Equilibria of collisionless systems

2rd part

Outlines

Relations between DFs and observables

The Jeans theorems

- Steady-state solutions of the Collisionless Boltzmann equation
- Symmetry and integrals of motion

Connections between DFs and orbits

Connections between barotropic fluids and ergodic stellar systems

Self-consitent spherical models with Ergodic DF

- DFs from mass distribution
  - The Eddington formula
  - Examples

The Collisionless Boltzmann equation in various coordinates

Generalized coordinates

$$\vec{p} = \frac{\partial L(\vec{q}, \vec{p})}{\partial \dot{\vec{q}}}$$

$$\boxed{\frac{\partial f}{\partial t} + \dot{\vec{q}} \cdot \frac{\partial f}{\partial \vec{q}} + \dot{\vec{p}} \cdot \frac{\partial f}{\partial \vec{p}} \equiv \frac{\partial f}{\partial t} + \frac{\partial f}{\partial \vec{q}} \cdot \frac{\partial H}{\partial \vec{p}} - \frac{\partial f}{\partial \vec{p}} \cdot \frac{\partial H}{\partial \vec{q}} = 0}$$

Cartesian coordinates

$$\begin{cases} p_x = \dot{x} = v_x \\ p_y = \dot{y} = v_y \\ p_z = \dot{z} = v_z \end{cases} \qquad H = \frac{1}{2}\left(v_x^2 + v_y^2 + v_z^2\right) + \Phi(x, y, z)$$

$$\boxed{\frac{\partial f}{\partial t} + \vec{v} \cdot \frac{\partial f}{\partial \vec{x}} - \frac{\partial \Phi}{\partial \vec{x}} \cdot \frac{\partial f}{\partial \vec{v}} = 0}$$

Spherical coordinates

$$\begin{cases} p_r = \dot{r} = v_r \\ p_\theta = r^2 \dot{\theta} = r v_\theta \\ p_\phi = r^2 \sin^2(\theta \dot{\phi}) = r \sin(\theta)\, v_\phi \end{cases} \qquad H = \frac{1}{2}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2 \sin^2(\theta)}\right) + \Phi(R, \theta, \phi)$$

$$\boxed{\frac{\partial f}{\partial t} + p_r \frac{\partial f}{\partial r} + \frac{p_\theta}{r^2} \frac{\partial f}{\partial \theta} + \frac{p_\phi}{r^2 \sin^2(\theta)} \frac{\partial f}{\partial \phi} - \left(\frac{\partial \Phi}{\partial r} - \frac{p_\theta^2}{r^3} - \frac{p_\phi^2}{r^3 \sin^2(\theta)}\right) \frac{\partial f}{\partial p_r} - \left(\frac{\partial \Phi}{\partial \theta} - \frac{p_\phi^2 \cos(\theta)}{r^2 \sin^3(\theta)}\right) \frac{\partial f}{\partial p_\theta} - \frac{\partial \Phi}{\partial \phi} \frac{\partial f}{\partial p_\phi} = 0}$$

Cylindrical coordinates

$$\begin{cases} p_R = \dot{R} = v_R \\ p_\phi = R^2 \dot{\phi} = R V_\phi \\ p_z = \dot{z} = v_z \end{cases} \qquad H = \frac{1}{2}\left(p_R^2 + \frac{p_\phi^2}{R^2} + p_z^2\right) + \Phi(R, \phi, z)$$

$$\boxed{\frac{\partial f}{\partial t} + p_R \frac{\partial f}{\partial R} + \frac{p_\phi}{R^2} \frac{\partial f}{\partial \phi} + p_z \frac{\partial f}{\partial z} - \left(\frac{\partial \Phi}{\partial R} - \frac{p_\phi^2}{R^3}\right) \frac{\partial f}{\partial p_R} - \frac{\partial \Phi}{\partial \phi} \frac{\partial f}{\partial p_\phi} - \frac{\partial \Phi}{\partial z} \frac{\partial f}{\partial p_z} = 0}$$

Relations between the DFs and observables

Relations between the DF and observables

$\boxed{f(\vec{w})}$

- $f(\vec{w})$: probability density in the phase space
- $f(\vec{w})\, d^6\vec{w}$: probability of finding 1 star in the phase space volume $[\vec{w}, \vec{w} + d\vec{w}]$

Distribution function in the configuration space

$$\boxed{\nu(\vec{x}) = \int d^3\vec{v}\; f(\vec{x},\vec{v})}$$

- $\nu(\vec{x})$: probability density in the configuration space

- $\nu(\vec{x})\, d^3\vec{x}$: probability of finding 1 star in the configuration space volume $[\vec{x}, \vec{x} + d\vec{x}]$

Distribution function in the configuration space

$$\boxed{n(\vec{x}) = N\nu(\vec{x}) = \int d^3\vec{v}\ \tilde{f}(\vec{x},\vec{v})}$$

- $n(\vec{x})$: number density of star in the configuration space
- $n(\vec{x})\, d^3\vec{x}$: probability of finding N stars in the configuration space volume $[\vec{x}, \vec{x}+d\vec{x}]$

Distribution function in the configuration space

$$\boxed{\rho(\vec{x}) = N \cdot m \cdot \nu(\vec{x}) = m \int d^3\vec{v}\ \tilde{f}(\vec{x}, \vec{v})}$$

m : mass of particles

- $\rho(\vec{x})$: mass density of star in the configuration space
- $\rho(\vec{x})\, d^3\vec{x}$: probability of finding a mass $M = N \cdot m$ in the configuration space volume $[\vec{x}, \vec{x} + d\vec{x}]$

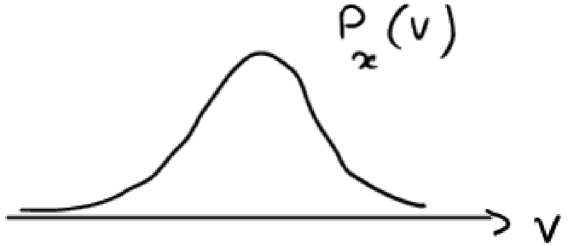
Distribution function in the velocity space

$$\boxed{P_{\vec{x}}(\vec{v}) = \frac{f(\vec{x},\vec{v})}{\nu(\vec{x})}}$$

$$\int P_{\vec{x}}(\vec{v})\, d^3\vec{v} = \frac{1}{\nu(\vec{x})} \underbrace{\int f(\vec{x},\vec{v})\, d^3\vec{v}}_{:=\nu(\vec{x})} = 1$$

$\equiv$ velocity distribution function (VDF)

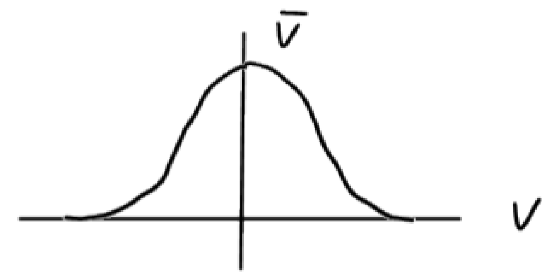
- $P_{\vec{x}}(\vec{v})$: probability density at the position $\vec{x}$ in the velocity space
- $P_{\vec{x}}(\vec{v})\, d^3\vec{v}$: probability of finding 1 star in $\vec{x}$ in the velocity space volume $[\vec{v}, \vec{v}+d\vec{v}]$



$\boxed{\text{Can be measured near the sun}}$

Mean velocity (first moment of the VDF)

$$\bar{\vec{V}}(\vec{x}) = \int \vec{V}\, P_x(\vec{V})\, d^3\vec{V} = \frac{1}{\nu(\vec{x})} \int \vec{V}\, f(\vec{x},\vec{V})\, d^3\vec{V}$$



- along one peculiar axis $\vec{n}$

$$\bar{V}_{\vec{n}}(\vec{x}) = \int \vec{V}\cdot\vec{n}\, P_x(\vec{V})\, d^3\vec{V} = \frac{1}{\nu(\vec{x})} \int \vec{V}\cdot\vec{n}\, f(\vec{x},\vec{V})\, d^3\vec{V}$$

- if $\vec{n} = \vec{e_i}$

$$\bar{V}_i(\vec{x}) = \int V_i\, P_x(\vec{V})\, d^3\vec{V} = \frac{1}{\nu(\vec{x})} \int V_i\, f(\vec{x},\vec{V})\, d^3\vec{V}$$

Velocity dispersion tensor (second moment of the VDF)

$$\sigma_{ij}^2 = \int (v_i - \bar{v}_i)(v_j - \bar{v}_j) \; P_{\vec{x}}(\vec{v}) \; d^3\vec{v}$$

$$= \frac{1}{\nu(\vec{x})} \int (v_i - \bar{v}_i)(v_j - \bar{v}_j) \; f(\vec{x}, \vec{v}) \, d^3\vec{v}$$

