

# Equilibria of collisionless systems

1<sup>st</sup> part

# Outlines

## Orbits in planar non-axisymmetric rotating potential

- Orbits not confined to Lagrange points

## Weak bars

- the Lindblad resonances
- orbit families in realistic bars

## The collisionless Boltzmann equation

- The distribution function (DF) of stellar systems
- The Collisionless Boltzmann equation
- Limitations

## Relations between DFs and observables

- Density, velocity distribution function, mean velocity, velocity dispersion

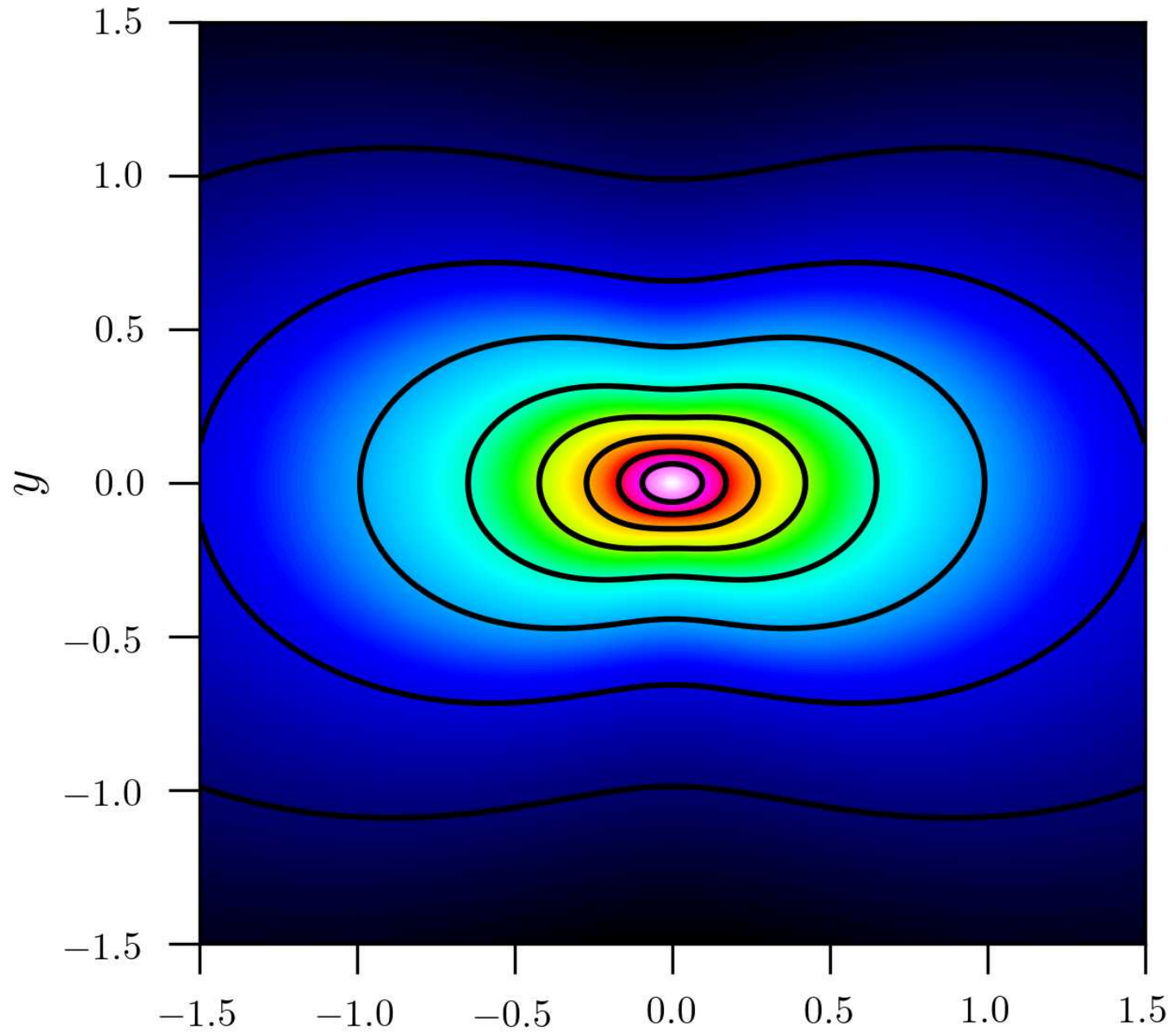
# **Stellar Orbits**

**Orbits not confined to  
Lagrange points**

Bar model : Logarithmic potential:  
( $V_0=0.1$   $R_c=0.1$   $q=0.8$ )

$$\Phi_{\log}(x, y) = \frac{1}{2} V_0^2 \ln \left( R_c^2 + x^2 + \left( \frac{y}{q} \right)^2 \right)$$

