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Solutions – Semaine 9

Exercice 1.
Soit K un corps et L une extension quadratique, i.e. $[L : K] = 2$.

1. Montrez que toute extension de K de degré 1 est égale à K.
2. Montrez qu'il existe un élément $\alpha \in L$ tel que $L = K(\alpha)$.
3. Soit K de caractéristique différente de 2. Montrez qu'il existe un élément $\delta \in L$ avec $\delta^2 = d \in K$ tel que $L = K(\delta) = K(\sqrt{d})$.
4. Soit M une extension de K et $\delta \in M \setminus K$ un élément avec $\delta^2 \in K$. Montrez que $K(\delta)$ est une extension quadratique de K.

Solution.

1. Let L' denote the field extension of K of degree 1. This means that L' is a field that contains K, and that has a K- vector space structure such that the dimension of L' as a K-vector space is 1.The K-subspace of L' generated by 1 is equal to K, and equal to L' as well, due to the dimension of L' over K being 1. Hence K and L' coincide.

2. We take any $\alpha \in L \setminus K$. Then we have the following field extensions, $K \subseteq K(\alpha) \subseteq L$. From this, it follows using multiplicativity of degrees that

$$\underbrace{[L : K]}_{=2} = [L : K(\alpha)] \cdot [K(\alpha) : K].$$

   Since we take $\alpha \notin K$, it holds that $K \neq K(\alpha)$, and hence by the first point, $[K(\alpha) : K] \neq 1$. From this, it follows using the equation above that $[K(\alpha) : K] = 2$. But that means that $[L : K(\alpha)] = 1$, from which it follows by the first point that $L = K(\alpha)$.

3. Since $L = K(\alpha)$, and $[L : K] = 2$, it holds that $\{1, \alpha\}$ forms a K-linear basis of $K(\alpha)$. This means in particular that α^2 is a K-linear combination of 1 and α. There exists $a, b \in K$ such that $\alpha^2 = b \cdot 1 + a \cdot \alpha \Leftrightarrow \alpha^2 - a\alpha - b = 0$. We define d to be $d = a^2 + 4b$, the discriminant of the quadratic equation. We now show that d is a square in $K(\alpha)$. We do so by multiplying the quadratic equation by 4 (note that the characteristic of K is not equal to 2), and completing the square, to find:

$$4\alpha^2 - 4a\alpha - 4b = 0 \Leftrightarrow (2\alpha - a)^2 - a^2 - 4b = 0 \Leftrightarrow (2\alpha - a)^2 = a^2 + 4b = d.$$

   Hence d is a square in $K(\alpha)$, and we let $\delta = 2\alpha - a \in K(\alpha) \setminus K$, with $\delta^2 = d$. By the second part of this exercise, it holds that $L = K(\delta) = K(\sqrt{d})$.

   Let us give an alternative proof that illuminates the role of the discriminant. Since the characteristic of K is different from 2, the well-known theory of quadratic equations with coefficients in $\mathbb{C}$ can be carried over verbatim to K to obtain the following: if $p(x) = ax^2 + bx + c \in K[x]$ is a degree 2 polynomial, then the roots ξ_1, ξ_2 of $p(x)$ in any extension F of K can be written

$$\xi_1 = \frac{-2b + \sqrt{\Delta(p)}}{2a}, \quad \xi_2 = \frac{-2b - \sqrt{\Delta(p)}}{2a}$$

   where $\Delta(p) = b^2 - 4ac$ and $\sqrt{\Delta(p)} \in F$ denotes a square root of $\Delta(p)$. Now observe that:

(a) $K(\xi_i) = K(\xi_1, \xi_2)$ for any $i = 1, 2$. We can write in $K(\xi_1)[x]$ that

$$p(x) = (x - \xi_1)q(x)$$

where necessarily $\deg q(x) = 1$. Thus $q(x) = x - \xi_2$, and so $\xi_2 \in K(\xi_1)$. Hence $K(\xi_1) = K(\xi_1, \xi_2)$, and by exchanging the roles of ξ_1 and ξ_2 we also obtain $K(\xi_2) = K(\xi_1, \xi_2)$.