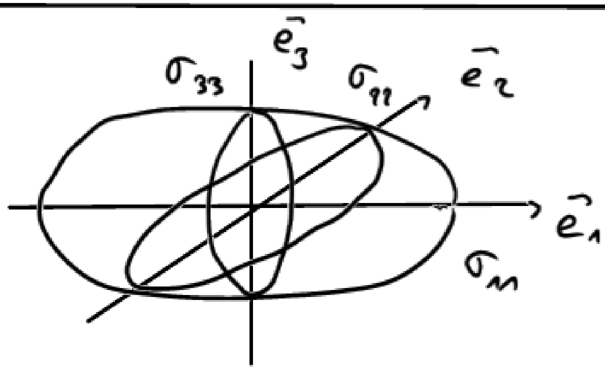
$$= \frac{1}{\nu(\vec{x})} \left[\int v_i v_j \; f(\vec{x}, \vec{v}) \, d^3\vec{v} - \left(\int v_i f(\vec{x}, \vec{v}) \, d^3 v \right) \left(\int v_j f(\vec{x}, \vec{v}) \, d^3 v \right) \right]$$

$$= \overline{v_i v_j} - \bar{v}_i \bar{v}_j$$

3 x 3 symmetric tensor

$\Rightarrow$ may be diagonalised

Describe an ellipsoid (velocity ellipsoid)



$$\sigma_{ij}^2 = \sigma_{ii}^2 \delta_{ij} = \begin{pmatrix} \sigma_{11} & 0 & 0 \\ 0 & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{pmatrix}$$

Equilibria of collisionless systems

The Jeans Theorems

Question :

$\boxed{\text{How can we obtain a steady-state solution of the collision-less Boltzmann equation ? } \boxed{\frac{\partial f}{\partial t} = 0}}$

$$\frac{d}{dt} f(\vec{q}, \vec{p}, \cancel{t}) = \cancel{\frac{\partial f}{\partial t}} + \boxed{\frac{\partial H}{\partial \vec{p}} \frac{\partial f}{\partial \vec{q}} - \frac{\partial H}{\partial \vec{q}} \frac{\partial f}{\partial \vec{p}} = 0}$$

In carthesian coordinates

$$\frac{\partial H}{\partial \vec{p}} = \vec{v}$$

$$\frac{\partial H}{\partial \vec{x}} = \vec{\nabla} \phi$$

$$\boxed{\vec{v} \frac{\partial f}{\partial \vec{x}} - \vec{\nabla} \phi \frac{\partial f}{\partial \vec{v}} = 0}$$

Back to the integrals of motion

The function $I(\vec{x}(t), \vec{v}(t))$ is an integral of motion if

$$\boxed{\frac{d}{dt} I(\vec{x}(t), \vec{v}(t)) = 0}$$

along the trajectory.

But

$$\frac{dI}{dt} = \frac{\partial I}{\partial \vec{x}} \dot{\vec{x}} + \frac{\partial I}{\partial \vec{v}} \dot{\vec{v}} = 0$$

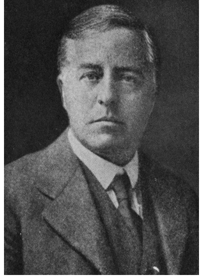
$$= \boxed{\frac{\partial I}{\partial \vec{x}} \vec{v} - \frac{\partial I}{\partial \vec{v}} \vec{\nabla}\phi = 0}$$

Similar to the Collisionless Boltzmann equation

If $I(\vec{x}, \vec{v})$ is an integral of motion

$I(\vec{x}, \vec{v})$ is a steady state solution of the Collisionless Boltzbann equation

James Jeans
1877-1947



Jeans theorems

I. Any steady-state solution of the collisionless Boltzmann equation depends on the phase-space coordinates only through integrals of motion.

II. Any function of integrals of motion yields a steady-state solution of the collisonless Boltzmann equation.

James Jeans
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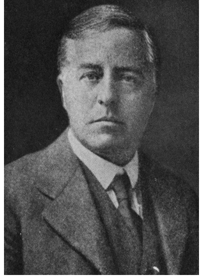

Jeans theorems

I. Any steady-state solution of the collisionless Boltzmann equation depends on the phase-space coordinates only through integrals of motion.

Demonstration:

If a function is a solution of the steady-state collisionless Boltzmann equation, then, it is an integral of motion, thus the function depends on the phase-space coordinates only through integrals of motion (itself !).

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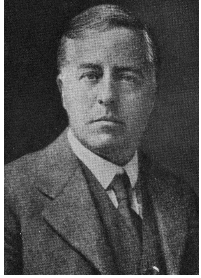

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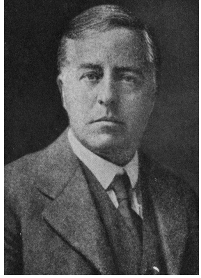
Demonstration:

Assume $$f(\vec{x},\vec{v}) = f(I_1(\vec{x},\vec{v}), I_2(\vec{x},\vec{v}), I_3(\vec{x},\vec{v}), ...)$$

$$\frac{\mathrm{d}}{\mathrm{d}t} f(\vec{x},\vec{v}) = \frac{\partial f}{\partial I_1}\underbrace{\frac{\mathrm{d}I_1}{\mathrm{d}t}}_{=0} + \frac{\partial f}{\partial I_1}\underbrace{\frac{\mathrm{d}I_2}{\mathrm{d}t}}_{=0} + \frac{\partial f}{\partial I_1}\underbrace{\frac{\mathrm{d}I_3}{\mathrm{d}t}}_{=0} + ... = 0$$



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Extremely useful to generate DFs

Demonstration:

Assume $$f(\vec{x},\vec{v}) = f(I_1(\vec{x},\vec{v}), I_2(\vec{x},\vec{v}), I_3(\vec{x},\vec{v}), ...)$$

$$\frac{\mathrm{d}}{\mathrm{d}t} f(\vec{x},\vec{v}) = \frac{\partial f}{\partial I_1} \underset{=0}{\frac{\mathrm{d}I_1}{\mathrm{d}t}} + \frac{\partial f}{\partial I_1} \underset{=0}{\frac{\mathrm{d}I_2}{\mathrm{d}t}} + \frac{\partial f}{\partial I_1} \underset{=0}{\frac{\mathrm{d}I_3}{\mathrm{d}t}} + ... = 0$$

Equilibria of collisionless systems

Symmetries and DFs

Choices of DFs and relations with the velocity moments

1. DFs that depend only on $\mathcal{H}$

(no particular symmetry except time
$\phi = \phi(\vec{x}, \cancel{t})$)

Ergodic distribution functions

Example

$$\begin{cases} H(\vec{x},\vec{v}) = \frac{1}{2} v^2 + \phi \\ f(\vec{x},\vec{v}) = f\left(\frac{1}{2} v^2 + \phi\right) \end{cases}$$

Note: the velocity dependency is only through v^2 (isothropic)

Mean velocity

$$\bar{\vec{v}}(\vec{x}) = \frac{1}{\nu(\vec{x})} \int \vec{v} \, f\left(\frac{1}{2}\vec{v}^2 + \phi(\vec{x})\right) d^3\vec{v} = 0$$

indeed

$$\bar{v}_x(\vec{x}) = \frac{1}{\nu(\vec{x})} \int_{-\infty}^{\infty} dv_z \int_{-\infty}^{\infty} dv_y \int_{-\infty}^{\infty} dv_x \, \underbrace{v_x}_{\text{odd}} \, \underbrace{f\left(\frac{1}{2}\vec{v}^2 + \phi(\vec{x})\right)}_{\text{even}} = 0$$

$$\boxed{1.\ \text{DFs that depend only on } \mathcal{H}}$$

Velocity dispersions

$$\sigma_{ij}^2 = \frac{1}{\nu(\vec{x})} \int \underbrace{(v_i - \cancel{\bar{v}_i}_{=0})(v_j - \cancel{\bar{v}_j}_{=0})}\, f\left(\tfrac{1}{2}v^2 + \phi(\vec{x})\right) d^3\vec{v}$$

$$= \delta_{ij}\sigma^2 \qquad \text{odd, except if } i=j \qquad (\sigma_{xx} = \sigma_{yy} = \sigma_{zz})$$

$$\sigma^2 = \frac{1}{\nu(\vec{x})} \int_{-\infty}^{\infty} v_z^2\, dv_z \int_{-\infty}^{\infty} dv_y \int_{-\infty}^{\infty} dv_x\, f\left(\tfrac{1}{2}v^2 + \phi(\vec{x})\right)$$

using spherical coord in velocity space :
$$\left\{ \begin{aligned} dv_x\, dv_y\, dv_z &= v^2 \sin\theta\, dv\, d\varphi\, d\theta \\ v_z^2 &= v^2 \cos^2\theta \end{aligned} \right.$$
$$v^2 = v_x^2 + v_y^2 + v_z^2$$

$$\boxed{\sigma^2 = 4\pi \frac{1}{\nu(\vec{x})} \int_0^{\infty} v^4 f\left(\tfrac{1}{2}v^2 + \phi(\vec{x})\right) dv}$$

$$\sigma_{ij} = \begin{pmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & \sigma \end{pmatrix}$$

isothropic system :

the velocity ellipsoid is a sphere

Note: The term "ergodic" denotes a system that uniformly explores its energy surface in phase space:

$$\boxed{\begin{array}{l}\Rightarrow \text{ the distribution function is uniform on the energy surface} \\[1em] \qquad\qquad \rho = \rho(E)\end{array}}$$

2. DFs that depend on H and $\vec{L}$

(spherical symmetry)