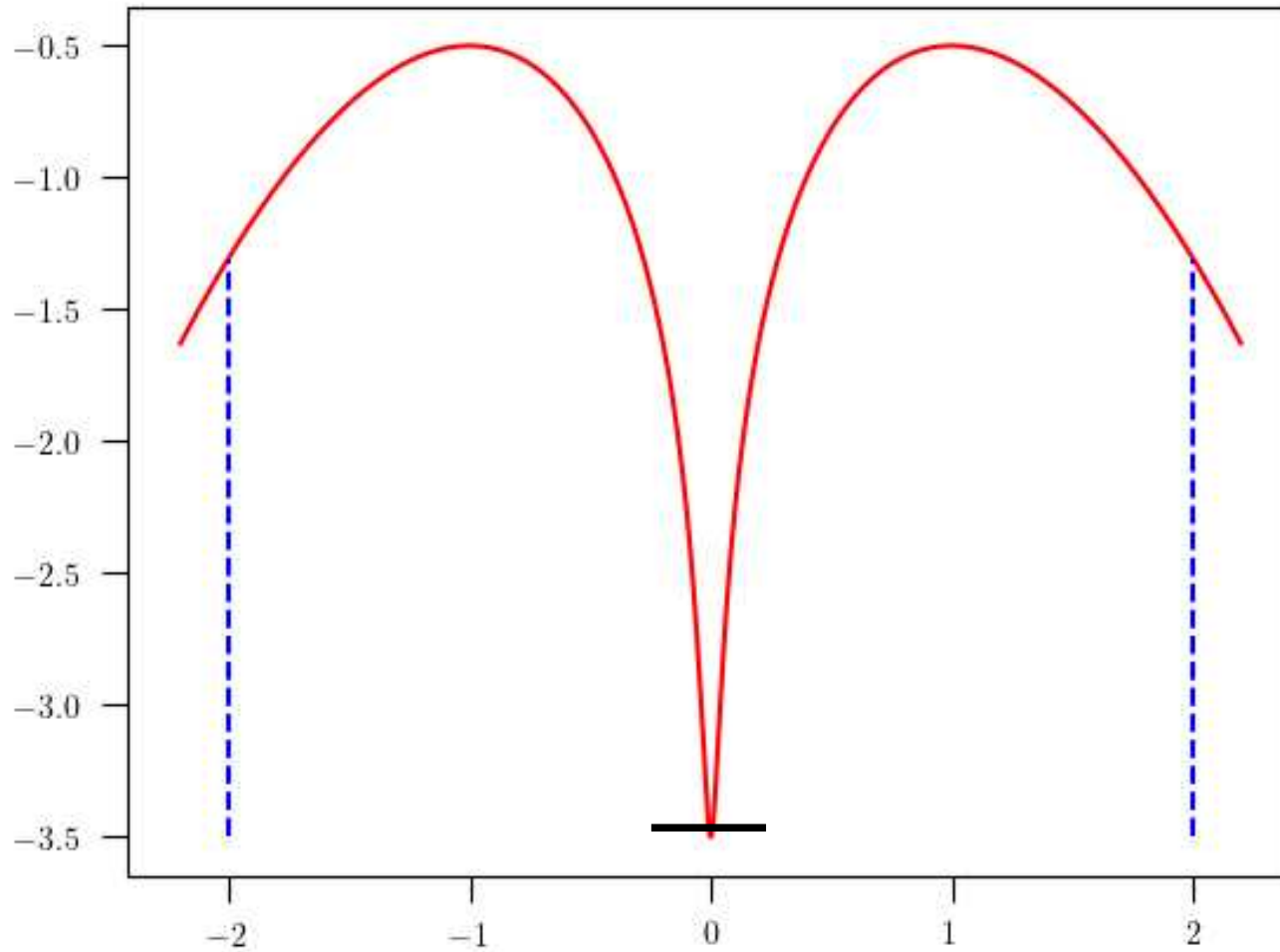
$\Omega_p \neq 0$



## Low energy orbits

$$R \ll R_{\text{corot}}$$

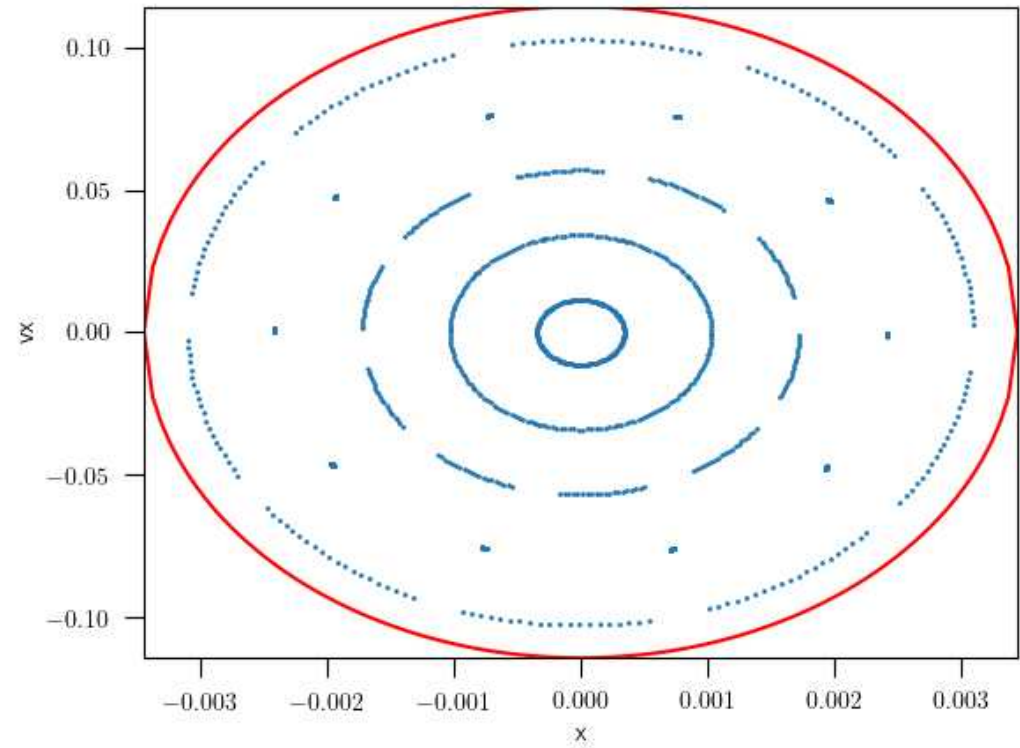
# Potential and energy



```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 --plotpotential
```

# Orbits around $L_3$

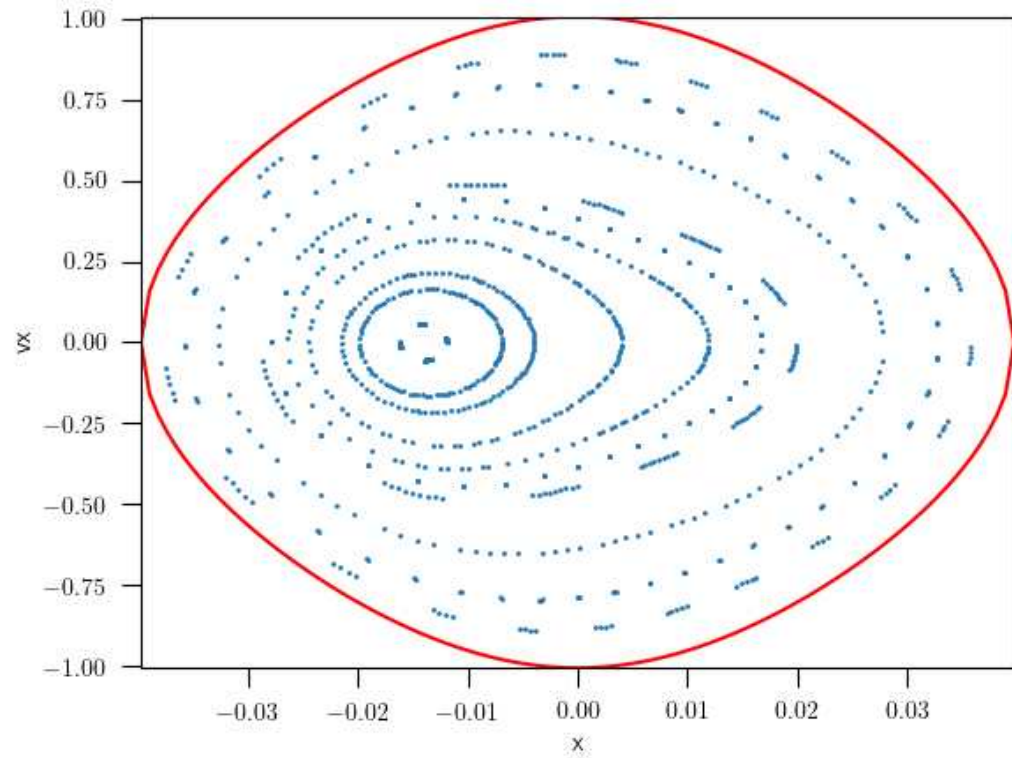
$$\Omega = 0$$



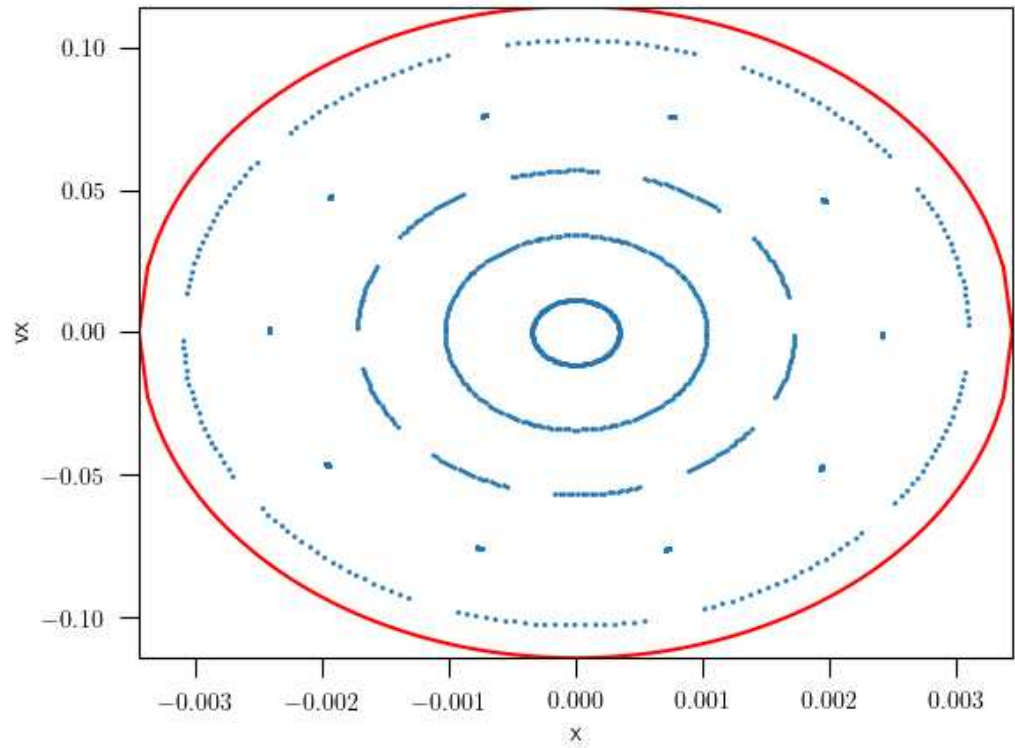
```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 0 -E -3.5
```

