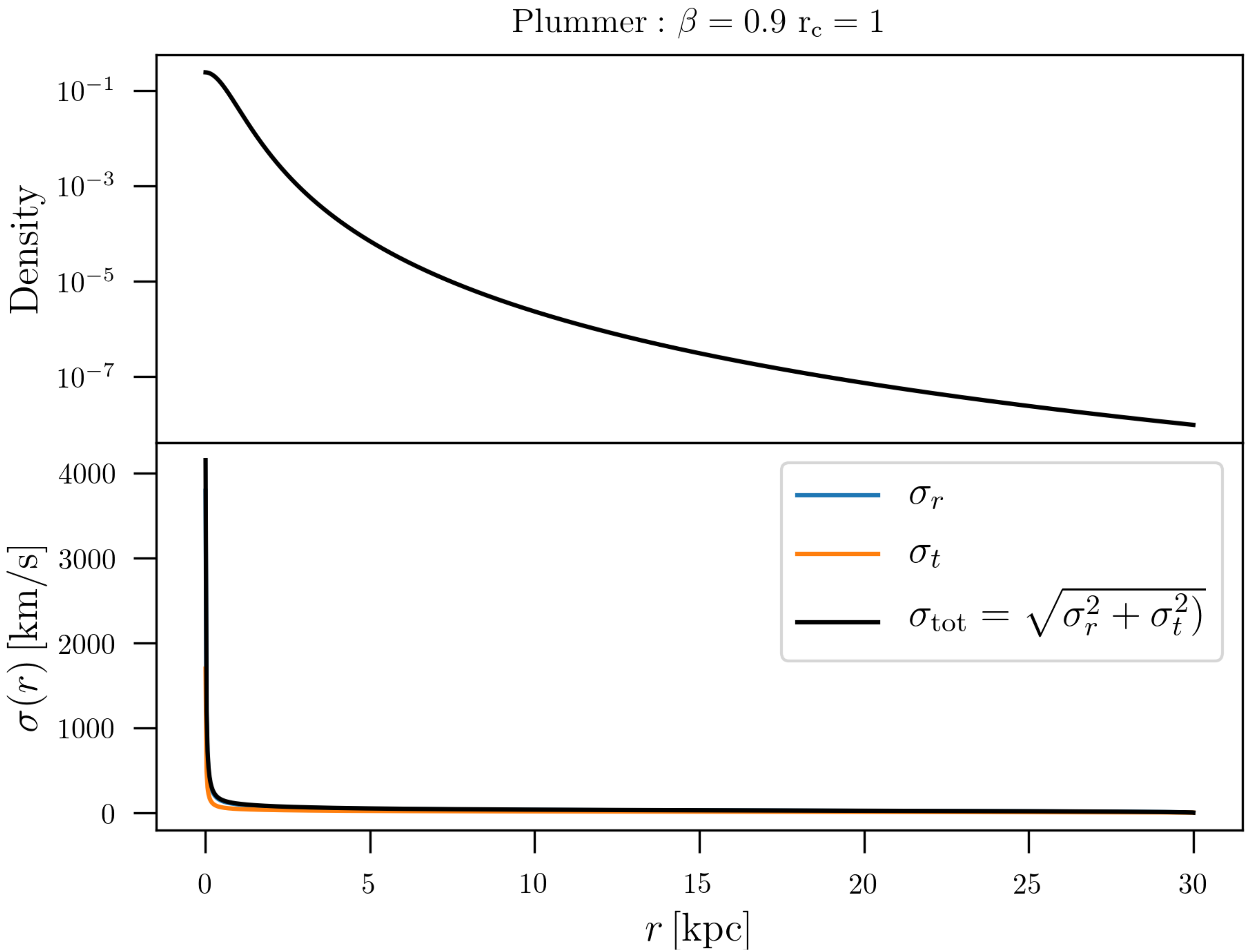
Play with the anisotropy parameter



Play with the anisotropy parameter

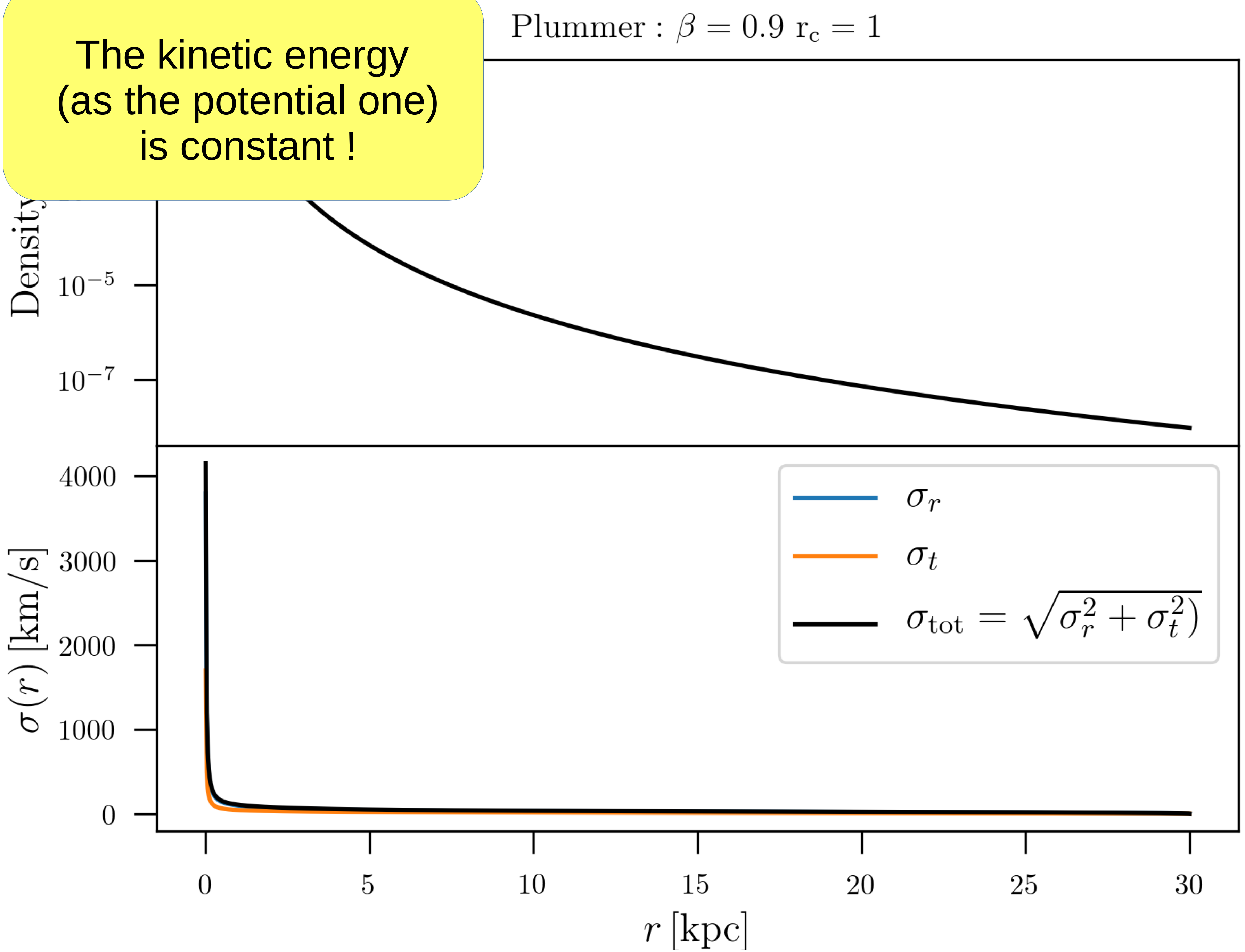


Play with the anisotropy parameter



Play with the anisotropy parameter

The kinetic energy
(as the potential one)
is constant !



Note on the pressure

For an ergodic system, defining $\boxed{P(\rho) = -\int_0^{\rho} d\rho' \, \rho' \, \frac{\partial \phi}{\partial \rho}(\rho')}$

leads to $\frac{\vec{\nabla} P}{\rho} = - \vec{\nabla} \phi$

Comparing the Jeans equations with Euler one suggests

$P = \rho \sigma^2$ but $\boxed{\rho \sigma^2(r) = \int_r^{\infty} dr' \, \rho(r') \, \frac{\partial \phi}{\partial r}}$

So, is

$$P(\rho) = - \int_0^{\rho} d\rho' \, \rho' \, \frac{\partial \phi}{\partial \rho}(\rho') \stackrel{?}{=} P(r) = \int_r^{\infty} dr' \, \rho(r') \, \frac{\partial \phi}{\partial r}$$

COMPLEMENT

$$P(\varrho) = -\int_0^{\varrho} d\varrho' \, \varrho' \, \frac{\partial \phi}{\partial \varrho}(\varrho') \qquad (1) \qquad\qquad P(r) = \int_r^{\infty} dr' \, \varrho(r') \, \frac{\partial \phi}{\partial r}$$