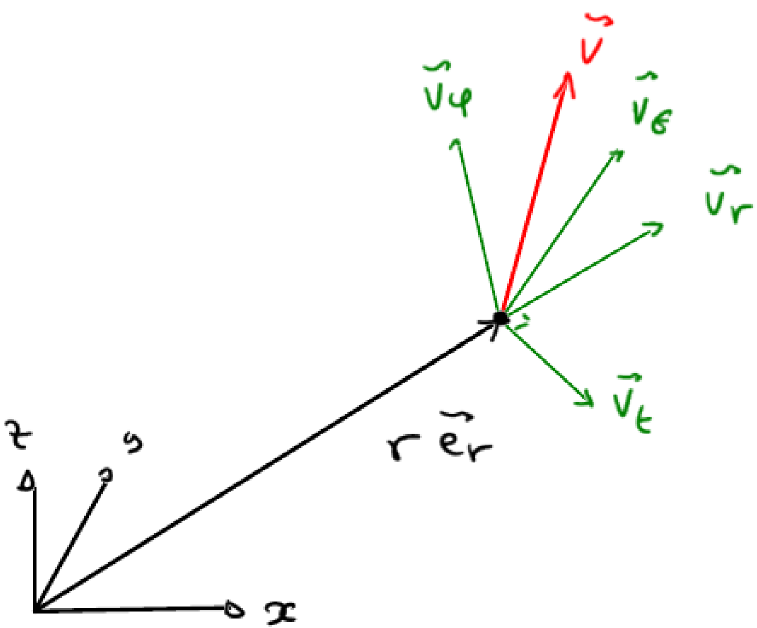
$\phi = \phi(r)$

We restrict our study to symmetric DFs : indep. of any direction

$f(\vec{x}, \vec{v}) = f(H, |\vec{L}|)$ $\quad \vec{L} \to |\vec{L}|$

We consider the system in spherical coordinates

$r \quad \theta \quad \varphi \qquad \vec{v}_r \quad \vec{v}_\theta \quad v_\varphi$

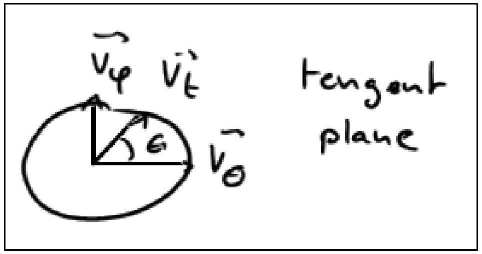


$\vec{v}_r$: radial velocity $\quad \vec{v}_r = (\vec{v} \cdot \vec{e}_r) \cdot \vec{e}_r$

$\vec{v}_t$: tangential velocity $\quad \vec{v}_t = |\vec{v} \times \vec{e}_r| \cdot \vec{e}_t$

$$\vec{v} = \vec{v}_r + \vec{v}_t$$

$$= \vec{v}_r + \vec{v}_\theta + \vec{v}_\varphi$$



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$$f(\vec{x},\vec{v}) = f(H, |\vec{L}|) \qquad \vec{L} \to |\vec{L}|$$

Mean velocities

$$\bar{v}_r = 0 \qquad \bar{v}_t = 0$$

EXERCICE

Velocity dispertions

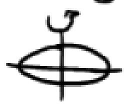
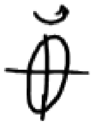
$$\sigma_r^2 \neq 0$$

$$\sigma_\theta^2 = \sigma_\varphi^2 \neq 0$$

Anisothropic system

$$\boxed{\sigma_r^2 \neq \sigma_\theta^2 = \sigma_\varphi^2}$$

The velocity ellipsoid is

oblate  or prolate 

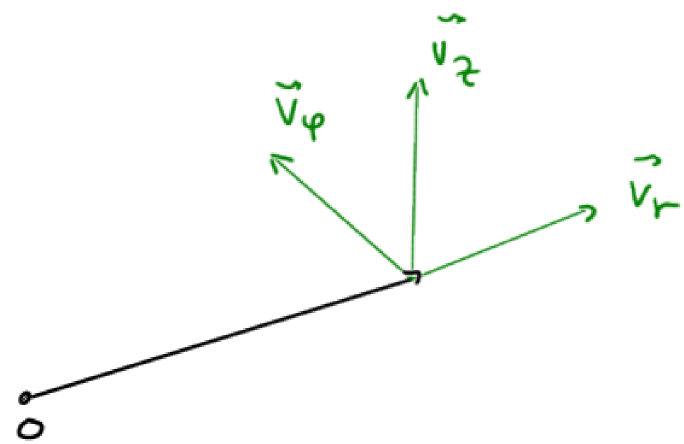
3. DFs that depend on H and L_z

(cylindrical symmetry)

$$\phi = \phi(R, |z|)$$

$$f(\vec{x}, \vec{v}) = f(H, L_z)$$

We consider the system in cylindrical coordinates



$r \quad \varphi \quad z \qquad \vec{v}_r \quad \vec{v}_\varphi \quad \vec{v}_z$

$\boxed{\text{3. DFs that depend on } H \text{ and } L_z}$

(cylindrical symmetry)

$$\phi = \phi(R, |z|)$$

$$f(\bar{x}, \bar{v}) = f(H, L_z)$$

Mean velocities

$$\bar{V}_R = 0 \qquad \bar{V}_z = 0 \qquad \bar{V}_\varphi \neq 0$$

Velocity dispersions

EXERCICE

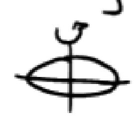
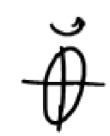
$$\sigma_\varphi^2 \neq 0$$

$$\sigma_R^2 = \sigma_z^2 \neq 0$$

Anisothropic system

$$\boxed{\sigma_\varphi^2 \neq \sigma_R^2 = \sigma_z^2}$$

The velocity ellipsoid is

oblate  or prolate 

Equilibria of collisionless systems

Connections between DFs and orbits

Example 1

1-D potential

$$\begin{cases} E = \frac{1}{2} v^2 + \phi(x) \\ v = \pm \sqrt{2(E - \phi(x))} \end{cases}$$

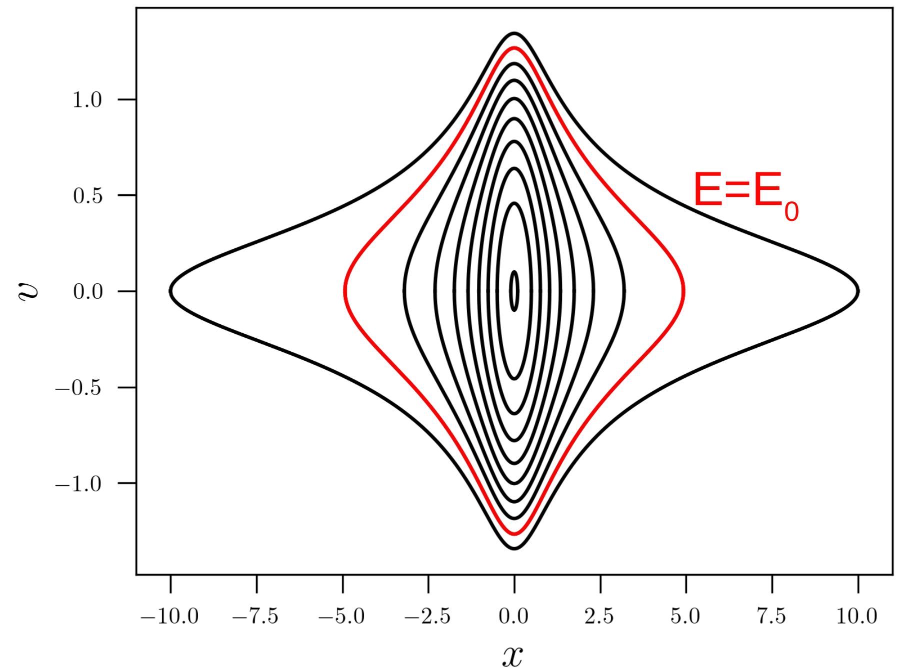
a) $f(x, v) = f(E) = \delta(E - E_0)$

$$\begin{cases} \infty & v = \pm \sqrt{2(E_0 - \phi(x))} \\ 0 & \text{instead} \end{cases}$$

b) $f(x, v) = f(E)$

$\downarrow$

give a weight to
orbits depending on
their energy



Example 2

3D sperical potential

$\rightarrow$ planar orbits described by $E, |\vec{L}|$

a) Ergodic DF : $f(\vec{x},\vec{v}) = f(E(\vec{x},\vec{v}))$

$$\begin{pmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & \sigma \end{pmatrix}$$

- model built-out of all orbits of all planes with a weight that depend on their energy

radial and circular orbits have the same weight

b) Non-ergodic DF : $f(\vec{x},\vec{v}) = f(E(\vec{x},\vec{v}), |\vec{L}|(\vec{x},\vec{v}))$

$$\boxed{\sigma_r^2 \neq \sigma_\theta^2 = \sigma_\varphi^2}$$

- model built-out of all orbits of all planes with a weight that depend only on their energy and angular momentum

radial and circular orbits are weighted differently

c) Non-ergodic DF: $f(\vec{x},\vec{v}) = f(E(\vec{x},\vec{v}), \vec{L}) = f_E(E)\, f_L(\vec{L})$

! not spherical

$$f_L(\vec{L}) \begin{cases} \neq 0 & \text{if } \vec{L} \parallel \vec{e}_z \\ = 0 & \text{instead} \end{cases}$$

$$\boxed{\sigma_\varphi^2 \neq \sigma_R^2 = \sigma_z^2}$$

- model built-out of orbits in the $z = 0$ plane with a weight that depend only on their energy and angular momentum

Questions

Why an ergodic DF with a priori no constraint on the symmetry of the potential leads to an isotropic velocity dispersion tensor ?

$$\Phi(r) \qquad f(H) \qquad \Longrightarrow \qquad \begin{pmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & \sigma \end{pmatrix}$$