# Orbits around $L_3$

$$\Omega = 1$$



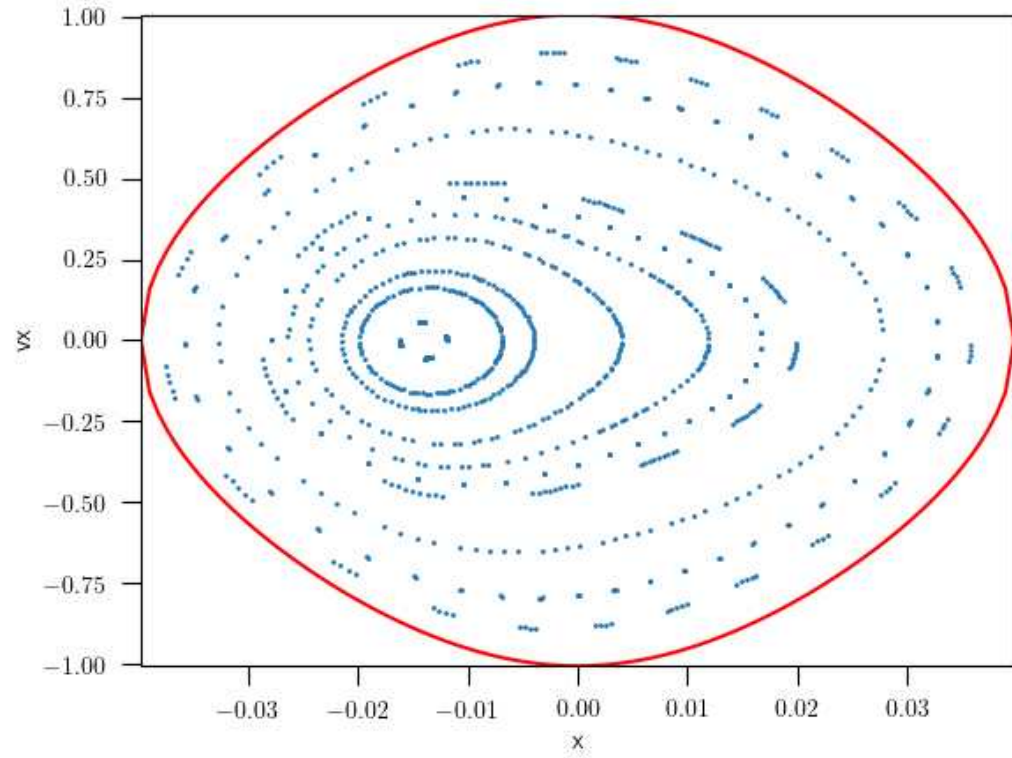
$$\Omega = 0$$



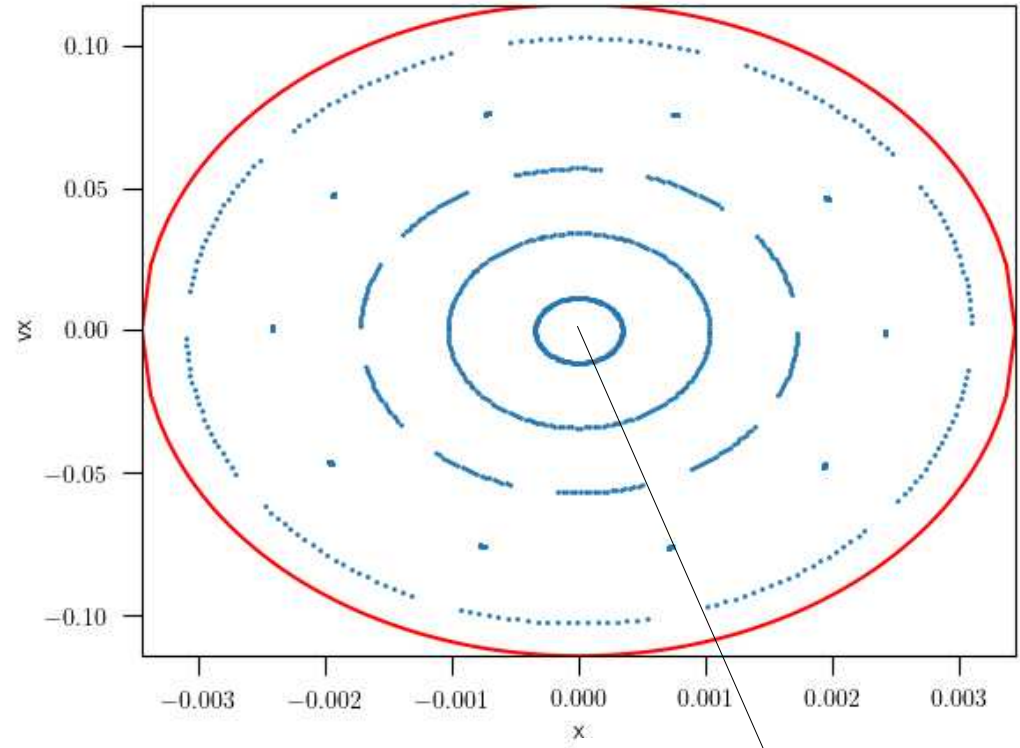
```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -3  
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 0 -E -3.5
```