<u>For a spherical system</u>

$$\varrho = \varrho(r) \qquad\qquad \phi = \phi(r)$$

$$d\varrho = \frac{d\varrho}{\partial r} dr \qquad\qquad \frac{\partial \phi}{\partial \varrho} = \frac{\partial \phi}{\partial r} \frac{\partial r}{\partial \varrho}$$

(1) becomes

$$-\int_{\infty}^{r} \cancel{\frac{\partial \varrho}{\partial r}}' dr' \, \varrho(r') \, \frac{\partial \phi}{\partial r}' \cancel{\frac{\partial r}{\partial \varrho}}' = \int_r^{\infty} dr' \, \varrho(r') \, \frac{\partial \phi}{\partial r}$$

R $\phi(\infty) = 0$

#

COMPLEMENT

Equilibria of collisionless systems

“Static” Jeans Equations for cylindrical systems

The Jeans equations for axisymmetric systems

Canonical momenta

$$\begin{cases} p_R = \dot{R} = v_R \\ p_\phi = R^2 \dot{\phi} = R v_\phi \\ p_z = \dot{z} = v_z \end{cases}$$

The static Collisionless Boltzmann Equation, for axisymmetric systems

$$\cancel{\frac{\partial f}{\partial t}} + p_R \frac{\partial f}{\partial R} + \cancel{\frac{p_\phi}{R^2} \frac{\partial f}{\partial \phi}} + p_z \frac{\partial f}{\partial z} - \left(\frac{\partial \Phi}{\partial R} - \frac{p_\phi^2}{R^3} \right) \frac{\partial f}{\partial p_R} - \cancel{\frac{\partial \Phi}{\partial \phi} \frac{\partial f}{\partial p_\phi}} - \frac{\partial \Phi}{\partial z} \frac{\partial f}{\partial p_z} = 0$$

Zeroth order moment of the Jeans Equations if $f = f(H, L_z) \Rightarrow \overline{v_R^2} = \overline{v_z^2}, \overline{v_R} = \overline{v_z} = 0$

$$\overline{v_r^2} = \sigma_r^2 \quad \overline{v_z^2} = \sigma_z^2$$

$$0 = 0$$

$$0 = 0$$

$$0 = 0$$

Canonical momenta

$$\begin{cases} p_R = \dot{R} = v_R \\ p_\phi = R^2 \dot{\phi} = R v_\phi \\ p_z = \dot{z} = v_z \end{cases}$$

The static Collisionless Boltzmann Equation, for axisymmetric systems

$$\cancel{\frac{\partial f}{\partial t}} + p_R \frac{\partial f}{\partial R} + \frac{p_\phi}{R^2} \cancel{\frac{\partial f}{\partial \phi}} + p_z \frac{\partial f}{\partial z} - \left(\frac{\partial \Phi}{\partial R} - \frac{p_\phi^2}{R^3} \right) \frac{\partial f}{\partial p_R} - \cancel{\frac{\partial \Phi}{\partial \phi}} \frac{\partial f}{\partial p_\phi} - \frac{\partial \Phi}{\partial z} \frac{\partial f}{\partial p_z} = 0$$

First order moment of the Jeans Equations

EXERCICE

$$\frac{\partial}{\partial R} \left(\nu \overline{v_R^2} \right) + \frac{\partial}{\partial z} \left(\nu \overline{v_R v_z} \right) + \nu \left(\frac{\overline{v_R^2} - \overline{v_\phi^2}}{R} + \frac{\partial \Phi}{\partial R} \right) = 0$$

$$\frac{1}{R} \frac{\partial}{\partial R} \left(R \nu \overline{v_R v_z} \right) + \frac{\partial}{\partial z} \left(\nu \overline{v_z^2} \right) + \nu \frac{\partial \Phi}{\partial z} = 0$$

$$\frac{1}{R^2} \frac{\partial}{\partial R} \left(R^2 \nu \overline{v_R v_\phi} \right) + \frac{\partial}{\partial z} \left(\nu \overline{v_z v_\phi} \right) = 0$$

The Jeans equations for axisymmetric systems

Canonical momenta

$$\begin{cases} p_R = \dot{R} = v_R \\ p_\phi = R^2\dot{\phi} = R v_\phi \\ p_z = \dot{z} = v_z \end{cases}$$

