Astrophysics IV, Dr. Yves Revaz EPFL

4th year physics Exercises week 9

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Astrophysics IV: Stellar and galactic dynamics

Solutions

Problem 1:

In two dimensions and in cylindrical coordinates, assuming a bar rotating with a pattern speed $\vec{\Omega}_b = \Omega_b \vec{e}_z$, the Lagrangian writes:

$$L\left(R, \dot{R}, \theta, \dot{\theta}\right) = \frac{1}{2}\dot{R}^2 + \frac{1}{2}\left(R\left(\dot{\theta} + \Omega_b\right)\right)^2 - \phi\left(R, \theta\right). \tag{1}$$

The equations of motion are derived using the Euler-Lagrange equation:

$$\begin{cases} \ddot{R} = R\left(\dot{\theta} + \Omega_b\right)^2 - \frac{\partial \phi}{\partial R} \\ \frac{d}{dt}\left(R^2\left(\dot{\theta} + \Omega_b\right)\right) = -\frac{\partial \phi}{\partial \theta} \end{cases}. \tag{2}$$

We assume a weak bar with:

$$\phi\left(R, \theta\right) = \phi_0\left(R\right) + \phi_1\left(R, \theta\right), \qquad \left|\frac{\phi_1}{\phi_0}\right| \ll 1, \tag{3}$$

where ϕ_0 represents the cylindrical symmetry, while ϕ_1 the perturbation. We then split the motion into two parts:

$$\begin{cases} R\left(t\right) = R_0 + R_1\left(t\right) \\ \theta\left(t\right) = \theta_0\left(t\right) + \theta_1\left(t\right) \end{cases} \tag{4}$$

with R_0 the radius of the guiding centre (circular orbit). The goal is then to develop the equations of motion at the first order and interpret both the zeroth and first-order terms.

To do so, we first need to Taylor expand the potential:

$$\phi\left(R, \theta\right) \cong \phi_0\left(R_0\right) + \phi_1\left(R_0, \theta\right) + \left.\frac{\partial \phi_0}{\partial R}\right|_{R_0}\left(R - R_0\right) + \left.\frac{\partial \phi_1}{\partial R}\right|_{R_0}\left(R - R_0\right) \tag{5}$$

$$+ \frac{1}{2}\left.\frac{\partial^2 \phi_0}{\partial R^2}\right|_{R_0}\left(R - R_0\right)^2 + \frac{1}{2}\left.\frac{\partial^2 \phi_1}{\partial R^2}\right|_{R_0}\left(R - R_0\right)^2, \tag{6}$$

Then, differentiating the potential with respect to R and θ, we get:

$$\begin{cases} \frac{\partial \phi}{\partial R} \cong \left.\frac{\partial \phi_0}{\partial R}\right|_{R_0} + \left.\frac{\partial \phi_1}{\partial R}\right|_{R_0} + \left.\frac{\partial^2 \phi_0}{\partial R^2}\right|_{R_0}\left(R - R_0\right) \\ \frac{\partial \phi}{\partial \theta} \cong \left.\frac{\partial \phi_1}{\partial \theta}\right|_{R_0} \end{cases}. \tag{7}$$

Note, we drop the $\left.\frac{\partial^2\phi_1}{\partial R^2}\right|_{R_0}(R-R_0)$ term, as $\phi_1 \ll \phi_0$.

Now, the goal is to introduce Eq. 3 and Eq. 7 in the equations of motion (1) and discuss the terms of different orders.

Zeroth order terms :

1. Radial equation

$$\ddot{R} = R\left(\dot{\theta}+\Omega_b\right)^2 - \frac{\partial\phi}{\partial R} \rightarrow R_0\left(\dot{\theta}_0+\Omega_b\right)^2 = \left.\frac{\partial\phi_0}{\partial R}\right|_{R_0}; \tag{8}$$

2. Azimuthal equation

$$\frac{d}{dt}\left(R^2\left(\dot{\theta}+\Omega_b\right)\right) = -\frac{\partial\phi}{\partial\theta} \rightarrow \dot{\theta}_0 = const. \tag{9}$$

Interpretation: in the absence of perturbation, the circular frequency at the radius R_0 is written:

$$\Omega^2\left(R_0\right) = \frac{1}{R_0}\left.\frac{\partial\phi_0}{\partial R}\right|_{R_0}, \tag{10}$$

thus, Eq. 8 leads to:

$$\dot{\theta}_0 + \Omega_b = \Omega\left(R_0\right) = \Omega_0, \tag{11}$$

and the angular frequency in the rotating rest frame is then:

$$\theta_0\left(t\right) = \left(\Omega_0 - \Omega_b\right)t. \tag{12}$$

First order terms :

1. Radial equation

$$\ddot{R} = R\left(\dot{\theta}+\Omega_b\right)^2 - \frac{\partial\phi}{\partial R} \rightarrow \ddot{R}_1 + R_1\left.\left(\frac{\partial^2\phi_0}{\partial R^2} - \Omega_b^2\right)\right|_{R_0} - 2R_0\dot{\theta}_1\Omega_0 = -\left.\frac{\partial\phi_1}{\partial R}\right|_{R_0}; \tag{13}$$

2. Azimuthal equation

$$\frac{d}{dt}\left(R^2\left(\dot{\theta}+\Omega_b\right)\right) = -\frac{\partial\phi}{\partial\theta} \rightarrow \ddot{\theta}_1 + 2\Omega_0\frac{\dot{R}_1}{R_0} = -\frac{1}{R_0^2}\left.\frac{\partial\phi_1}{\partial\theta}\right|_{R_0}. \tag{14}$$

To move forward, we have to guess some specific potential. We assume a perturbation of the type:

$$\phi_1\left(R,\theta\right) = \phi_b\left(R\right)\cos\left(m\theta\right). \tag{15}$$

where $m = 2$ corresponds to a bar.
Note also that any other perturbation can be obtained by summing over m.