(b) $K(\xi_1, \xi_2) = K\left(\sqrt{\Delta(p)}\right)$. Indeed $\sqrt{\Delta(p)} = 2a(\xi_1 - \xi_2)$ so the inclusion $\supseteq$ holds. Also it follows from the formulae for ξ_1 and ξ_2 that $\subseteq$ holds.

So we obtain that $K(\xi_1) = K(\xi_2) = K(\xi_1, \xi_2) = K\left(\sqrt{\Delta(p)}\right)$ as subfields of F. Taking $F = L$ and $p(x) = m_{\alpha,K}$, we obtain an alternative proof of the exercise.

4. From the definition of δ, it immediately follows that $\{1, \delta\}$ forms a K-linear basis of $K(\delta)$ as a K-vector space. By definition, $[K(\delta) : K]$ is the dimension of $K(\delta)$ as a K-vector space, which is 2.

Exercice 2. 1. Soit L une extension de K avec $[L : K]$ impair. Montrer que $K(\alpha) = K(\alpha^2)$ pour tout $\alpha \in L \setminus K$.

2. Soient $p, q \in \mathbb{Z}$ deux nombres premiers distincts. Montrez que $\sqrt{p} \notin \mathbb{Q}(\sqrt{q})$ et $\sqrt{q} \notin \mathbb{Q}(\sqrt{p})$. Calculez $[\mathbb{Q}(\sqrt{p}, \sqrt{q}) : \mathbb{Q}]$.

3. Soit L une extension de K et soient $\alpha, \beta \in L$ des éléments tels que $[K(\alpha) : K] = m$ et $[K(\beta) : K] = n$ sont premiers entre eux. Montrer que $[K(\alpha, \beta) : K] = mn$.

Solution.

1. We have the following field extensions,

$$K \subset K(\alpha^2) \subset K(\alpha) \subset L.$$

By multiplicavitiy of degrees, it follows that

$$[L : K] = [L : K(\alpha)] \cdot [K(\alpha) : K(\alpha^2)] \cdot [K(\alpha^2) : K].$$

Since the degree of the field extension L over K is odd, it follows that the degrees on the right hand side of the equality above are odd as well. We now look at the extension $K(\alpha)$ over $K(\alpha^2)$. The degree of this extension is at most 2, since the polynomial $x^2 - \alpha^2 \in K(\alpha^2)[x]$ vanishes at α. But since the degree needs to be odd, it follows that it is 1. Hence $K(\alpha) = K(\alpha^2)$.

2. We first show that $\sqrt{p} \notin \mathbb{Q}(\sqrt{q})$. If $\sqrt{p}$ is contained in $\mathbb{Q}(\sqrt{q})$, then there are $r, s \in \mathbb{Q}$ such that $\sqrt{p} = r + s\sqrt{q}$. From this, it follows that

$$p = (r + s\sqrt{q})^2 = (r^2 + s^2 q) + (2rs)\sqrt{q}.$$

Using the fact that $p \in \mathbb{Q}$, we compare the right hand side and left hand side, and note that $2rs = 0$. If $r = 0$, then $p = s^2 q$ which is a contradiction with p, q prime and distinct.

If $s = 0$, then $\sqrt{p} = r \Rightarrow p = r^2$, which is a contradiction to p prime.

It follows that $\sqrt{p} \notin \mathbb{Q}(\sqrt{q})$. The same argument, with the roles of p and q reversed shows that $\sqrt{q} \notin \mathbb{Q}(\sqrt{p})$.

We now compute the degree of the field extension $\mathbb{Q}(\sqrt{p}, \sqrt{q})$ over $\mathbb{Q}$. We have the following extensions of fields,

$$\mathbb{Q} \subset \mathbb{Q}(\sqrt{p}) \subset \mathbb{Q}(\sqrt{p}, \sqrt{q}).$$

From mutliplicativity of degrees it follows that

$$[\mathbb{Q}(\sqrt{p},\sqrt{q}):\mathbb{Q}] = [\mathbb{Q}(\sqrt{p},\sqrt{q}):\mathbb{Q}(\sqrt{p})]\cdot[\mathbb{Q}(\sqrt{p}):\mathbb{Q}].$$

We calculate both degrees on the right hand side separately. Firstly, $[\mathbb{Q}(\sqrt{p}):\mathbb{Q}] = 2$. This holds because $\sqrt{p}\notin\mathbb{Q}$. The polynomial $x^2-p\in\mathbb{Q}[x]$ vanishes at $\sqrt{p}$, and combining Gauss III with Eisenstein for the prime p, it follows that the polynomial is irreducible over $\mathbb{Q}$. Hence it is the minimal polynomial, and the degree is 2.