The static Collisionless Boltzmann Equation, for axisymmetric systems

$$\cancel{\frac{\partial f}{\partial t}} + p_R \frac{\partial f}{\partial R} + \cancel{\frac{p_\phi}{R^2}\frac{\partial f}{\partial \phi}} + p_z \frac{\partial f}{\partial z} - \left(\frac{\partial \Phi}{\partial R} - \frac{p_\phi^2}{R^3}\right)\frac{\partial f}{\partial p_R} - \cancel{\frac{\partial \Phi}{\partial \phi}\frac{\partial f}{\partial p_\phi}} - \frac{\partial \Phi}{\partial z}\frac{\partial f}{\partial p_z} = 0$$

First order moment of the Jeans Equations if $f = f(H, L_z) \Rightarrow \overline{v_R^2} = \overline{v_z^2}, \overline{v_R} = \overline{v_z} = 0$

$$\overline{v_r^2} = \sigma_r^2 \;\; \overline{v_z^2} = \sigma_z^2$$

$$\frac{\partial}{\partial R}\left(\nu \sigma_R^2\right) + \nu \left(\frac{\sigma_R^2 - \overline{v_\phi^2}}{R} + \frac{\partial \Phi}{\partial R}\right) = 0$$

$$\frac{\partial}{\partial z}\left(\nu \sigma_z^2\right) + \nu \frac{\partial \Phi}{\partial z} = 0 \quad \Rightarrow \quad \boxed{\sigma_R^2(R,z) = \sigma_z^2(R,z) = \frac{1}{\nu(R,z)}\int_z^\infty \mathrm{d}z' \nu(R,z') \frac{\partial \Phi}{\partial z'}}$$

$$0 = 0 \quad \Rightarrow \quad \boxed{\overline{v_\phi^2}(R,z) = \sigma_R^2 + \frac{R}{\nu(R,z)}\frac{\partial}{\partial R}\left(\nu \sigma_R^2\right) + R\frac{\partial \Phi}{\partial R}}$$

Jeans equations for axisymmetric systems

$f = f(H, L_z)$

Equations for σ_R, σ_z, $\overline{v_\phi^2}$

$$\boxed{\sigma_R^2 = \sigma_z^2 = \frac{1}{\nu} \int_z^\infty dz' \, \nu(R, z') \frac{\partial \phi}{\partial z'}}$$

$$\boxed{\overline{v_\phi^2}(R,z) = \sigma_R^2 + \frac{R}{\nu} \frac{\partial}{\partial R} \left(\nu \sigma_R^2 \right) + R \frac{\partial \phi}{\partial R}}$$

Interpretation

$$\boxed{\overline{V_\phi^2}(R,z) = \sigma_R^2 + \frac{R}{\nu}\frac{\partial}{\partial R}\left(\nu\sigma_R^2\right) + R\frac{\partial\phi}{\partial R}}$$

In the plane $z=0$

- $R\dfrac{\partial\phi}{\partial R} = V_c^2$
- $\overline{V_\phi^2} = \sigma_\phi^2 + \overline{V}_\phi^2$

$$\boxed{\overline{V}_\phi^2 = V_c^2 - \sigma_\phi^2 + \sigma_R^2 + \frac{R}{\nu}\frac{\partial}{\partial R}\left(\nu\sigma_R^2\right)}$$

1 Equation, 2 Unknowns $\overline{V}_\phi$ σ_ϕ

⚠ This equation involves different energies ⚠

Interpretation

$$\boxed{\overline{V_\phi}^2 = V_c^2 - \sigma_\phi^2 + \sigma_R^2 + \frac{R}{\nu}\frac{\partial}{\partial R}\left(\nu\sigma_R^2\right)}$$

1. if $\sigma_\phi = \sigma_R = 0$ $\quad(\Rightarrow \sigma_z = 0)$ as $\sigma_R = \sigma_z$ $\quad$! ok if $\nu \sim \delta(z)$ = razor thin disk

$$\boxed{\overline{V_\phi}^2 = V_c^2}$$

The mean azimuthal velocity is the circular velocity

The disk is "super cold"