Assuming $\theta_1 \ll \theta_0$, the gradients of Eq. 7 can now be written as:

$$\begin{cases} \frac{\partial \phi_1}{\partial R} = \frac{\phi_b}{\partial R} \cos\left(m\theta\right) \approx \frac{\partial \phi_b}{\partial R} \cos\left(m\theta_0\right) = \frac{\partial \phi_b}{\partial R} \cos\left(m\left(\Omega_0 - \Omega_b\right)t\right) \\ \frac{\partial \phi_1}{\partial \theta} = -\phi_b\left(R\right) \sin\left(m\theta\right) m \approx -\phi_b\left(R\right) m \sin\left(m\left(\Omega_0 - \Omega_b\right)t\right) \end{cases} . \tag{16}$$

Introducing those gradients in Eq 13 and 2, we get:

$$\begin{cases} \ddot{R}_1 + R_1 \left(\frac{\partial^2 \phi_0}{\partial R^2} - \Omega^2\right)\Big|_{R_0} - 2R_0\dot{\theta}_1\Omega_0 = -\left.\frac{\partial \phi_b}{\partial R}\right|_{R_0} \cos\left(m\left(\Omega_0 - \Omega_b\right)t\right) \\ \ddot{\theta}_1 + 2\Omega_0 \frac{\dot{R}_1}{R_0} = \frac{m\phi_b(R_0)}{R_0^2} \sin\left(m\left(\Omega_0 - \Omega_b\right)t\right) \end{cases} . \tag{17}$$

At this stage, it is possible to integrate $\ddot{\theta}_1$ over time:

$$\dot{\theta}_1 = -2\Omega_0 \frac{R_1}{R_0} - \frac{\phi_b\left(R_0\right)}{R_0^2\left(\Omega_0 - \Omega_b\right)} \cos\left(m\left(\Omega_0 - \Omega_b\right)t\right) + const, \tag{18}$$

and replacing it in the equation for $\ddot{R}_1$ we find:

$$\ddot{R}_1 + \kappa_0^2 R_1 = -\left[\frac{\partial \phi_b}{\partial R} + \frac{2\Omega_0\phi_b}{R\left(\Omega_0 - \Omega_b\right)}\right]_{R_0} \cos\left(m\left(\Omega_0 - \Omega_b\right)t\right) + const. \tag{19}$$

Note that we have used the radial epicycle frequency:

$$\kappa_0^2 = \left(\frac{\partial^2 \phi}{\partial R^2} + 3\Omega^2\right)\Big|_{R_0} . \tag{20}$$

The general solution is a harmonic oscillator of frequency κ_0 driven at frequency $m\left(\Omega_0 - \Omega_b\right)$.

Using Eq. 12 we find:

$$R_1\left(\theta_0\right) = C_1 \cos\left(\frac{\chi_0\theta_0}{\Omega_0 - \Omega_b} + \alpha\right) - \left[\frac{\partial \phi_b}{\partial R} + \frac{2\Omega_0\phi_b}{R\left(\Omega_0 - \Omega_b\right)}\right]_{R_0} \frac{\cos\left(m\theta_0\right)}{\chi_0^2 - m^2\left(\Omega_0 - \Omega_b\right)^2}, \tag{21}$$

with C_1 and α arbitrary constants.

Problem 2:

We start from the Hamiltonian of a 1-D harmonic oscillator, assuming a frequency ω:

$$H = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\omega^2 x^2$$

The equations of motion (Hamilton equations) are:

$$\begin{cases} \dot{x} = \dot{x} \\ \ddot{x} = -\omega^2 x \end{cases}$$

with the general solution:

$$\begin{cases} x(t) = A\cos(\omega t + \alpha) + B\sin(\omega t + \alpha) \\ \dot{x}(t) = -A\omega\sin(\omega t + \alpha) + B\omega\cos(\omega t + \alpha) \end{cases}$$

Taking the square, we get:

$$\begin{cases} x^2(t) = A^2\cos^2(\omega t+\alpha) + B^2\sin^2(\omega t+\alpha) + 2AB\cos^2(\omega t+\alpha)\sin^2(\omega t+\alpha) \\ \dot{x}^2(t) = A^2\omega^2\sin^2(\omega t+\alpha) + B^2\omega^2\cos^2(\omega t+\alpha) - 2AB\omega^2\cos^2(\omega t+\alpha)\sin^2(\omega t+\alpha) \end{cases}$$

and thus:

$$x(t)^2 + \frac{\dot{x}^2(t)}{w^2} = A^2 + B^2 \; .$$

which is the equation of an ellispe of ellipticity w (if $w < 1$) or $1/w$ (if $w > 1$).
Assuming $\alpha = 0$ we have:

$$\begin{cases} A = x_0 \\ B = v_0/\omega \end{cases}$$

and the evolution is equivalent to multiplying the initial position and velocity by a matrix (time operator):

$$\begin{pmatrix} x(t) \\ \dot{x}(t) \end{pmatrix} = \begin{pmatrix} \cos(\omega t) & \frac{1}{\omega}\sin(\omega t) \\ -\omega\sin(\omega t) & \cos(\omega t) \end{pmatrix} \cdot \begin{pmatrix} x_0 \\ \dot{x}_0 \end{pmatrix}$$

The geometrical interpretation of the matrix is to apply a rotation with some deformation (if $w = 1$, this is a pure rotation). However, as the determinant of the matrix is 1, the area is conserved and thus the density of the phase space is conserved.