# Orbits around $L_3$

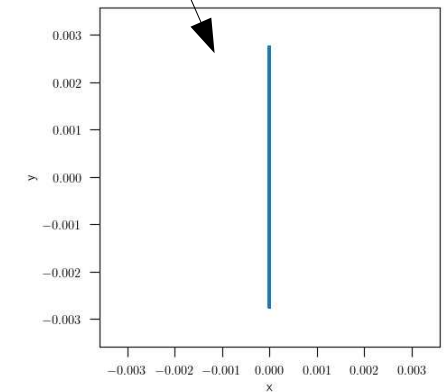
$$\Omega = 1$$



$$\Omega = 0$$

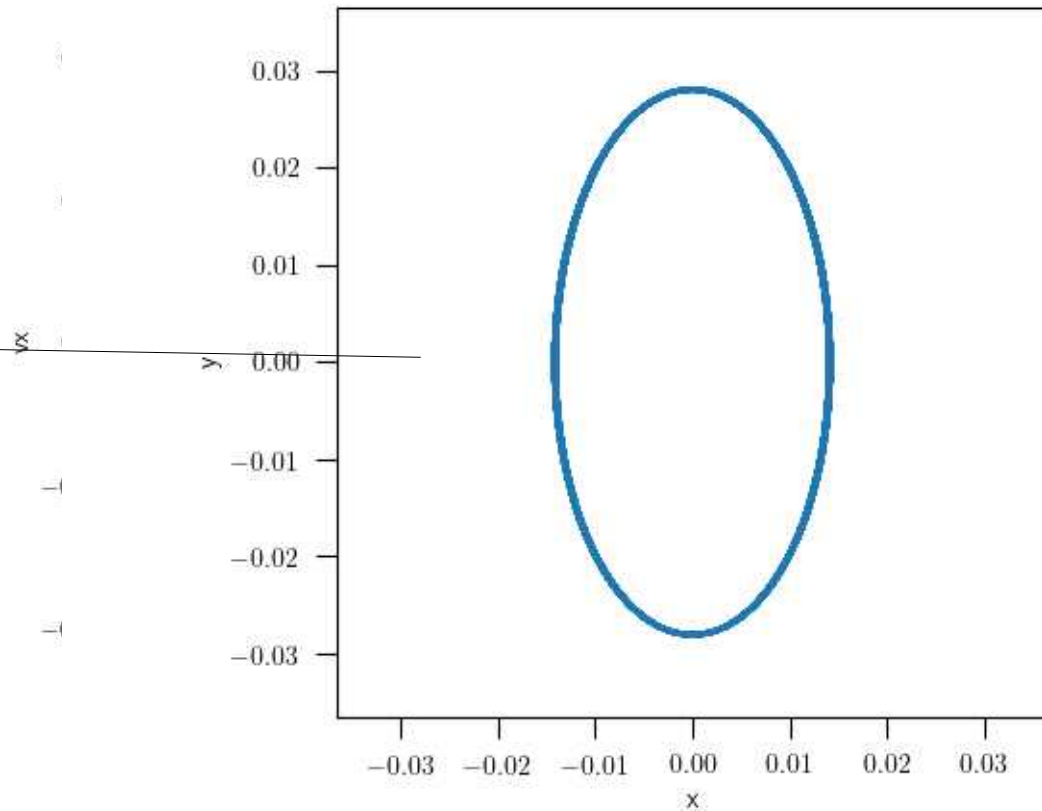
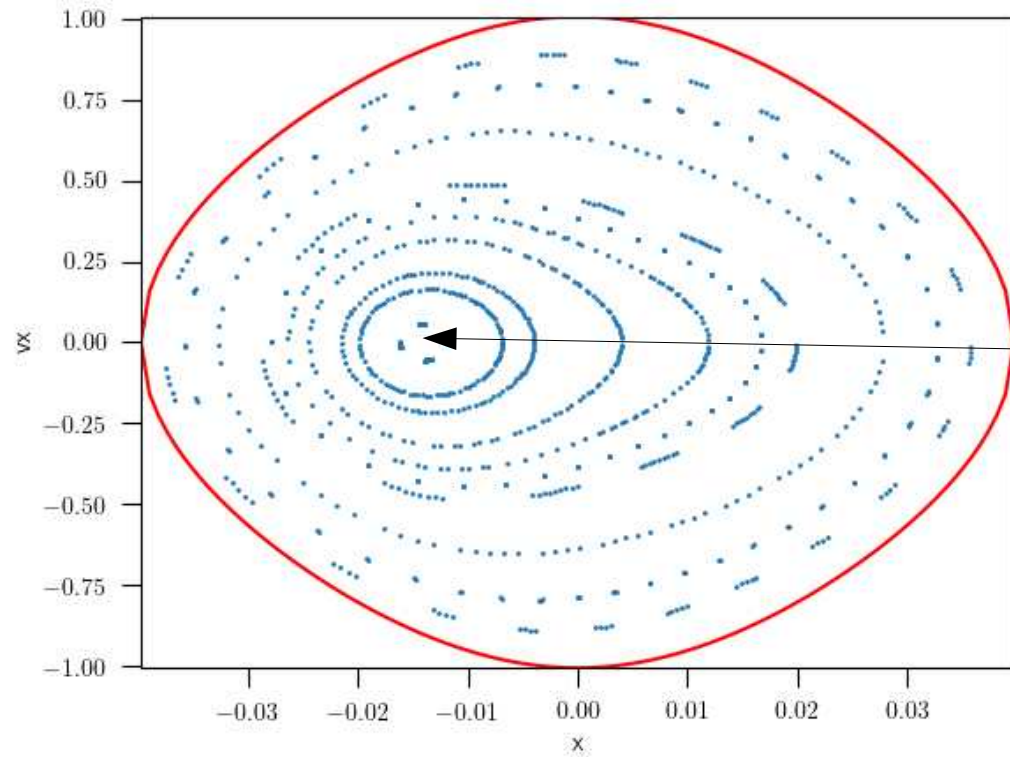


```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -3  
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 0 -E -3.5
```



# Short axis (Y) orbits (periodic)

$$\Omega = 1$$



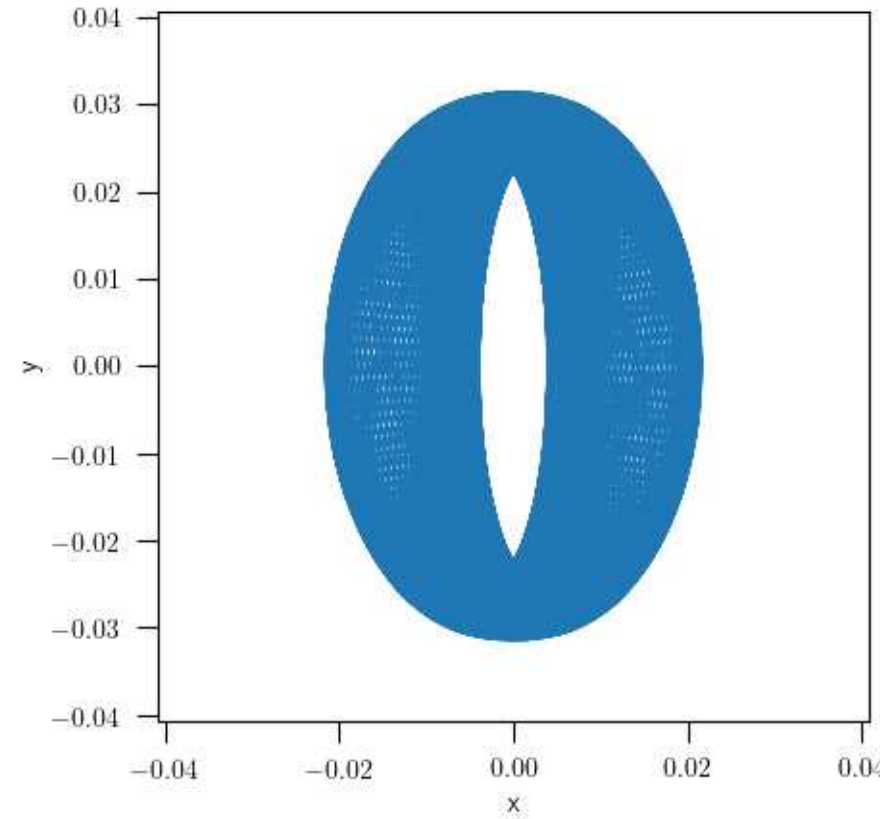
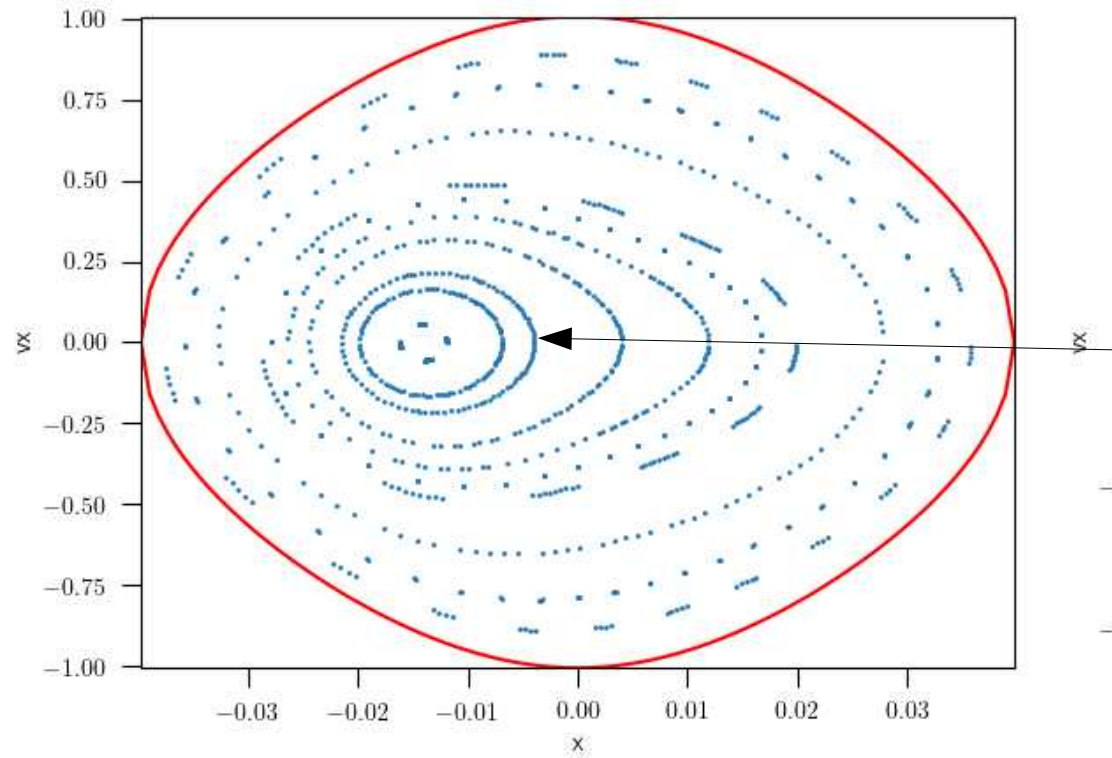
X4

```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -3  
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -3 --x -0.014
```