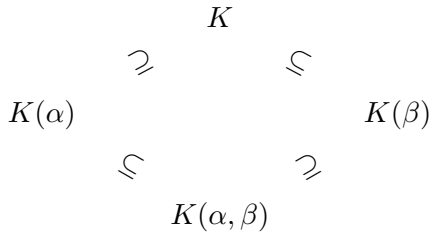
Secondly, $[\mathbb{Q}(\sqrt{p},\sqrt{q}):\mathbb{Q}(\sqrt{p})] = 2$. This holds because $\sqrt{q}\notin\mathbb{Q}(\sqrt{p})$. Therefore, the degree of the extension is not equal to 1. Furthermore, the degree of the extension is at most 2, since $\sqrt{q}^2 = q\in\mathbb{Q}$, and hence $\sqrt{q}^2\in\mathbb{Q}(\sqrt{p})$. Combining these restrictions, the degree of the extension is equal to 2, and hence the product of the two extensions is 4, meaning that $[\mathbb{Q}(\sqrt{p},\sqrt{q}):\mathbb{Q}] = 4$.

3. We have the following extension of fields, $K\subset K(\alpha)\subset K(\alpha,\beta)$. Using multiplicativity of degrees, it follows that

$$[K(\alpha,\beta):K] = [K(\alpha,\beta):K(\alpha)]\cdot[K(\alpha):K].$$

From this, it follows that $m = [K(\alpha):K]$ divides $[K(\alpha,\beta):K]$. The same argument for the extension of fields $K\subset K(\beta)\subset K(\alpha,\beta)$ shows that n divides $[K(\alpha,\beta):K]$. Using the fact that m and n are coprime, it follows that mn divides $[K(\alpha,\beta):K]$. This means that the degree of the field extension is a multiple of mn. We show that it is equal to mn by considering the first field extension again, $K\subset K(\alpha)\subset K(\alpha,\beta)$. Since $[K(\beta):K] = n$, it holds in particular that the degree of the field extension $K(\alpha,\beta)$ over $K(\alpha)$ is at most n. Hence $[K(\alpha,\beta):K]$ is at most nm. On the other hand, as we have seen above, it is at least mn, from which we conclude that it is exactly mn.

The two field extensions are illustrated below.



Exercice 3.
Soit $K = \mathbb{Q}(\sqrt{3}+\sqrt{7})$. Montrez que $[K:\mathbb{Q}] = 4$.

Solution. It holds that $\mathbb{Q}(\sqrt{3}+\sqrt{7})\subseteq\mathbb{Q}(\sqrt{3},\sqrt{7})$. We show that indeed it holds that $\mathbb{Q}(\sqrt{3}+\sqrt{7}) = \mathbb{Q}(\sqrt{3},\sqrt{7})$. For this, it is enough to show that $\sqrt{3}\in\mathbb{Q}(\sqrt{3}+\sqrt{7})$ and $\sqrt{7}\in\mathbb{Q}(\sqrt{3}+\sqrt{7})$. We denote $K = \mathbb{Q}(\sqrt{3}+\sqrt{7})$. It holds that $(\sqrt{3}+\sqrt{7})^3 = 24\sqrt{3}+16\sqrt{7}\in K$. With this, and using that $-16\sqrt{3}-16\sqrt{7}\in K$, it follows that their sum is contained in K as well,

$$(24\sqrt{3}+16\sqrt{7})+(-16\sqrt{3}-16\sqrt{7}) = 8\sqrt{3}.$$

Now using that $\frac{1}{8}\in K$, and $8\sqrt{3}\in K$ we deduce that their product $\sqrt{3}\in K$. From $\sqrt{3}\in K$, it immediately follows that $\sqrt{7}\in K$ as well, since $\sqrt{7} = (\sqrt{3}+\sqrt{7})-\sqrt{3}$. This shows that indeed $K = \mathbb{Q}(\sqrt{3},\sqrt{7})$.