Equilibria of collisionless systems

Connections between barotropic fluids and ergodic stellar systems

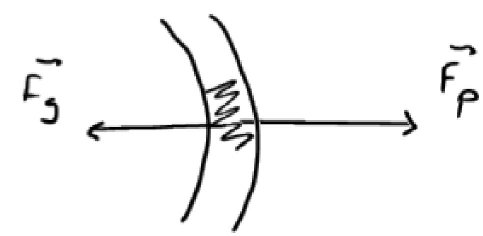
Connections between fluids and stellar systems

In fluid dynamics, the properties of a fluid at rest in a potential is obtained through the Euler equation

$$\boxed{\frac{d\vec{v}}{dt} = \underbrace{-\frac{\vec{\nabla} P}{\rho}}_{\text{pressure force}} \underbrace{- \vec{\nabla}\phi}_{\text{gravity}}}$$

At rest $\quad \frac{d\vec{v}}{dt} = 0$

$$\boxed{\frac{\vec{\nabla} P}{\rho} = - \vec{\nabla}\phi}$$



In 1-D (isothropic case)

$$\boxed{\frac{1}{\rho}\frac{\partial P}{\partial r} = - \frac{\partial \phi}{\partial r}}$$

Eqvation of state (EoS) $\qquad P = P(\rho, T)$

$P = P(\rho)$: barotropic (depends only on the density)

$P = K \rho^{\gamma}$: polytropic

$P = \frac{k_B T}{m} \rho$: isotherm (T = cte)

Together with the hydrostatic eqvation,

$$\boxed{\frac{1}{\rho} \frac{\partial P}{\partial r} = - \frac{\partial \phi}{\partial r}}$$

this relates $\underline{\rho(r) \text{ with } \phi(r)}$.

Self - gravity

The Poisson equation

$$\boxed{\vec{\nabla}^2 \phi = 4\pi G \rho}$$

This constraints the potential $\phi(r)$

or equivalently the density $\rho(r)$

Indeed :

$$\boxed{\frac{1}{\rho}\frac{\partial P}{\partial r} = -\frac{\partial \phi}{\partial r}} \quad + \quad P(\rho) \quad + \quad \boxed{\vec{\nabla}^2 \phi = 4\pi G \rho}$$

$\Rightarrow$ diff. equation for $\phi(r)$ or $\rho(r)$

Note An ergodic DF is such that the velocity dispertion is isothropic

$$\left({}^{\sigma}\sigma_{\sigma} \right) \equiv \text{similar to a gaseous system}$$

Idea : define a function $P(\rho)$ (an equivalent of the pressure) which is such that :

$$\boxed{\frac{\vec{\nabla} P}{\rho} = - \vec{\nabla} \phi} \qquad \frac{1}{\rho} \frac{\partial P}{\partial r} = - \frac{\partial \phi}{\partial r}$$

if spherical

If we find $P(\rho)$ for our stellar system, its density will be the same than the one of a gaseous system as the "pressure" will be equivalent.

Ergodic DF

$$f(\vec{x}, \vec{v}) = f\left(\frac{1}{2}\vec{v}^2 + \phi(\vec{x})\right)$$

Density

$$\rho(\vec{x}) = \int d^3v \; f(\vec{x}, \vec{v})$$

$$= \int d^3v \; f\left(\frac{1}{2}\vec{v}^2 + \phi(\vec{x})\right)$$

as ρ depends on $\vec{x}$ only through ϕ, we can write:

$\rho = \rho(\phi)$ and assuming it to be bijective

$$\boxed{\phi = \phi(\rho)}$$

we can then compute $\dfrac{\partial \phi}{\partial \rho}$

Lets define the function $p(\rho)$

$$\boxed{P(\rho) = -\int_0^{\rho} d\rho' \, \rho' \, \frac{\partial \phi}{\partial \rho}(\rho')}$$

Differentiating gives

$$\frac{\partial P}{\partial \rho}(\rho) = -\rho \frac{\partial \phi}{\partial \rho}(\rho)$$

with $\rho = \rho(\vec{x})$

$$\frac{\partial P}{\partial \rho} = \vec{\nabla} P \cdot \frac{\partial \vec{x}}{\partial \rho}, \qquad \frac{\partial \phi}{\partial \rho} = \vec{\nabla} \phi \cdot \frac{\partial \vec{x}}{\partial \rho}$$

it becomes:

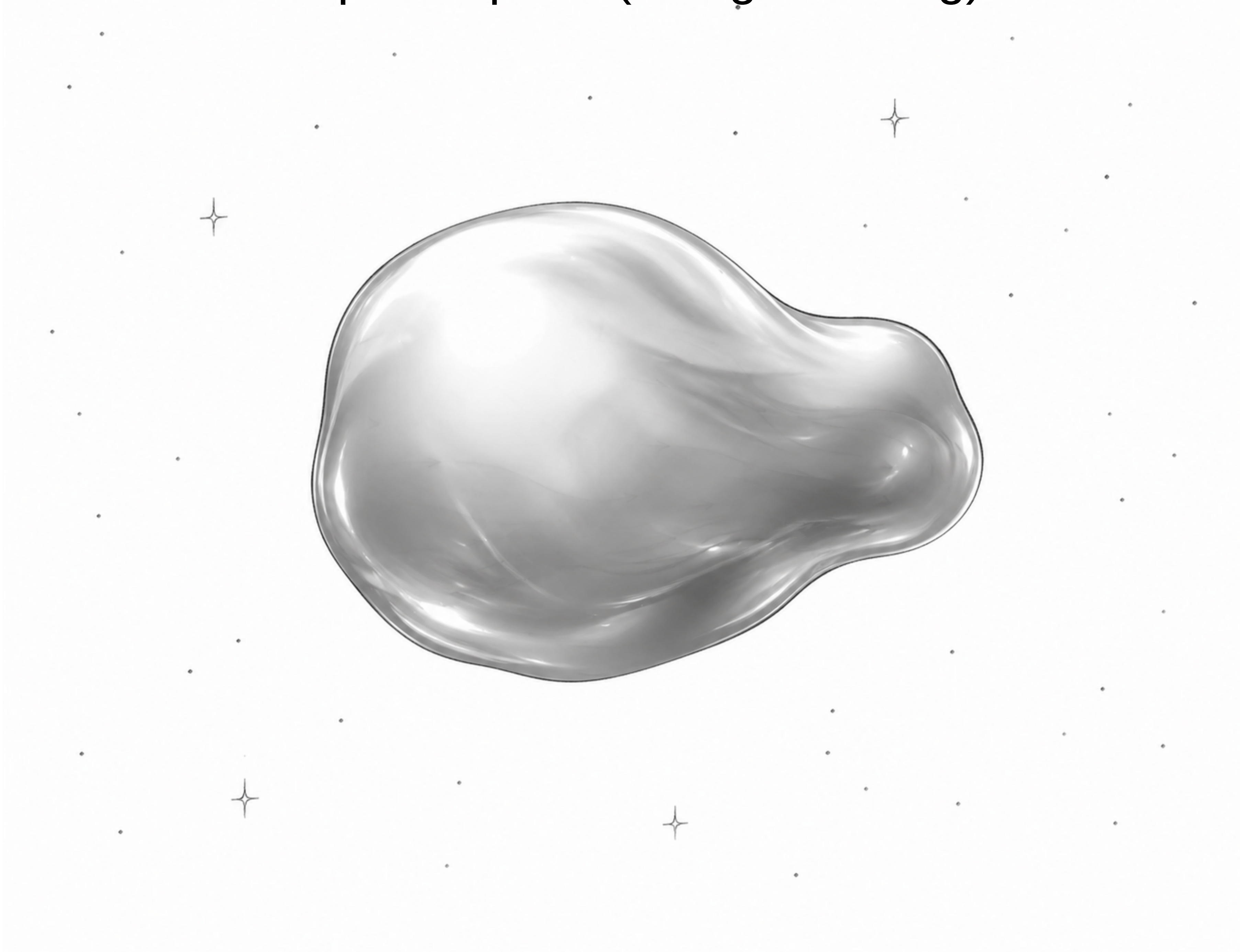
$$\boxed{\frac{\vec{\nabla} P}{\rho} = -\vec{\nabla} \phi}$$

Which is the equation of equilibrium for a barotropic fluid.

Conclusion

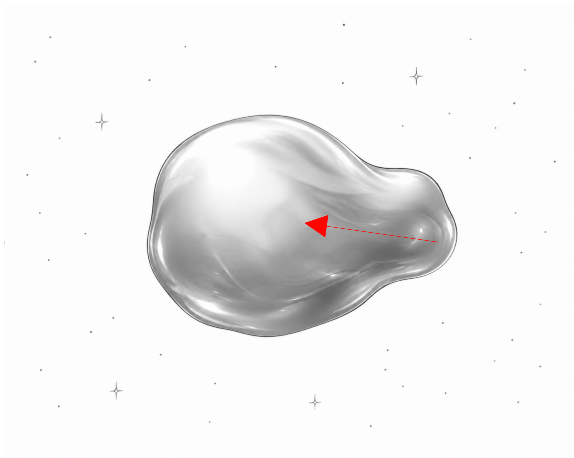
I. An ergodic stellar system is analog to a gasous barothrope.