Apparition of a periodic loop orbit  
(replace the radial orbit, perpendicular to the bar), clockwise rotation

# Short axis (Y) orbits (periodic)

$$\Omega = 1$$



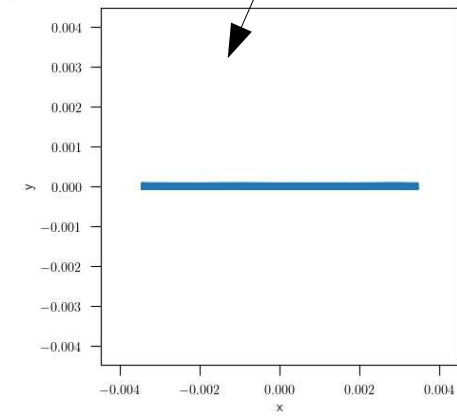
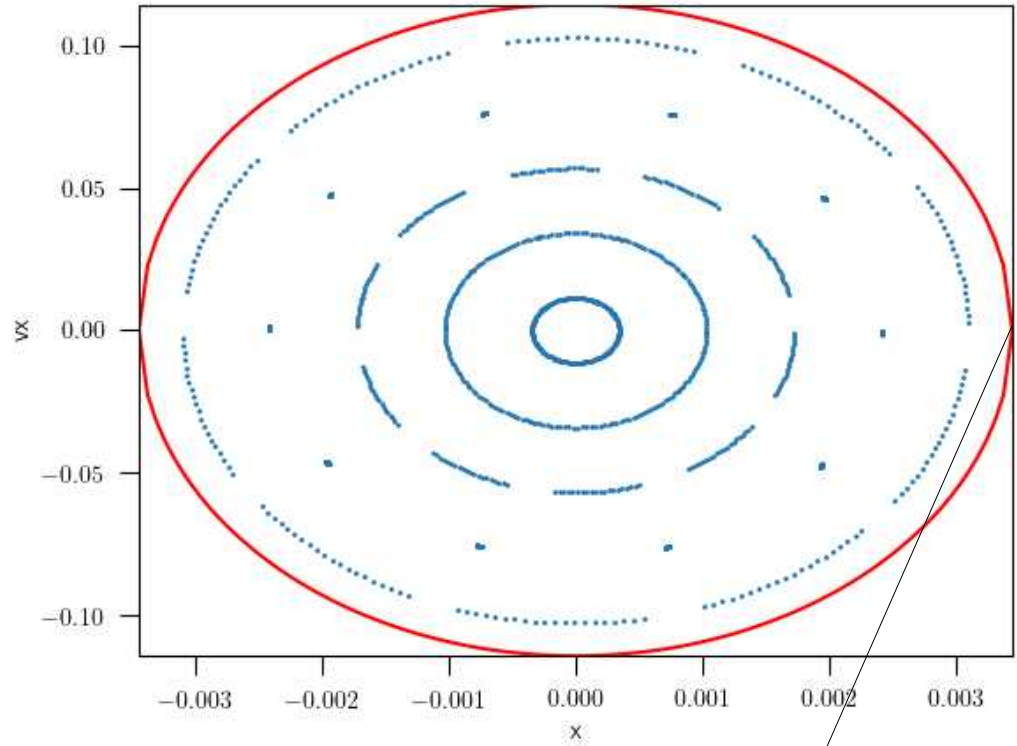
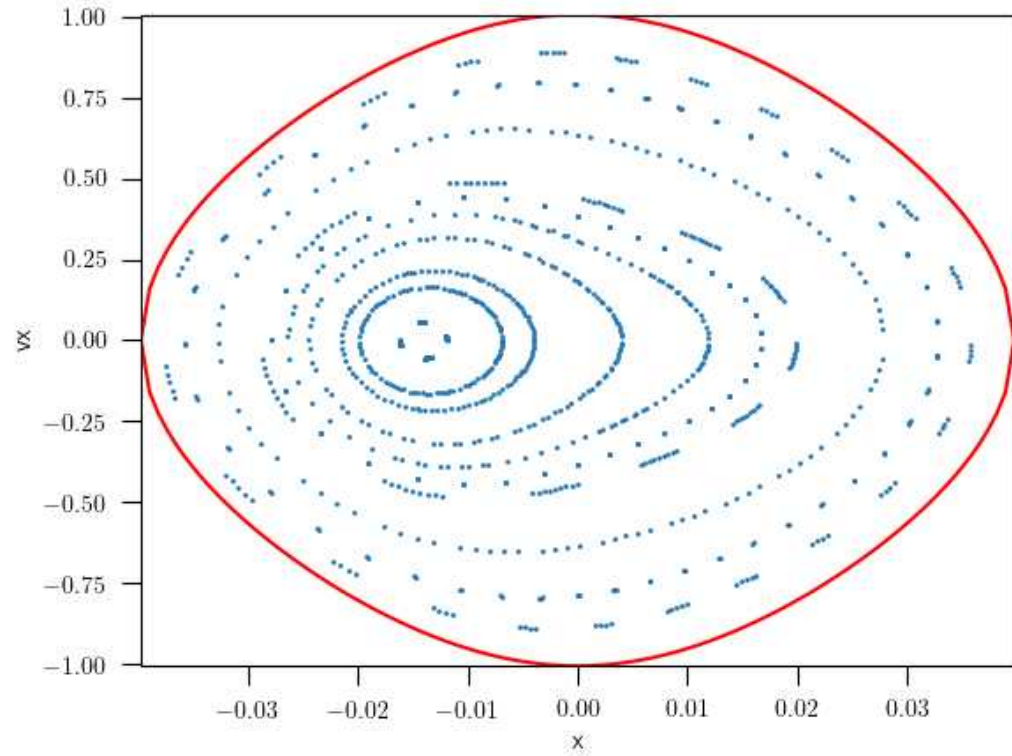
X4

```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -3  
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```

# Orbits around $L_3$

$\Omega = 1$

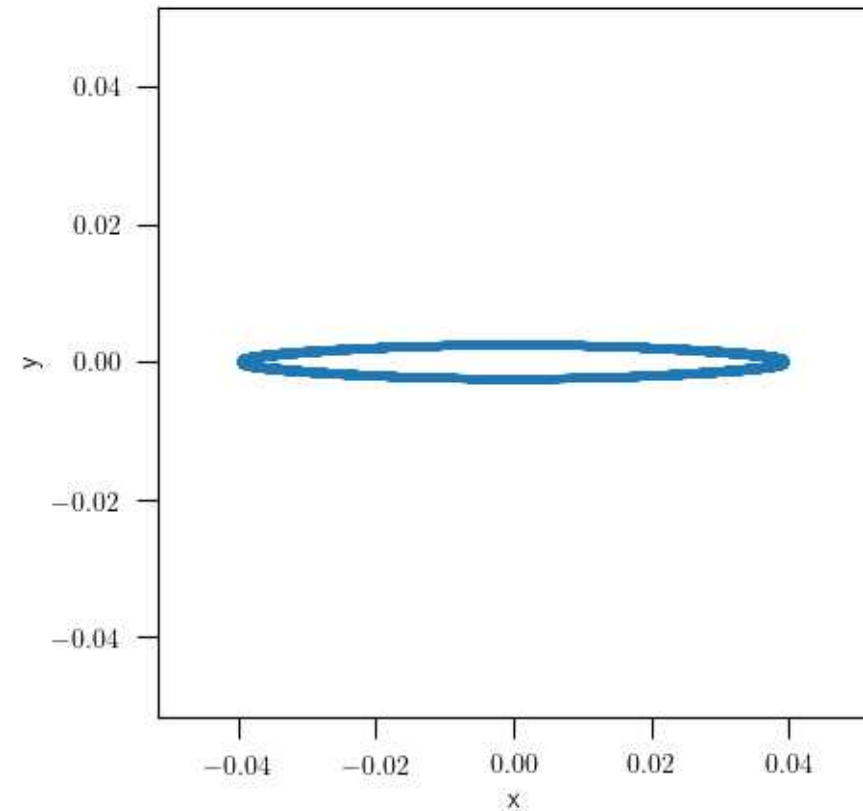
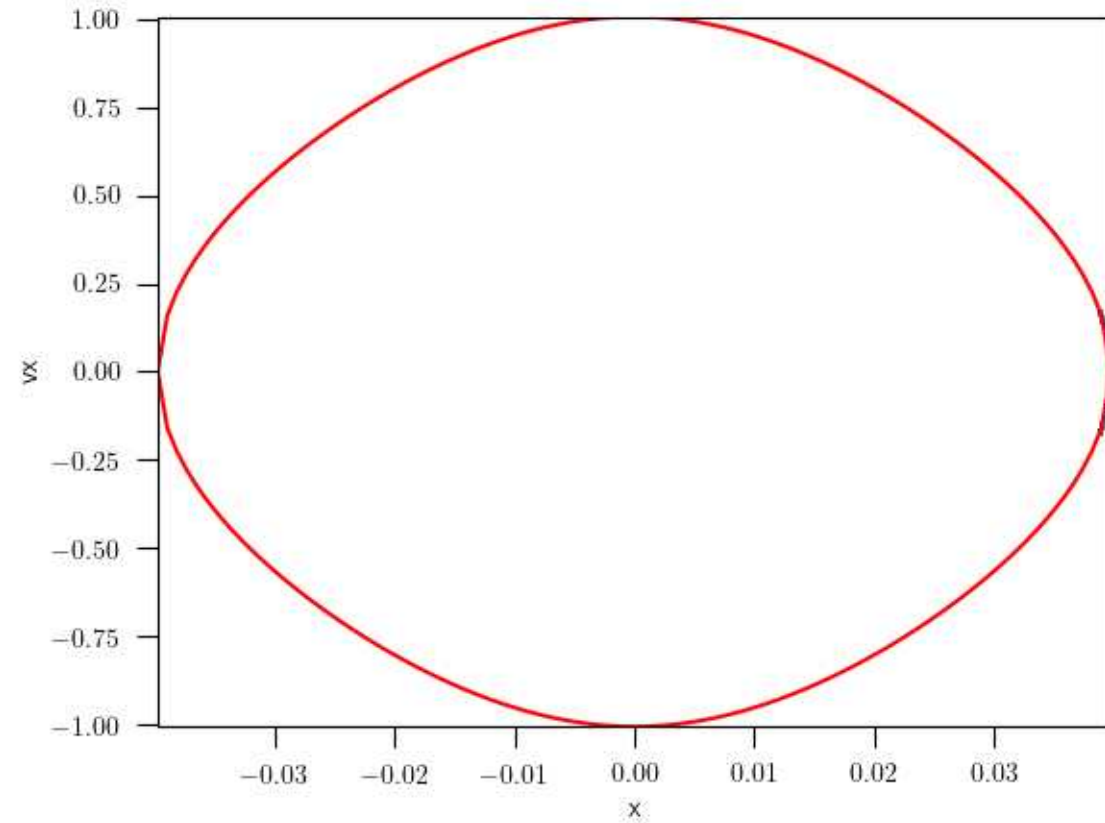
$\Omega = 0$