The degree of the field extension $[\mathbb{Q}(\sqrt{3},\sqrt{7}):\mathbb{Q}]$ is by definition the dimension of $\mathbb{Q}(\sqrt{3},\sqrt{7})$ as a $\mathbb{Q}$-vector space. Using the previous exercise, it follows that the degree is 4. $\{1,\sqrt{3},\sqrt{7},\sqrt{3}\sqrt{7}\}$ forms a basis of this vector space.

Exercice 4.
Dans tous les cas suivants, calculez le degré de l'extension.

1. $[\mathbb{R}(e^{2i\pi/p}) : \mathbb{R}]$ pour p un nombre premier;
2. $[\mathbb{Q}(\alpha) : \mathbb{Q}]$ pour α une racine de $t^{42} + t^{41} + \cdots + t^2 + t + 1$;
3. $[\mathbb{Q}(i, \sqrt[5]{13}) : \mathbb{Q}]$;
4. $[\mathbb{F}_3(\alpha) : \mathbb{F}_3]$ où α est une racine de $t^4 - t^3 - t^2 - t - [1]_3 \in \mathbb{F}_3[t]$ La réponse peut changer en fonction de la racine considérée.
5. $[\mathbb{Q}(\sqrt{14 + 6\sqrt{5}}, \sqrt{3}) : \mathbb{Q}]$ (on pourra calculer $(3 + \sqrt{5})^2$ pour commencer);
6. $[\mathbb{Q}(\sqrt[6]{7}) : \mathbb{Q}((\sqrt[6]{7})^2)\,]$;
7. $[\mathbb{F}_2(\alpha) : \mathbb{F}_2(\alpha^2)\,]$ où α est une racine de $t^3 + t + [1]_2 \in \mathbb{F}_2[t]$.

Solution.

1. If $p = 2$, then $e^{2i\pi/2} = -1$, which is contained in $\mathbb{R}$, and hence $\mathbb{R}(e^{2i\pi/p}) = \mathbb{R}$. From this, it follows that the degree of the extension is equal to 1.

   For $p \neq 2$, it holds that $e^{2i\pi/p}$ is a complex number, and not contained in $\mathbb{R}$. We know that $[\mathbb{C} : \mathbb{R}] = 2$. It follows that $\mathbb{R}(e^{2i\pi/p}) = \mathbb{C}$, and hence $[\mathbb{R}(e^{2i\pi/p}) : \mathbb{R}] = [\mathbb{C} : \mathbb{R}] = 2$.

2. By definition, α vanishes over $t^{42} + t^{41} + \cdots + t^2 + t + 1$. Furthermore, using the fact that 43 is prime, and an example from the lecture (changing variable from $t \mapsto t + 1$ and then use Einsenstein with p), it follows that $t^{42} + t^{41} + \cdots + t^2 + t + 1$ is irreducible over $\mathbb{Q}$. Hence we get that $m_{\alpha,\mathbb{Q}} = t^{42} + t^{41} + \cdots + t^2 + t + 1$, and so $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 42$.

3. First, we note that we have the following field extensions, $\mathbb{Q} \subseteq \mathbb{Q}(\sqrt[5]{13}) \subseteq \mathbb{Q}(\sqrt[5]{13}, i)$. We can calculate the degree of the extension $\mathbb{Q}(\sqrt[5]{13}, i)$ over $\mathbb{Q}$ using multiplicativity of degrees. It holds that
   $$[\mathbb{Q}(\sqrt[5]{13}, i) : \mathbb{Q}] = [\mathbb{Q}(\sqrt[5]{13}, i) : \mathbb{Q}(\sqrt[5]{13})] \cdot [\mathbb{Q}(\sqrt[5]{13}) : \mathbb{Q}].$$
   First, we calculate $[\mathbb{Q}(\sqrt[5]{13}) : \mathbb{Q}]$. The polynomial $x^5 - 13$ vanishes at $\sqrt[5]{13}$. Furthermore, the polynomial is irreducible over $\mathbb{Q}$: By Gauss III, it is equivalent to showing that the polynomial is irreducible over $\mathbb{Z}$. We can apply Eisensteins criterion with $p = 13$, form which irreducibility over $\mathbb{Z}$ follow. Therefore, $m_{\sqrt[5]{13},\mathbb{Q}} = x^5 - 13$, and the degree of the field extension is 5.