$\sigma_R = \sigma_z = \sigma_\phi = 0$

2. if $\sigma_R = 0$, $\sigma_\phi \neq 0$ $\quad(\Rightarrow \sigma_z = 0)$ $\quad$! ok if $\nu \sim \delta(z)$ = razor thin disk

$$\boxed{\overline{V_\phi}^2 = V_c^2 - \sigma_\phi^2}$$

But $\sigma_r = 0 \Rightarrow$ only circular orbits

1. $\overline{V_\phi}^2 = V_c^2 \Rightarrow \sigma_\phi = 0$ ⚠

2. $\overline{V_\phi}^2 = 0 \Rightarrow$ counter rotating disk with

$\overline{V_\phi} = \frac{1}{2}\left(V_c - V_c\right) = 0$

$\sigma_\phi^2 = \frac{1}{2}\left(V_c^2 + V_c^2\right) = V_c^2$

COMPLEMENT

Interpretation

$$\boxed{\overline{V_\phi}^2 = V_c^2 - \sigma_\phi^2 + \sigma_R^2 + \frac{R}{\nu}\frac{\partial}{\partial R}\left(\nu \sigma_R^2\right)}$$

3. if $\sigma_R = \sigma_\phi \neq 0$ ("Ergodic")

$$\bar{V}_\phi^2 = R\frac{\partial \phi}{\partial R} + \frac{R}{\nu}\frac{\partial}{\partial R}\left(\nu \sigma_R^2\right)$$

$$\frac{1}{R}\bar{V}_\phi^2 = \frac{\partial \phi}{\partial R} + \frac{1}{\nu}\frac{\partial}{\partial R}\left(\nu \sigma_R^2\right)$$

$$\boxed{\underbrace{\frac{1}{\nu}\frac{\partial}{\partial R}\left(\nu \sigma_R^2\right)}_{\sim \frac{\vec{\nabla} P}{\rho} \text{ "pressure" force}} = \underbrace{-\frac{\partial \phi}{\partial R}}_{\vec{F}_{grav} \text{ gravit. force}} + \underbrace{\frac{\bar{V}_\phi^2}{R}}_{\text{centrifugal force}}}$$

Equilibrium in the rotating frame $\Omega = \frac{\bar{V}_\phi}{R}$

$F_c = \Omega^2 R$
$= \frac{V^2}{R}$

$\Omega = \frac{V}{R}$

COMPLEMENT

Interpretation

$$\boxed{\overline{V_\phi}^2 = V_c^2 - \sigma_\phi^2 + \sigma_R^2 + \frac{R}{\nu}\frac{\partial}{\partial R}\left(\nu\sigma_R^2\right)}$$

4. if $\sigma_\phi = 0$, $\sigma_r \neq 0$ (radial orbits)

$$0 = V_c^2 + \sigma_R^2 + \frac{R}{\nu}\frac{\partial}{\partial R}\left(\nu\sigma_R^2\right)$$

$$\frac{1}{\nu}\frac{\partial}{\partial R}\left(\nu\sigma_R^2\right) + \frac{\sigma_R^2}{R} = -\frac{\partial\phi}{\partial R}$$

Nearly identical to the spherical case.

$$\frac{1}{\nu}\frac{\partial}{\partial r}\left(\nu\sigma_r^2\right) + \frac{2\sigma_r^2}{r} = \frac{\partial\phi}{\partial r}$$

How to close the equation? i.e., chose σ_ϕ?