```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -3  
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 0 -E -3.5
```

# Long axis (X) orbits (periodic)

$$\Omega = 1$$



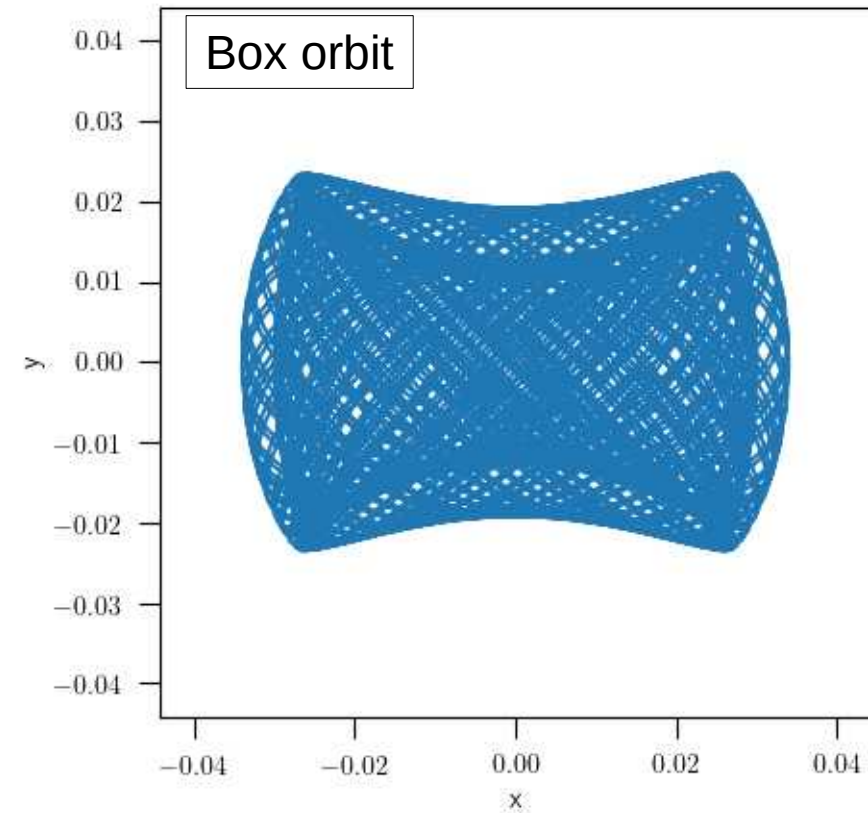
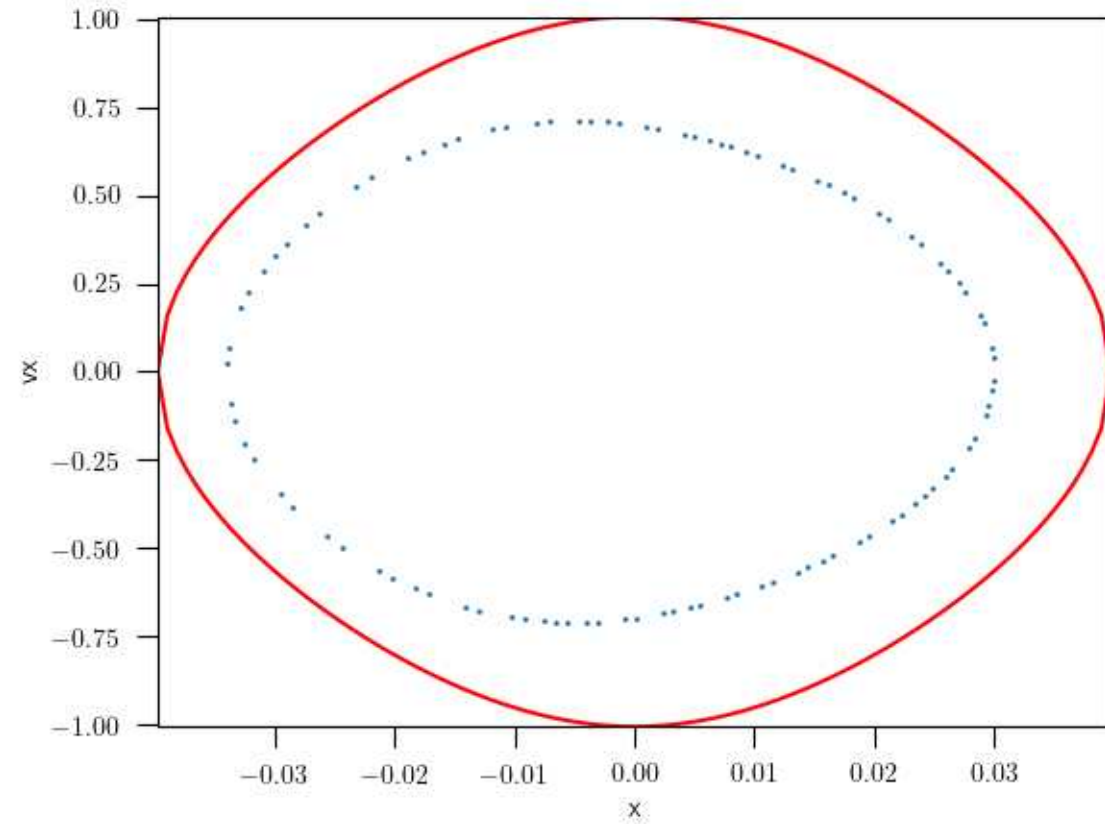
**x1**