   Secondly, we calculate $[\mathbb{Q}(\sqrt[5]{13}, i) : \mathbb{Q}(\sqrt[5]{13})]$. Since $\mathbb{Q} \subseteq \mathbb{R}$, and $\sqrt[5]{13} \in \mathbb{R}$, it follows that $\mathbb{Q}(\sqrt[5]{13}) \subseteq \mathbb{R}$. Hence $i \notin \mathbb{Q}(\sqrt[5]{13})$. Using that i is a root of $x^2 + 1$, we get that the degree of i over $\mathbb{Q}(\sqrt[5]{13})$ is 2, and hence $[\mathbb{Q}(\sqrt[5]{13}, i) : \mathbb{Q}(\sqrt[5]{13})] = 2$.

   By the formula above, it follows that
   $$[\mathbb{Q}(\sqrt[5]{13}, i) : \mathbb{Q}] = [\mathbb{Q}(\sqrt[5]{13}, i) : \mathbb{Q}(\sqrt[5]{13})] \cdot [\mathbb{Q}(\sqrt[5]{13}) : \mathbb{Q}] = 2 \cdot 5 = 10.$$

4. There are two possibilities. The first possibility is that α is the root $\alpha = [1]_3$. In that case, $\mathbb{F}_3(\alpha) = \mathbb{F}_3$, and hence $[\mathbb{F}_3(\alpha) : \mathbb{F}_3] = 1$. We can therefore write the polynomial $t^4 - t^3 - t^2 - t - [1]_3 = (t - [1]_3)(t^3 - t + [1]_3)$. If $\alpha \neq [1]_3$, then α is a root of the polynomial $t^3 - t + [1]_3$. But this polynomial is irreducible over $\mathbb{F}_3$, since neither $[0]_3, [1]_3$ or $[2]_3$ is a root of $t^3 - t + [1]_3$. We conclude with the fact that $m_{\alpha,\mathbb{F}_3} = t^3 - t + [1]_3$, and hence $[\mathbb{F}_3(\alpha) : \mathbb{F}_3] = 3$.

5. We note that $(3+\sqrt{5})^2 = 14+6\sqrt{5} \Rightarrow 3+\sqrt{5} = \sqrt{14 + 6\sqrt{5}}$. Therefore, $\mathbb{Q}(\sqrt{14 + 6\sqrt{5}}, \sqrt{3}) = \mathbb{Q}(3 + \sqrt{5}, \sqrt{3}) = \mathbb{Q}(\sqrt{5}, \sqrt{3})$. It follows that $[\mathbb{Q}(\sqrt{5}, \sqrt{3}) : \mathbb{Q}] = 4$. $\{1, \sqrt{3}, \sqrt{5}, \sqrt{3}\sqrt{5}\}$ forms a basis of $\mathbb{Q}(\sqrt{5}, \sqrt{3})$ as a $\mathbb{Q}$-vector space.

6. We calculate the degree of the extension using multiplicativity of degrees for the extension $\mathbb{Q} \subseteq \mathbb{Q}((\sqrt[6]{7})^2) \subseteq \mathbb{Q}(\sqrt[6]{7})$, from which it follows that

$$[\mathbb{Q}(\sqrt[6]{7}) : \mathbb{Q}] = [\mathbb{Q}(\sqrt[6]{7}) : \mathbb{Q}((\sqrt[6]{7})^2)] \cdot [\mathbb{Q}((\sqrt[6]{7})^2) : \mathbb{Q}].$$

We first calculate $[\mathbb{Q}(\sqrt[6]{7}) : \mathbb{Q}]$. The polynomial $x^6 - 7 \in \mathbb{Q}[x]$ is zero for $\sqrt[6]{7}$. Furthermore, by Gauss III, it is irreducible if it is irreducible over $\mathbb{Z}$. Applying Eisenstein with $p = 7$, this holds. Hence $m_{\sqrt[6]{7},\mathbb{Q}} = x^6 - 7$, and the degree of the field extension is 6.