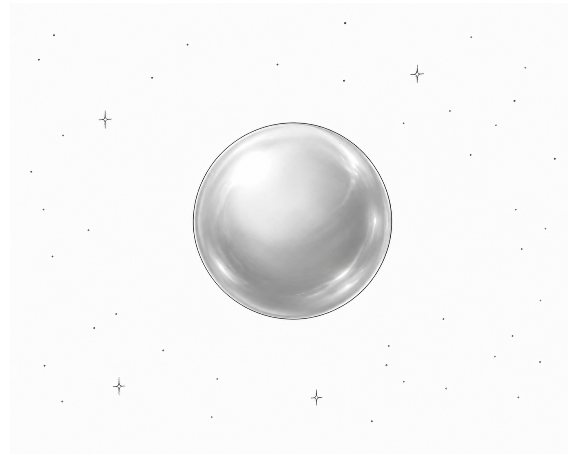
A gasous barothrope in space (self-gravitating)

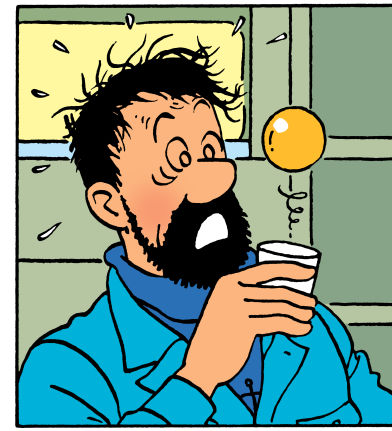


A gasous barothrope in space (self-gravitating)

As an isolated finite, static, self-gravitating barotropic fluid must be spherical. (Lichtenstein's theorem)







assymmetric → net force
→ oscillations → non static

→ static → symmetric

As a stellar system with an ergodic DF satisfies the same equations, it must be spherical

Conclusion

I. An ergodic stellar system is analog to a gasous barothrope.

II. An ergodic isolated stellar system is <u>spherical.</u>

Self-consistent spherical models with ergodic DFs

Distribution function for spherical systems (Ergodic DFs)

isothropic velocity field

Goal provide a <u>self-consistent</u> model for a spherical stellar system

ex: - elliptical galaxy
- galaxy cluster
- globular cluster

<u>self-consistent</u> = the density obtained from the DF is the one that generates the potential

i.e. is a solution of the Poisson equation

$$\boxed{\rho(\vec{x}) = N m \int d^3v \, f(\underbrace{\tfrac{1}{2}v^2 + \phi(\vec{x})}_{H(\vec{x},\vec{v})}) \equiv \frac{1}{4\pi G} \nabla^2\phi(\vec{x})}$$

<u>assumptions</u> : only one type of stars (one stellar population)

i.e. all stars are modeled through the same DF.

Distribution function for spherical systems

- Method 1

  - from $\rho(r)$ $\phi(r)$ $\longrightarrow$ get $f(\varepsilon) \equiv f\left(\frac{1}{2}v^2 + \phi(r)\right)$

- Method 2

  - assume $f(\varepsilon)$ $\longrightarrow$ get $\rho(r)$

Spherical systems definded by DFs

Equilibria of collisionless systems

DFs from mass distribution

Determination of the DF from the mass distribution

- We assume that $\rho(r)$ and $\phi(r)$ are known funtions related together by the Poisson equation : $\boxed{\nabla^2\phi = 4\pi G\rho}$

- The density is related to the DF by :

isothropic
→ spherical coords
in velocity space
$d^3V = V^2 dV\, dV_\varphi\, dV_\theta$

$$\rho(r) = \underbrace{N\cdot m}_{M}\, \nu(r) = m\int_0^\infty \tilde{f}(E)\, d^3V$$

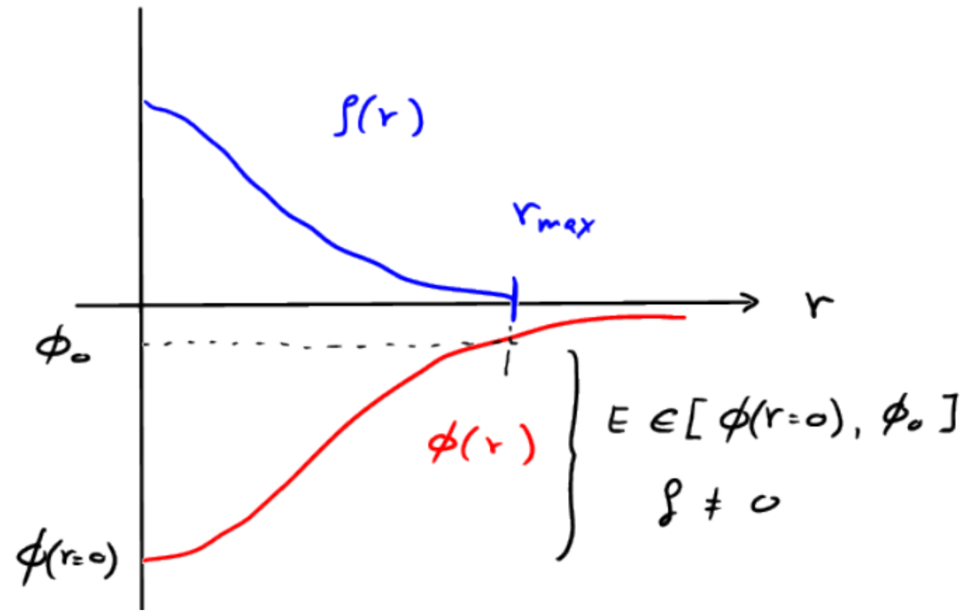
$$m\int_0^\infty dv\, 4\pi\, v^2 \tilde{f}\left(\tfrac{1}{2}v^2 + \phi(r)\right)$$

We are thus looking for DFs f that satisfy : $f = N\tilde{f}$

$$\boxed{\nu(r) = 4\pi\int_0^\infty dV\, v^2 f\left(\tfrac{1}{2}v^2 + \phi(r)\right)}$$

Density and potential

- $\rho(r)$ $\qquad \rho(r > r_{max}) = 0$
- $\phi(r)$ $\qquad$ no limit

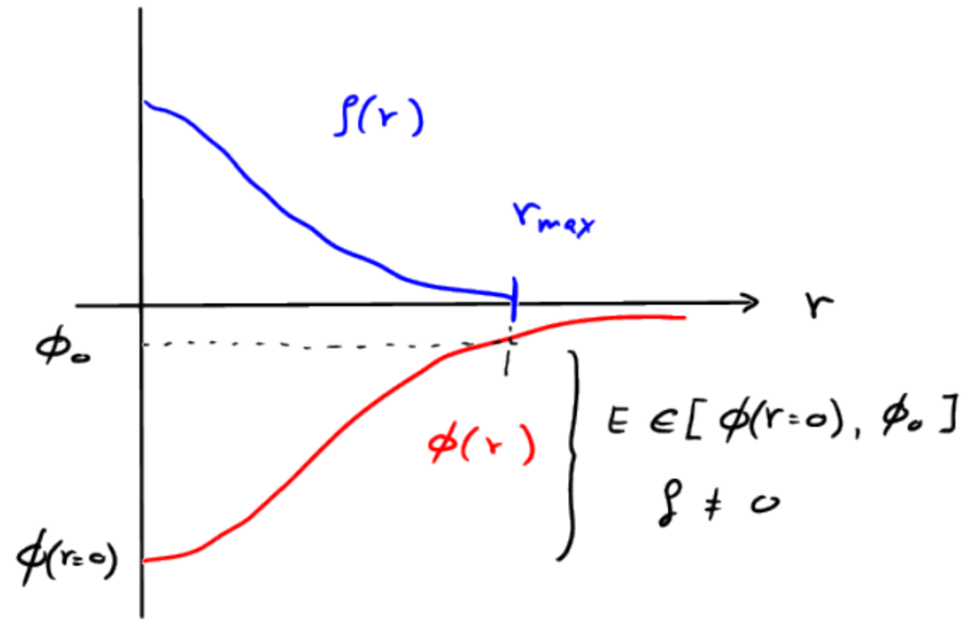


Goal: find $\rho = \rho(E)$ with

$\rho = 0$ if $r > r_{max}$

Density and potential

- $\rho(r)$ $\qquad$ $\rho(r > r_{max}) = 0$
- $\phi(r)$ $\qquad$ no limit



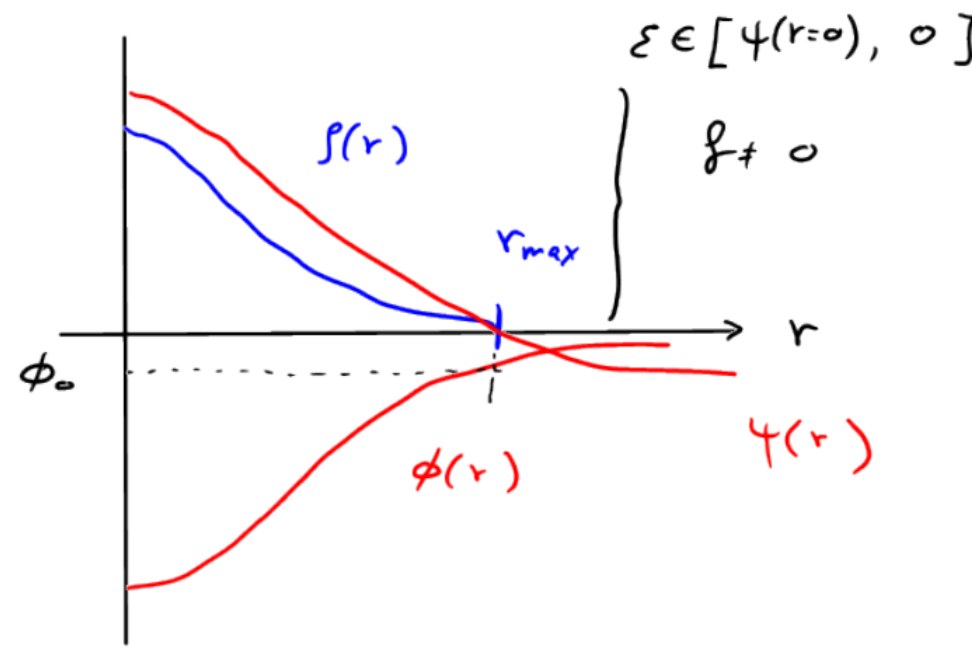
$$\boxed{\begin{array}{l}\text{Goal: find } \rho = \rho(\varepsilon) \text{ with} \\ \qquad \rho = 0 \quad \text{if} \quad r > r_{max}\end{array}}$$