- Assume that stars are near circular orbits

$$\begin{cases} \ddot{x} = -\kappa^2 x \\ \ddot{y} = -\kappa^2 y \end{cases}$$

oscillations around the guiding center

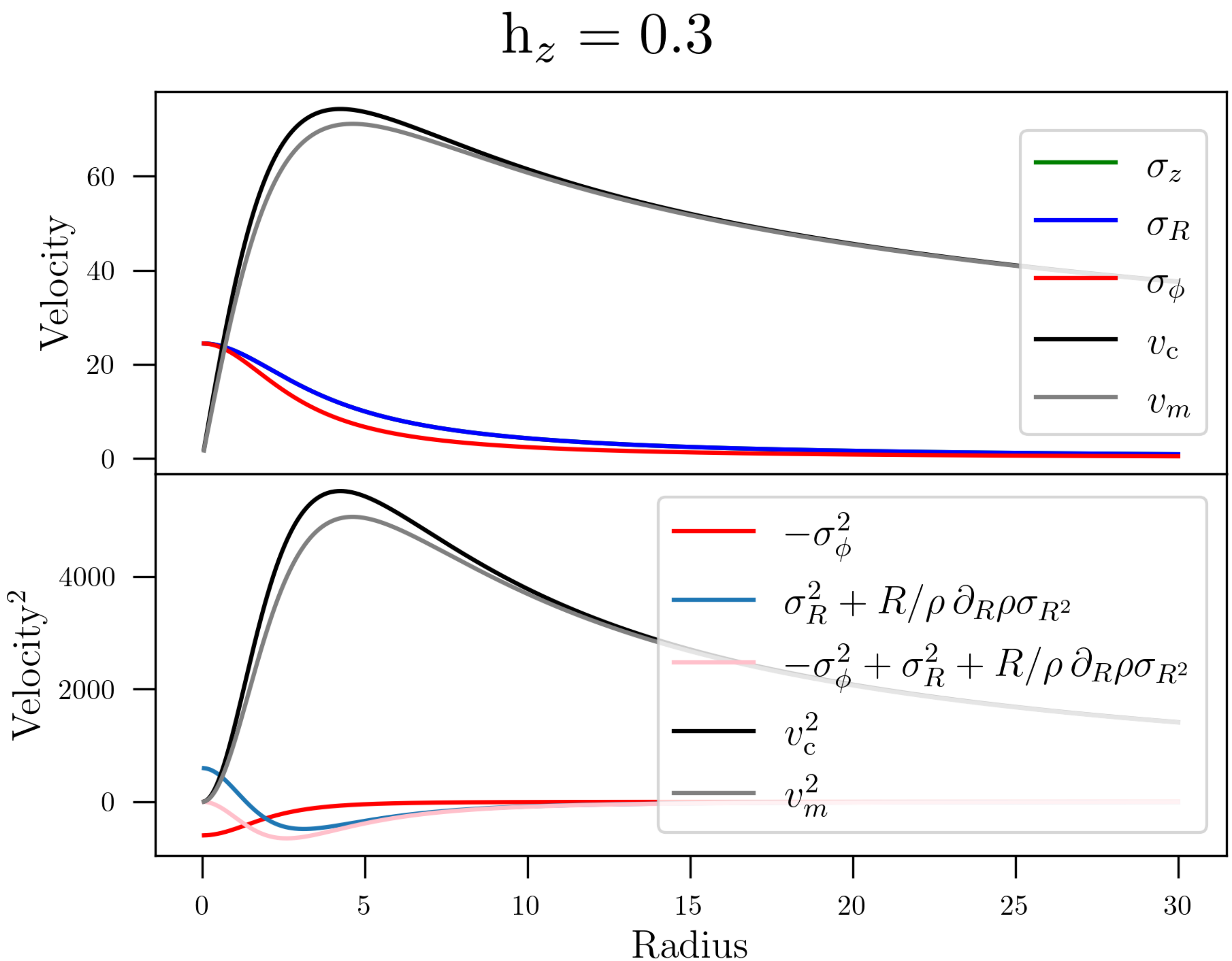
$$\begin{cases} \dot{x}(t) = -X\kappa \sin(\kappa t + \alpha) \\ \dot{y}(t) = -Y\kappa \cos(\kappa t + \alpha) \end{cases} \qquad Y = \frac{2\Omega_s}{\kappa} X$$

$$\sigma_r^2 \equiv \sigma_x^2 = \frac{1}{2\pi} \int_0^{\frac{2\pi}{\kappa}} X^2 \kappa^2 \sin^2(\kappa t + \alpha)\, dt = \frac{X^2 \kappa^2}{2}$$

$$\sigma_\phi^2 \equiv \sigma_Y^2 = \frac{1}{2\pi} \int_0^{\frac{2\pi}{\kappa}} Y^2 \kappa^2 \cos^2(\kappa t + \alpha)\, dt = \frac{Y^2 \kappa^2}{2}$$