Secondly, we calculate $[\mathbb{Q}((\sqrt[6]{7})^2) : \mathbb{Q}]$. It holds that $(\sqrt[6]{7})^2 = \sqrt[3]{7}$. The polynomial $x^3 - 7 \in \mathbb{Q}[x]$ is zero for $\sqrt[3]{7}$. Furthermore, by Gauss III, it is irreducible if it is irreducible over $\mathbb{Z}$. Applying Eisenstein with $p = 7$, this holds. Hence $m_{\sqrt[3]{7},\mathbb{Q}} = x^3 - 7$, and the degree of the field extension is 3.

Using the formula above, we get that $[\mathbb{Q}(\sqrt[6]{7}) : \mathbb{Q}((\sqrt[6]{7})^2)] = 2$.

7. We apply the same technique as in the exercise above, noting that we have an extension as follows, $\mathbb{F}_2 \subseteq \mathbb{F}_2(\alpha^2) \subseteq \mathbb{F}_2(\alpha)$, and hence

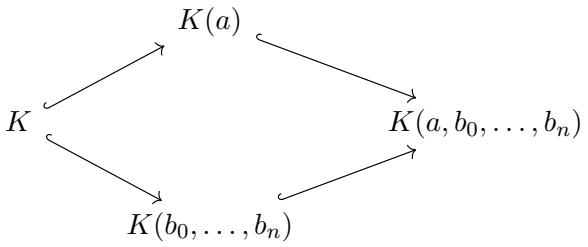
$$[\mathbb{F}_2(\alpha) : \mathbb{F}_2] = [\mathbb{F}_2(\alpha) : \mathbb{F}_2(\alpha^2)] \cdot [\mathbb{F}_2(\alpha^2) : \mathbb{F}_2].$$

On the left hand side, the degree is equal to 3, since $m_{\alpha,\mathbb{F}_2} = t^3 + t + [1]_2$. Hence on the right hand side, one of the factors is 1, and the other one is three. We note that $[\mathbb{F}_2(\alpha^2) : \mathbb{F}_2]$ can not be 1, since $\alpha^2 \notin \mathbb{F}_2$. If α^2 was contained in $\mathbb{F}_2$, then the polynomial $t^2 - \alpha^2 \in \mathbb{F}_2[t]$ vanishes at α, which contradicts the fact that $[\mathbb{F}_2(\alpha) : \mathbb{F}_2] = 3$. Therefore, $[\mathbb{F}_2(\alpha^2) : \mathbb{F}_2] = 3$, and so $[\mathbb{F}_2(\alpha) : \mathbb{F}_2(\alpha^2)] = 1$.

Exercice 5.
Soient $K \subset L \subset F$ des extensions de corps. Si $K \subset L$ et $L \subset F$ sont algébriques, montrez qu'il en est de même pour $K \subset F$.

Solution. Soient $K \subset L \subset F$ comme dans l'énoncé. Pour montrer que F est algébrique sur K, il suffit de montrer que chaque $a \in F$ est algébrique sur K. Puisque a est algébrique sur L, il existe $b_0, \ldots, b_n \in L$ tels que $m_{a,L}(t) = \sum_{i=0}^{n} b_i t^i$. En particulier, a est algébrique sur le sous-corps $K(b_0, \ldots, b_n)$.

Nous allons comparer les deux chaînes d'extensions suivantes :



On prétend que les degrés

$$[K(a, b_0, \ldots, b_n) : K(b_0, \ldots, b_n)] \quad \text{et} \quad [K(b_0, \ldots, b_n) : K]$$

sont finis. C'est le cas du premier par construction. Pour le second, par la formule de multiplication des degrés on se réduit à montrer que chaque

$$[K(b_0, \ldots, b_{i+1}) : K(b_0, \ldots, b_i)]$$

est fini. C'est le cas puisque b_{i+1} est algébrique sur K, donc a fortiori sur $K(b_0, \ldots, b_i)$. On peut ainsi appliquer la propriéré de multiplicativité des degrés pour obtenir

$$[K(a, b_0, \ldots, b_n) : K] = [K(a, b_0, \ldots, b_n) : K(b_0, \ldots, b_n)] \cdot [K(b_0, \ldots, b_n) : K] < \infty.$$

On en déduit que l'extension intermédiaire $K \subset K(a) \subset K(a, b_0, \ldots, b_n)$ est de degré fini sur K (il s'agit simplement d'algèbre linéaire : un sous-espace vectoriel d'un espace de dimension finie, est également de dimension finie). Donc a est algébrique sur K car $[K(a) : K]$ est finie.