<u>Idea</u> new variables

relative potential

$$\left\{\begin{array}{l} \psi = -(\phi - \phi_0) = -\phi + \phi_0 \\ \varepsilon = -(H - \phi_0) = -H + \phi_0 \end{array}\right.$$

relative energy $\qquad = \psi - \frac{1}{2}v^2$



$$\boxed{\rho \rightarrow \rho(\varepsilon)} \qquad \left\{\begin{array}{ll} \varepsilon > 0 & \rho > 0 \\ \varepsilon \leq 0 & \rho = 0 \end{array}\right.$$

Back to the density

$$\boxed{\nu(r) = 4\pi \int_0^\infty dv\, v^2 \underbrace{f\left(\tfrac{1}{2} v^2 + \phi(r)\right)}_{f(\varepsilon) \,\equiv\, f(\psi - \frac{1}{2} v^2)}}$$

<u>But</u> $\quad f(\varepsilon) = 0 \quad$ if $\quad \varepsilon \leq 0 \quad$ i.e $\quad \psi - \frac{1}{2} v^2 < 0$

i.e $\quad v > \sqrt{2\psi}$

So, we can limit the integral to:

$[\,0 \,,\, \sqrt{2\psi}\,]$

$$\boxed{\nu(r) = 4\pi \int_0^{\sqrt{2\psi}} dv\, v^2 f\left(\psi - \tfrac{1}{2} v^2\right)}$$

Now, lets integrate over $\mathcal{E}$, rather than v

as $\quad \mathcal{E} = \psi - \frac{1}{2} v^2$

$$v = \sqrt{2(\psi - \mathcal{E})} \quad \text{and} \quad dv = \frac{-1}{\sqrt{2(\psi - \mathcal{E})}} \, d\mathcal{E}$$

$$\nu(r) = 4\pi \int_0^{\sqrt{2\psi}} dv \, v^2 f\left(\psi - \tfrac{1}{2} v^2\right)$$

becomes

$$\nu(r) = 4\pi \int_{\psi \, \binom{v=0}{\mathcal{E}=\psi}}^{0 \, \binom{v=\sqrt{2\psi}}{\mathcal{E}=0}} d\mathcal{E} \;\; 2(\psi - \mathcal{E}) \, f(\mathcal{E}) \; \frac{-1}{\sqrt{2(\psi - \mathcal{E})}}$$

$$= 4\pi \int_0^{\psi} d\mathcal{E} \; \sqrt{2(\psi - \mathcal{E})} \; f(\mathcal{E})$$

- if Ψ is a monotonic function of ν (typical potential)

$$\Psi(r) \rightarrow r(\Psi) \quad \Rightarrow \quad \nu(r) = \nu(r(\Psi)) = \nu(\Psi)$$

and thus

$$\frac{1}{\sqrt{8}\,\pi}\,\nu(\Psi) = \int_0^{\Psi} d\mathcal{E}\,\sqrt{\Psi - \mathcal{E}}\; f(\mathcal{E})$$

Derivating with respect to Ψ (not trivial), we get

$$\boxed{\frac{1}{\sqrt{8}\,\pi}\,\frac{\partial \nu(\Psi)}{\partial \Psi} = \int_0^{\Psi} d\mathcal{E}\,\frac{f(\mathcal{E})}{\sqrt{\Psi - \mathcal{E}}}}$$

Abel integral

Solution : Eddington formula

$$f(\varepsilon) = \frac{1}{\sqrt{8}\,\pi^2} \frac{d}{d\varepsilon} \left[\int_0^{\varepsilon} \frac{d\psi}{\sqrt{\varepsilon - \psi}} \frac{d\nu}{d\psi} \right]$$

or

$$f(\varepsilon) = \frac{1}{\sqrt{8}\,\pi^2} \left[\int_0^{\varepsilon} \frac{d\psi}{\sqrt{\varepsilon - \psi}} \frac{d^2\nu}{d\psi^2} + \frac{1}{\sqrt{\varepsilon}} \left(\frac{d\nu}{d\psi} \right)_{\psi=0} \right]$$

Note : $f(\varepsilon) > 0$ only if $\displaystyle\int_0^{\varepsilon} \frac{d\psi}{\sqrt{\varepsilon - \psi}} \frac{d\nu}{d\psi}$

is an increasing function of ε !

How using this formula ? $\boxed{f(\mathcal{E}) = \frac{1}{\sqrt{8}\,\pi^2} \frac{d}{d\mathcal{E}} \left[\int_0^{\mathcal{E}} \frac{d\Psi}{\sqrt{\mathcal{E} - \Psi}} \frac{d\nu}{d\Psi} \right]}$

- We start from a given $\rho(r)$, $\phi(r)$

① get r_{max} and compute $\phi_0 = \phi(r_{max})$

② a) get $\nu(r) = \rho(r)/M$

$\Psi(r) = -\phi(r) + \phi_0$

b) and $\nu = \nu(\Psi)$ if $\Psi(r)$ may be inverted

③ if $\frac{\partial \nu}{\partial \Psi}$ is analytical , compute $f(\mathcal{E})$ (Eddington's formula)

④ $f(x, v) = f(\mathcal{E}) = f(\phi_0 - \mathcal{E}) = f(\frac{1}{2} v^2 + \phi)$

Note ②a and ③ may be performed numerically

Example : Hernquist model

- $$\rho(r) = \frac{\rho_0}{(r/a)(1 + r/a)}$$

$$M(r) = 2\pi \rho_0 a^3 \frac{(r/a)^2}{(1 + r/a)^2}$$

$$M = M(\infty) = 2\pi \rho_0 a^3$$

- $$\phi(r) = -2\pi G \rho_0 \frac{a^2}{(1 + r/a)}$$

The density is non-zero at $r = \infty \Rightarrow \phi_0 = 0$

$$\psi(r) = -\phi(r)$$

- inverting $\phi(r)$, we have

$$r/a = \frac{2\pi G \rho_0 a^2}{\psi} - 1 \underset{M = 2\pi \rho_0 a^3}{=} \frac{GM}{\psi a} - 1 \underset{\tilde{\psi} := \frac{\psi}{GM} a}{=} \frac{1}{\tilde{\psi}} - 1$$

we can now express ν as $\nu(\psi)$, eliminating r/a

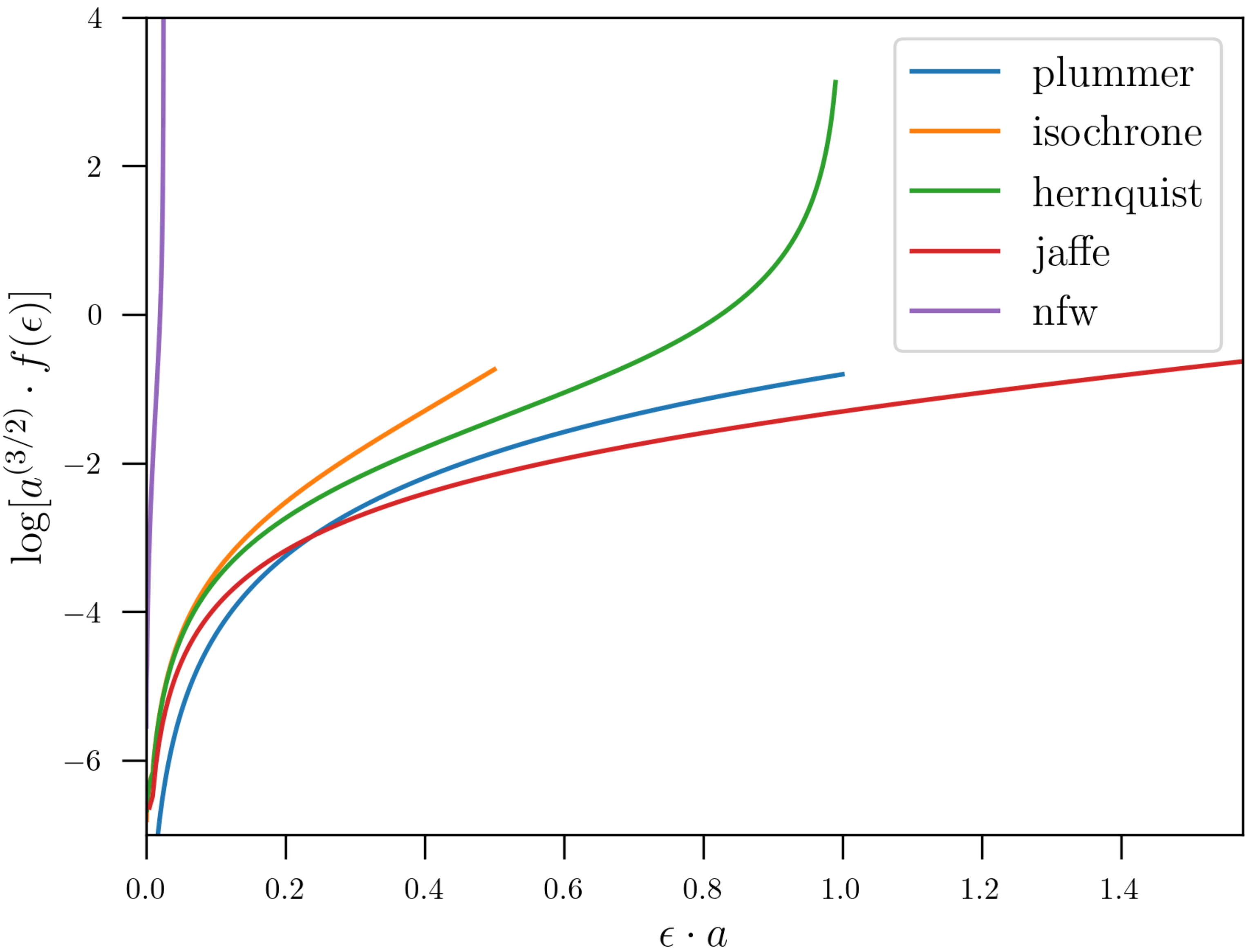
$$\nu(\psi) = \frac{\rho}{M} = \frac{1}{2\pi a^3} \frac{\tilde{\psi}^4}{1-\tilde{\psi}^4}$$