```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -3 --x 0.03975
```

Apparition of a periodic loop orbit  
(replace the radial orbit, parallel  
to the bar), anti-clockwise rotation

# Long axis (X) orbits (non periodic)

$$\Omega = 1$$

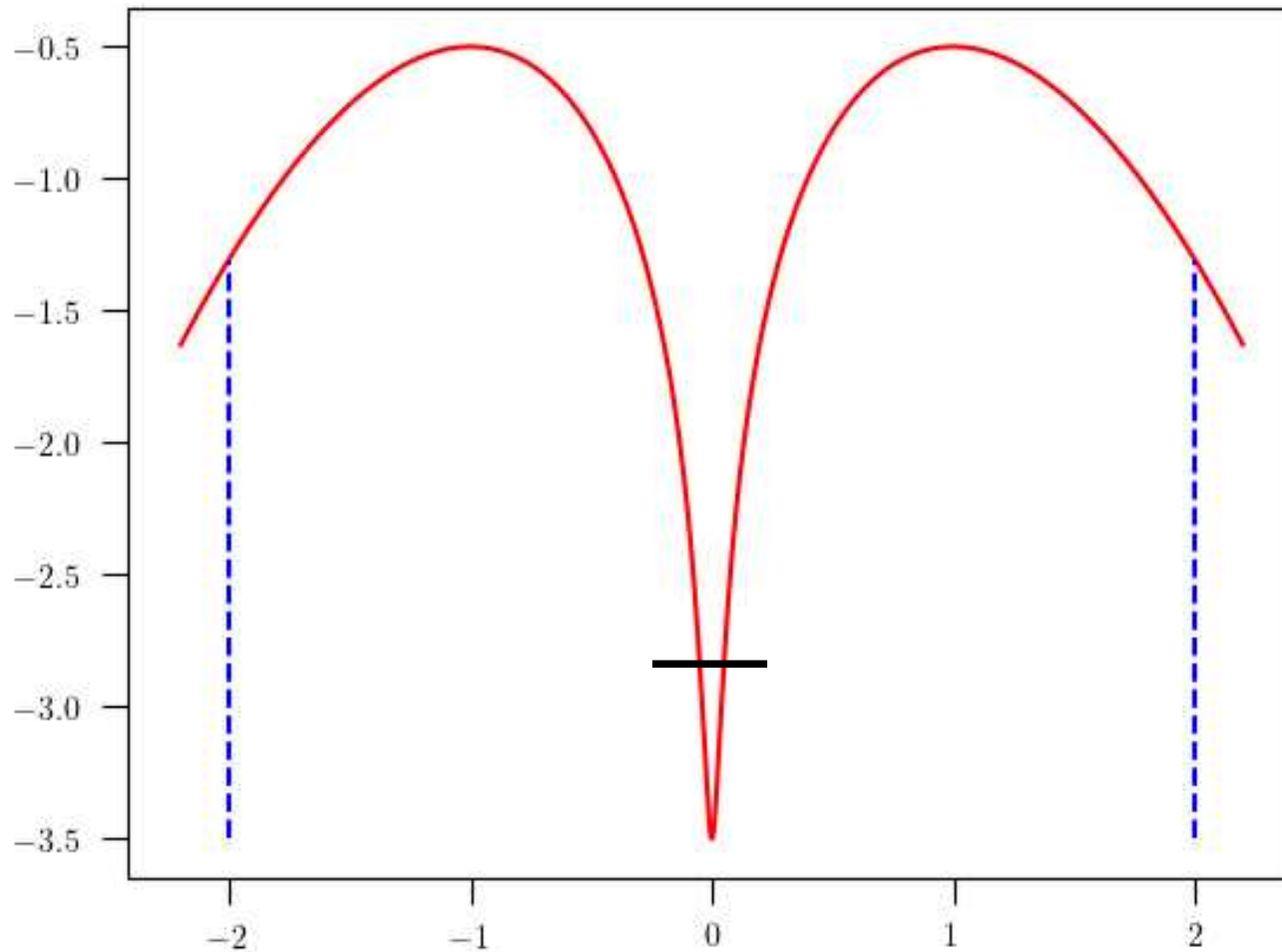


x1

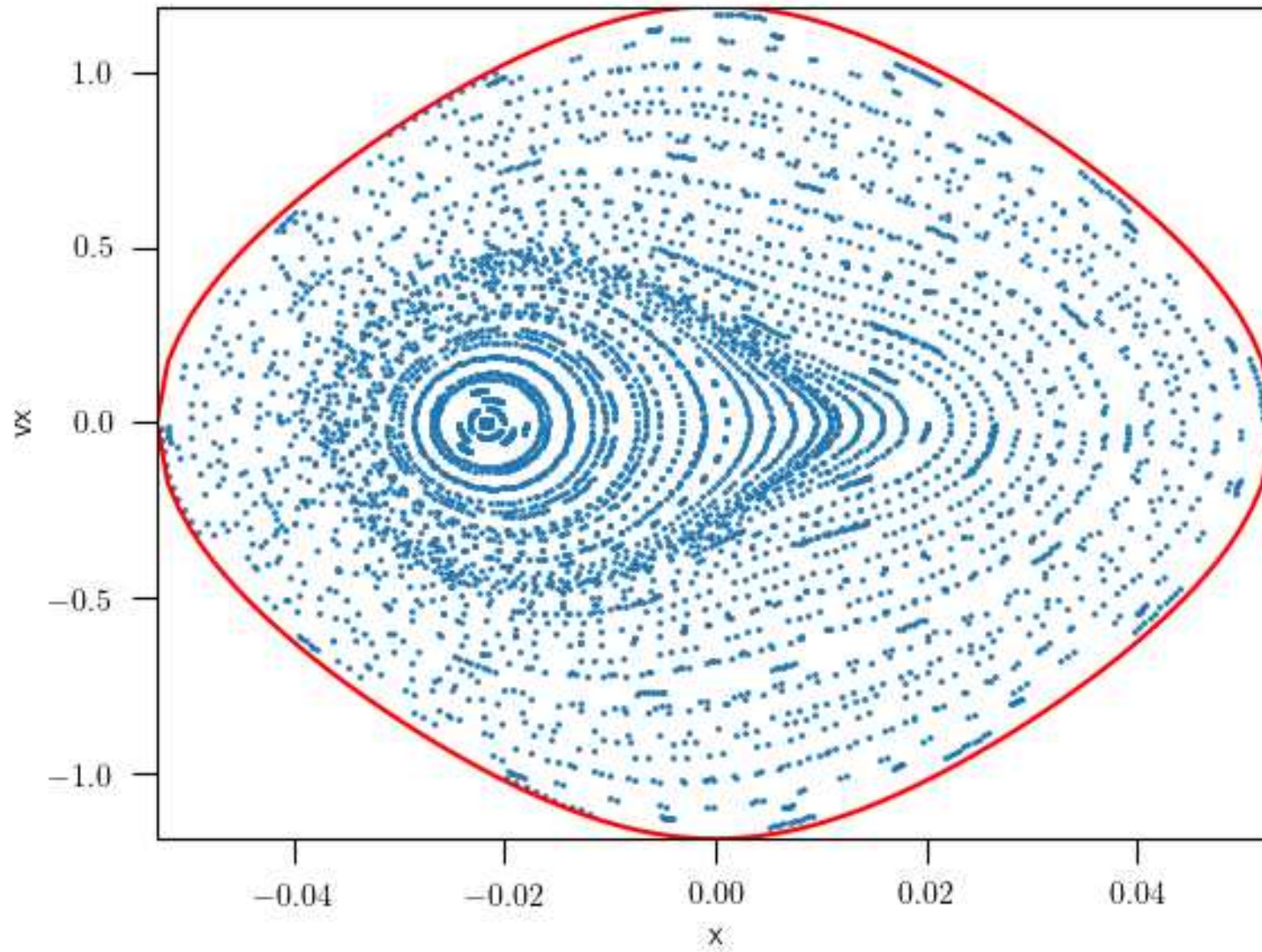