Exercice 6.
Soit $\mathbb{Q}(x)$ le corps de fractions de l'anneau polynomial $\mathbb{Q}[x]$, et considérons

$$s := \frac{x^3+2}{x} \in \mathbb{Q}(x).$$

On a les extensions successives $\mathbb{Q} \subset \mathbb{Q}(s) \subset \mathbb{Q}(x)$.

1. Montrez que $\mathbb{Q}(x)$ est une extension algébrique de $\mathbb{Q}(s)$.
2. Calculez $[\mathbb{Q}(s) : \mathbb{Q}]$ et $[\mathbb{Q}(x) : \mathbb{Q}(s)]$.

Solution. Dans $\mathbb{Q}(x)$ on a la relation $x^3 - sx + 2 = 0$, ce qui montre que x est une racine du polynôme $t^3 - st + 2 \in \mathbb{Q}(s)[t]$. Ainsi $\mathbb{Q}(x) = \mathbb{Q}(s, x)$ est une extension algébrique de $\mathbb{Q}(s)$. On prétend que $\mathbb{Q}(s)$ est une extension transcendante de $\mathbb{Q}$. Si ce n'était pas le cas, alors par multiplicativité des degrés l'extension $\mathbb{Q} \subset \mathbb{Q}(x)$ serait également algébrique, ce qui est absurde. Donc $[\mathbb{Q}(s) : \mathbb{Q}] = \infty$.

Calculons ensuite le degré de $\mathbb{Q}(x)$ sur $\mathbb{Q}(s)$. On prétend que $t^3 - st + 2$ est irréductible dans $\mathbb{Q}(s)[t]$, et il s'ensuivra que $[\mathbb{Q}(x) : \mathbb{Q}(s)] = 3$.

Par le lemme de Gauss III, il suffit de montrer que ce polynôme est irréductible dans $\mathbb{Q}[s][t]$. Comme ce polynôme est monique, par le critère qui permet de vérifier qu'un polynôme est irréductible dans $B[t]$ pour l'image de ce polynôme par tout morphisme induit par un morphisme $\mathbb{Q}[s] \to B$ où B est un domaine, il suffit de montrer que la réduction modulo s, à savoir $t^3+2 \in \mathbb{Q}[t]$, est irréducible. Par Gauss III encore, il suffit de montrer que $t^3 + 2 \in \mathbb{Z}[t]$ est irréductible, et cela se vérifie en appliquant le critère d'Eisenstein.

Voici une autre méthode pour montrer que ce polynôme est irréductible. Si ce polynôme n'est pas irréductible, puisqu'il est de degré 3 il doit admettre une racine dans $\mathbb{Q}(s)$. Puisque s est transcendant sur $\mathbb{Q}$, on peut traiter s comme une variable indépendante et oublier qu'elle a été définie en fonction de x. Supposons donc qu'il existe $p(s), q(s) \in \mathbb{Q}[s]$ tels que

$$\frac{p^3}{q^3} - s\frac{p}{q} + 2 = 0.$$

On obtient donc

$$p\left[p^2 - sq^2\right] = -2q^3 \quad \text{dans } \mathbb{Q}[s].$$

Distinguons deux cas :

1. p est un polynôme constant, qu'on peut sans perte de généralité prendre égal à 1. Dans ce cas $1 - sq^2 = -2q^3$. Le terme constant de $1 - sq^2$ vaut 1, tandis que celui de $-2q^3$ vaut $-2b^3$ où b est le coefficient constant de q. Donc $b \in \mathbb{Q}$ est une racine cubique de $-1/2$, ce qui est impossible. Donc p ne peut être constant.
2. p n'est pas constant. Puisque p divise le membre de gauche, il doit aussi diviser $-2q^3$, et donc q^3. En particulier p et q ne sont pas premiers entre eux. Or on peut sans perte de généralité les supposer premiers entre eux, on a donc une contradiction.

On obtient ainsi que $t^3 - st + 2$ est irréductible dans $\mathbb{Q}(s)$, ce qui conclut.