then
$$\frac{\partial \nu(\psi)}{\partial \psi} = \frac{1}{2\pi a^2 GM} \frac{\tilde{\psi}^3(4-3\tilde{\psi})}{(1-\tilde{\psi})^2}$$

And the DF becomes, using $\tilde{\mathcal{E}} = -\frac{\mathcal{E} a}{GM}$

$$\boxed{\begin{aligned} f(\mathcal{E}) &= \frac{\sqrt{2}}{(2\pi)^3 (GM)^2 a} \int_0^{\mathcal{E}} \frac{d\psi}{\sqrt{\mathcal{E}-\psi}} \frac{2\tilde{\psi}^2(6-8\tilde{\psi}+3\tilde{\psi}^2)}{(1-\tilde{\psi})^3} \\ &= \frac{1}{\sqrt{2}(2\pi)^3 (GMa)^{3/2}} \frac{\sqrt{\tilde{\mathcal{E}}}}{(1-\tilde{\mathcal{E}})^2} \left[(1-2\tilde{\mathcal{E}})(8\tilde{\mathcal{E}}^2-8\tilde{\mathcal{E}}-3) + \frac{3 \arcsin(\sqrt{\tilde{\mathcal{E}}})}{\sqrt{\tilde{\mathcal{E}}(1-\tilde{\mathcal{E}})}} \right] \end{aligned}}$$

Note: Proceeding similary, it is possible to compute the DF for others spherical potentials



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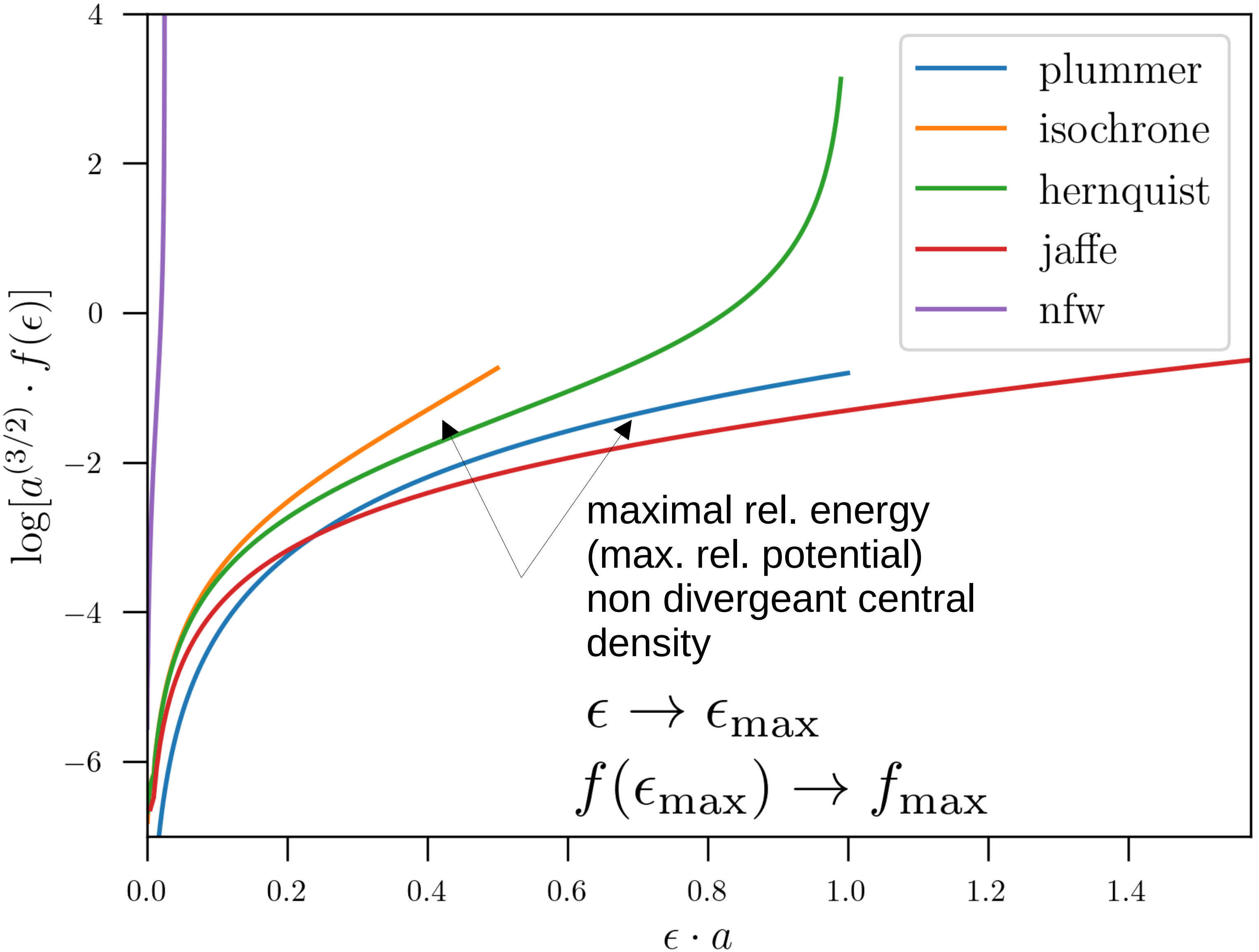

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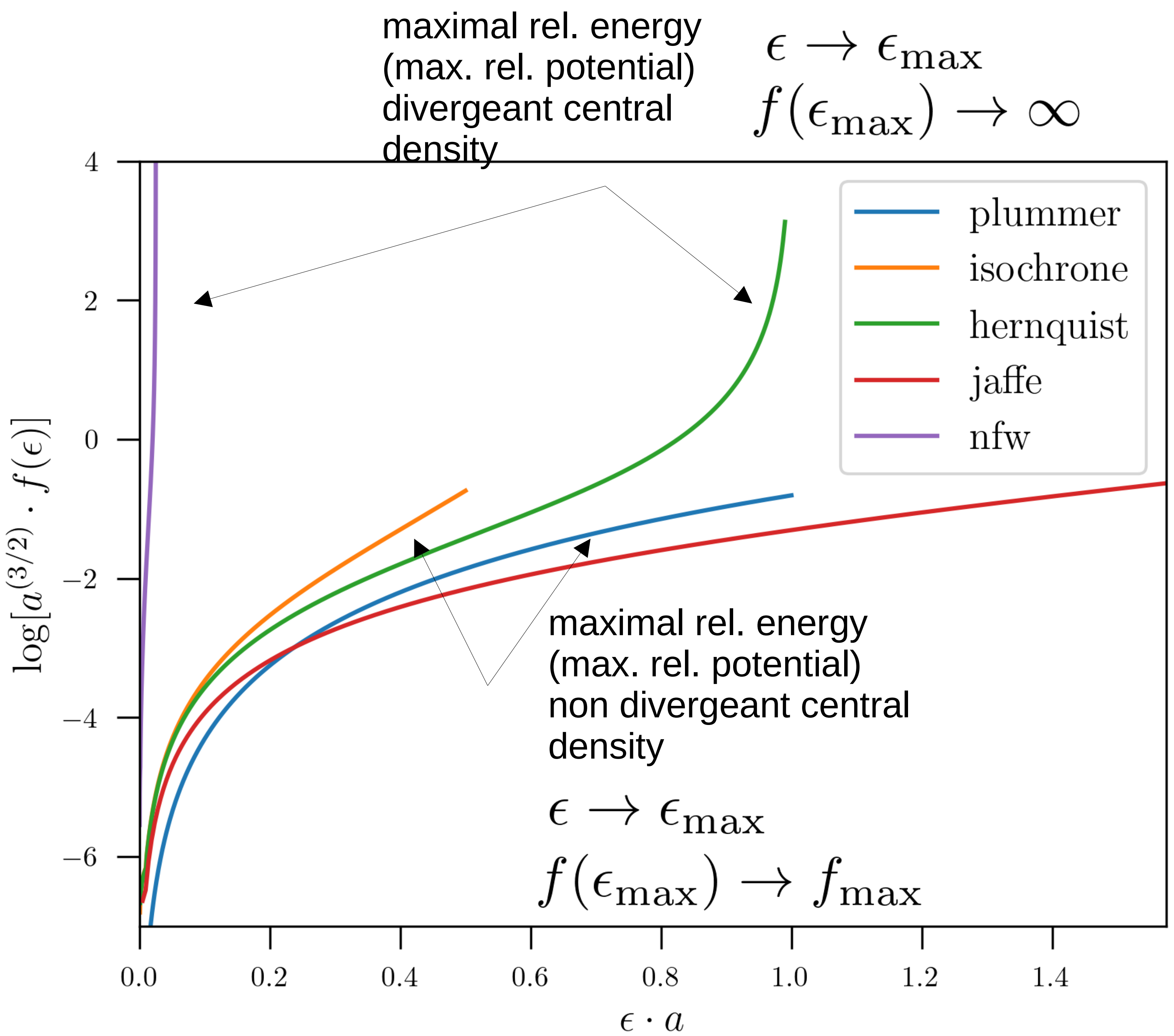