thus $\boxed{\sigma_\phi^2 = \frac{\kappa^2}{4\Omega_s^2} \sigma_r^2}$

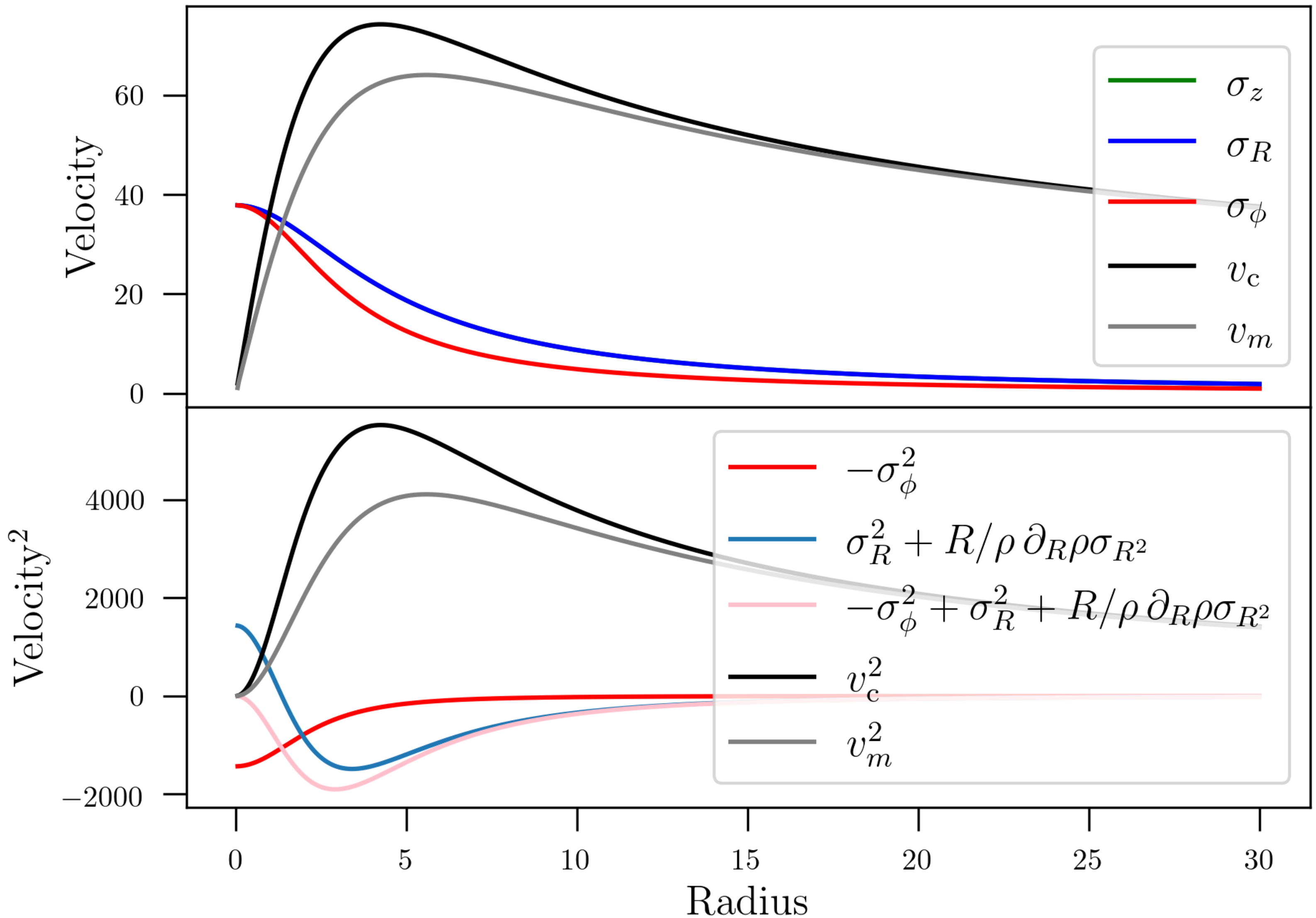
Jeans Moments and rotation curve for a Miyamoto-Nagai disk



$$\sigma_z^2 = \frac{1}{\nu}\int_z^\infty \mathrm{d}z'\nu\frac{\partial\Phi}{\partial z'} \qquad \sigma_R^2 = \sigma_z^2 \qquad \frac{\sigma_\phi^2}{\sigma_R^2} = \frac{\kappa^2}{4\Omega^2} \qquad \overline{v_\phi}^2 = v_c^2 - \sigma_\phi^2 + \sigma_R^2 + \frac{R}{\nu}\frac{\partial}{\partial R}\left(\nu\sigma_R^2\right)$$