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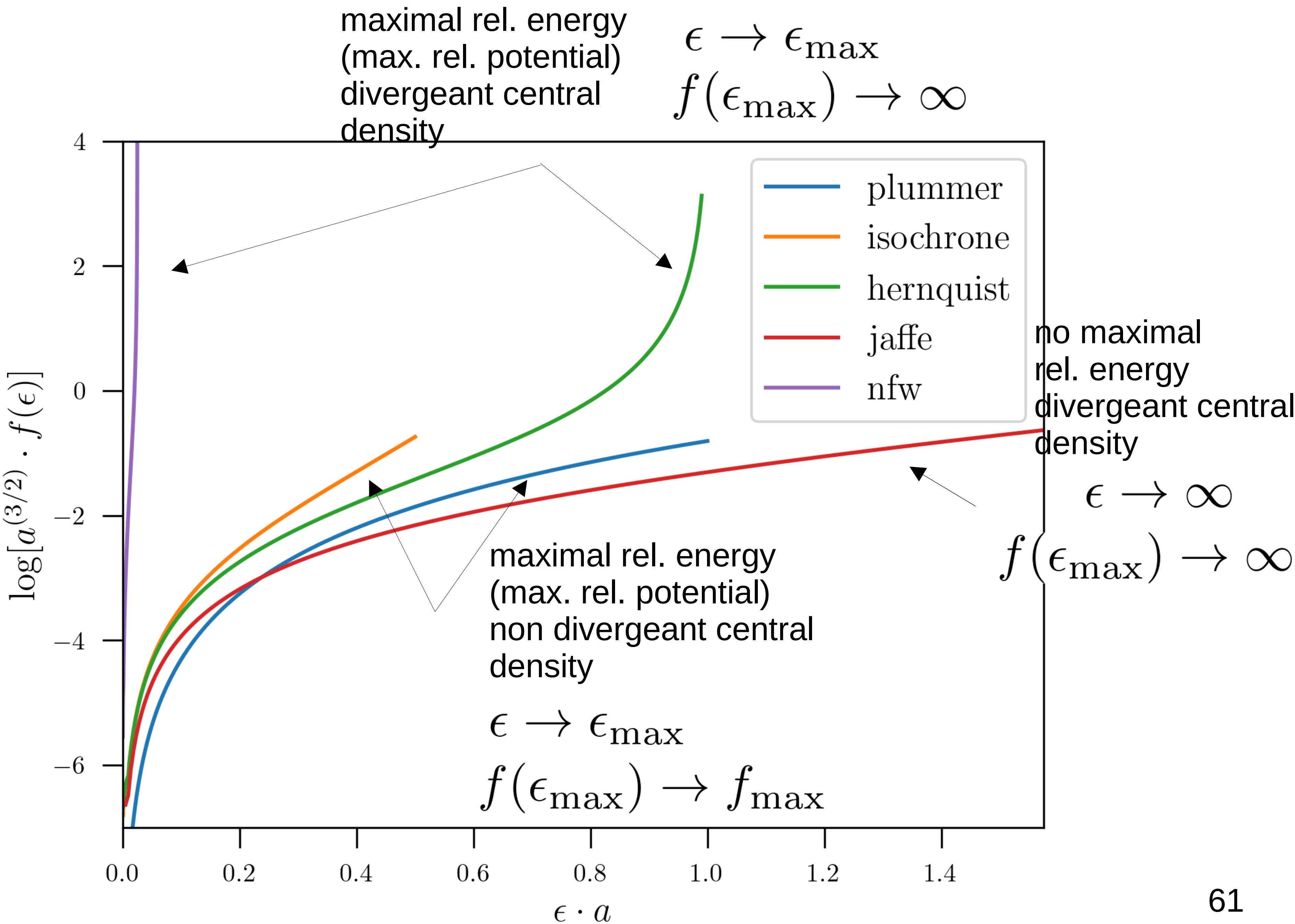


Plummer model

$$\Phi(r) = -\frac{GM}{\sqrt{r^2 + b^2}}$$

$$\rho(r) = \frac{3M}{4\pi b^3} \left(1 + \frac{r^2}{b^2}\right)^{-5/2}$$

Isochrone model

$$\Phi(r) = -\frac{GM}{b + \sqrt{r^2 + b^2}}$$

$$\rho(r) = M \frac{3(b + \sqrt{b^2 + r^2})(b^2 + r^2) - r^2(b + 3\sqrt{b^2 + r^2})}{4\pi(b + \sqrt{b^2 + r^2})^3 (b^2 + r^2)^{3/2}}$$

Jaffe model

$$\Phi(r) = -4\pi G \rho_0 a^2 \ \ln(1 + a/r)$$

$$\rho(r) = \frac{\rho_0}{(r/a)^2 (1 + r/a)^2}$$

Hernquist model

$$\Phi(r) = -4\pi G \rho_0 a^2 \ \frac{1}{2(1 + r/a)}$$

$$\rho(r) = \frac{\rho_0}{(r/a)(1 + r/a)^3}$$

Question: What is the number of stars with an energy between $[E, E + dE]$?

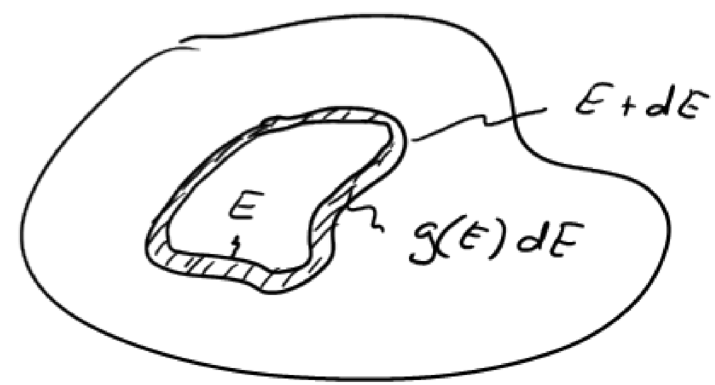
$$\boxed{N(E)\, dE = f(E) \cdot dw(E)}$$

density of the phase space

volume of the phase space occupied by stars with energies $\in [E, E+dE]$

Lets define $g(E)$ such that

$$\boxed{g(E)\, dE = dw(E)}$$



$$g(E) = \int dw \; \delta\big(H(w) - E\big)$$

Thus : $\boxed{N(E)\, dE = f(E)\, g(E)\, dE}$

For an ergodic DF (spherical model)

$$g(E) = \int d^3x\, d^3v \; \delta\left(\tfrac{1}{2}v^2 + \phi - E\right)$$

$$= \int 4\pi r^2 dr \; 4\pi v^2 dv \; \delta\left(\tfrac{1}{2}v^2 + \phi - E\right)$$

$$= (4\pi)^2 \int_0^{r_m(E)} dr\, r^2 \underbrace{\int dv\, v^2\, \delta\left(\tfrac{1}{2}v^2 + \phi - E\right)}_{*}$$

4 $\quad \xi = \frac{1}{2} v^2 \qquad v^2 = 2\xi$

$$dV = \frac{1}{V}\, d\xi = \frac{1}{\sqrt{2\xi}}\, d\xi$$

$$\int d\xi \sqrt{2\xi}\; \delta(\xi + \phi - E) = 2\sqrt{E - \phi}$$

And thus

$$\boxed{g(E) = 4\pi^2 \int_0^{r_m(E)} dr\, r^2 \sqrt{2(E - \phi)}}$$

Question :

$\boxed{\text{For a fixed radius } r, \text{ what is the energy distribution function ?}}$

$$\mathcal{L}_r(E) \sim \frac{d}{dr}\Big(f(E)\, g(E) \Big)$$

$\mathcal{L}_r(E)\,dr \sim f(E)\, g(E)$

$$\sim \frac{d}{dr}\left(\int_0^{r_m(E)} dr\, r^2 \sqrt{2(\phi - E)}\, r^2 \right) f(E)$$

$$\boxed{\mathcal{L}_r(E) \sim r^2 \sqrt{2(\phi - E)}\, f(E)}$$

The End