Jeans Moments and rotation curve for a Miyamoto-Nagai disk

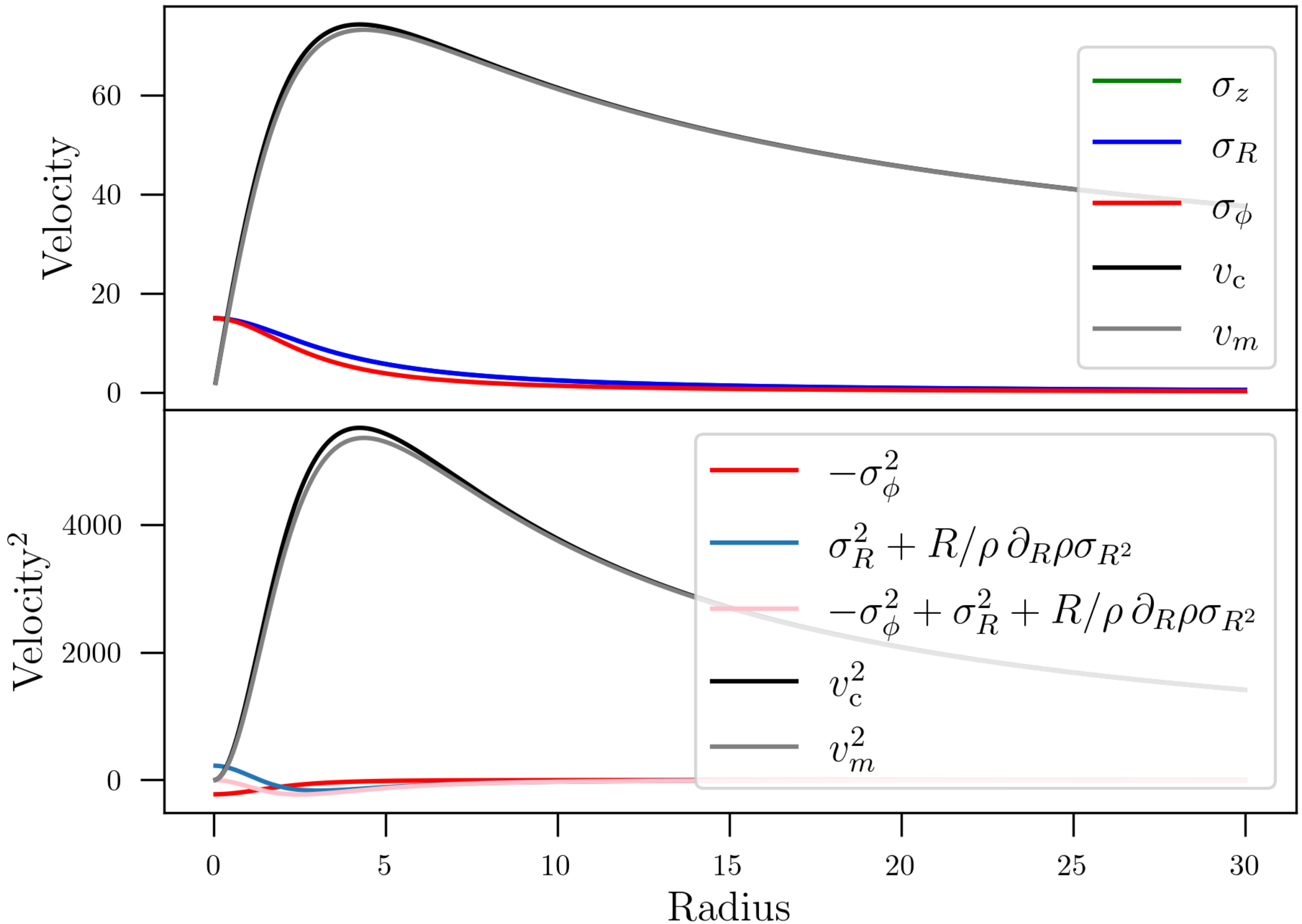
$$\mathrm{h}_z = 1.0$$



$$\sigma_z^2 = \frac{1}{\nu}\int_z^\infty \mathrm{d}z'\,\nu\frac{\partial\Phi}{\partial z'} \qquad \sigma_R^2 = \sigma_z^2 \qquad \frac{\sigma_\phi^2}{\sigma_R^2} = \frac{\kappa^2}{4\Omega^2} \qquad \overline{v_\phi}^2 = v_c^2 - \sigma_\phi^2 + \sigma_R^2 + \frac{R}{\nu}\frac{\partial}{\partial R}\left(\nu\sigma_R^2\right)$$

Jeans Moments and rotation curve for a Miyamoto-Nagai disk

$$\mathrm{h}_z = 0.1$$



$$\sigma_z^2 = \frac{1}{\nu}\int_z^\infty \mathrm{d}z'\nu\frac{\partial\Phi}{\partial z'} \qquad \sigma_R^2 = \sigma_z^2 \qquad \frac{\sigma_\phi^2}{\sigma_R^2} = \frac{\kappa^2}{4\Omega^2} \qquad \overline{v_\phi}^2 = v_c^2 - \sigma_\phi^2 + \sigma_R^2 + \frac{R}{\nu}\frac{\partial}{\partial R}\left(\nu\sigma_R^2\right)$$

The End