```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -3 --x 0.03
```

# Increasing the energy

$$E = -2.8$$

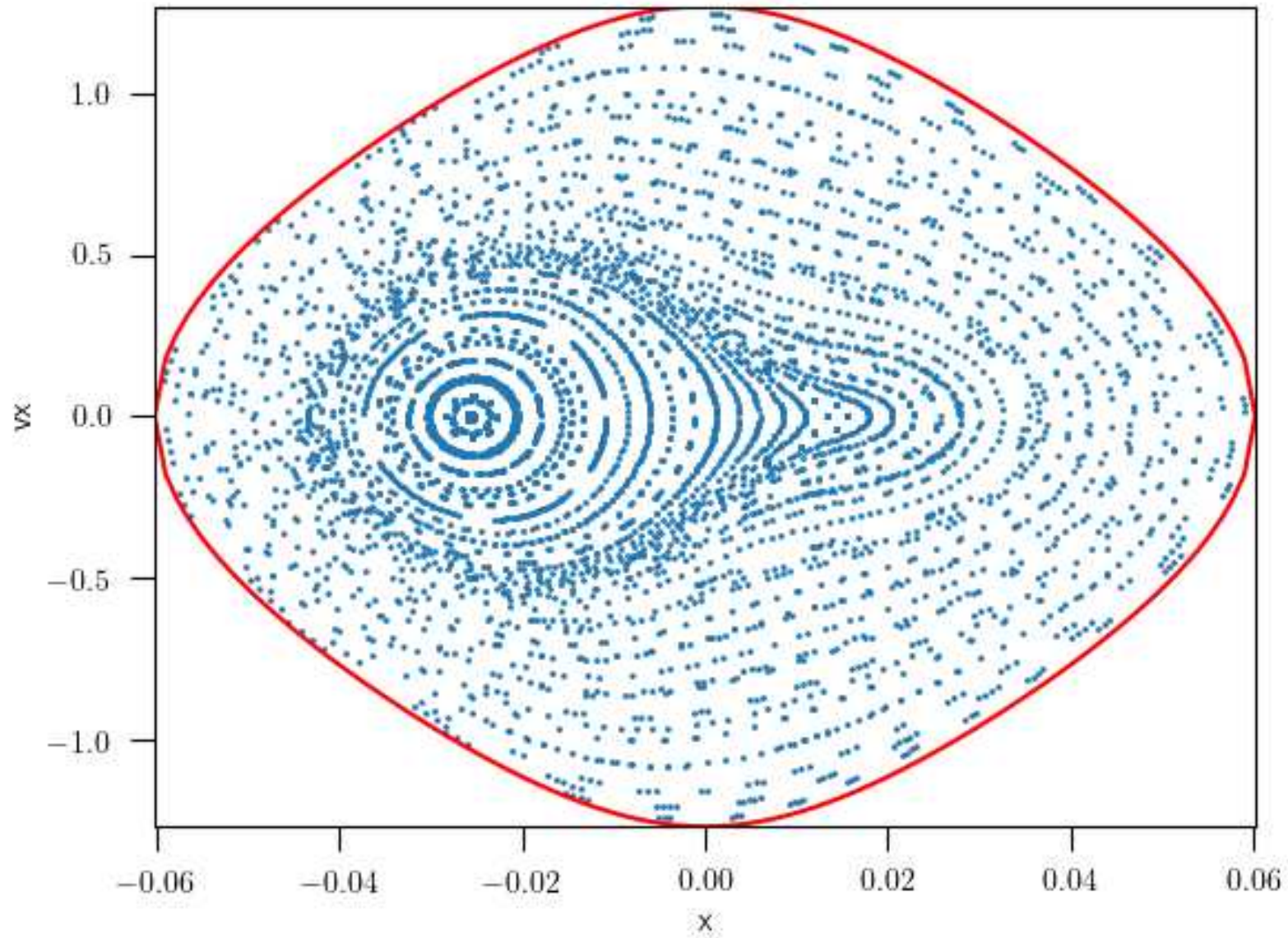


$$E = -2.8$$



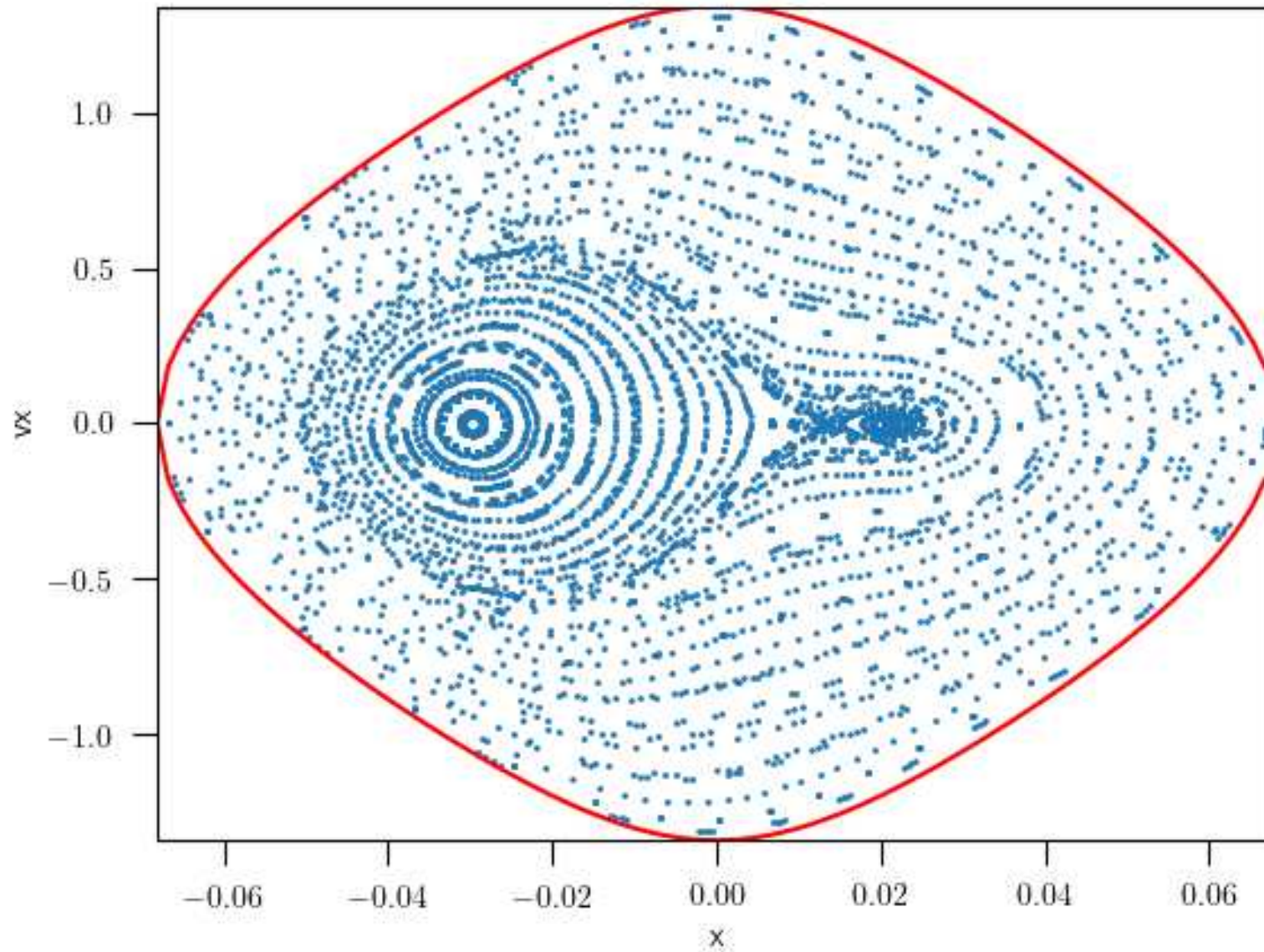
```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -2.8 --norbits 50
```

$$E = -2.7$$



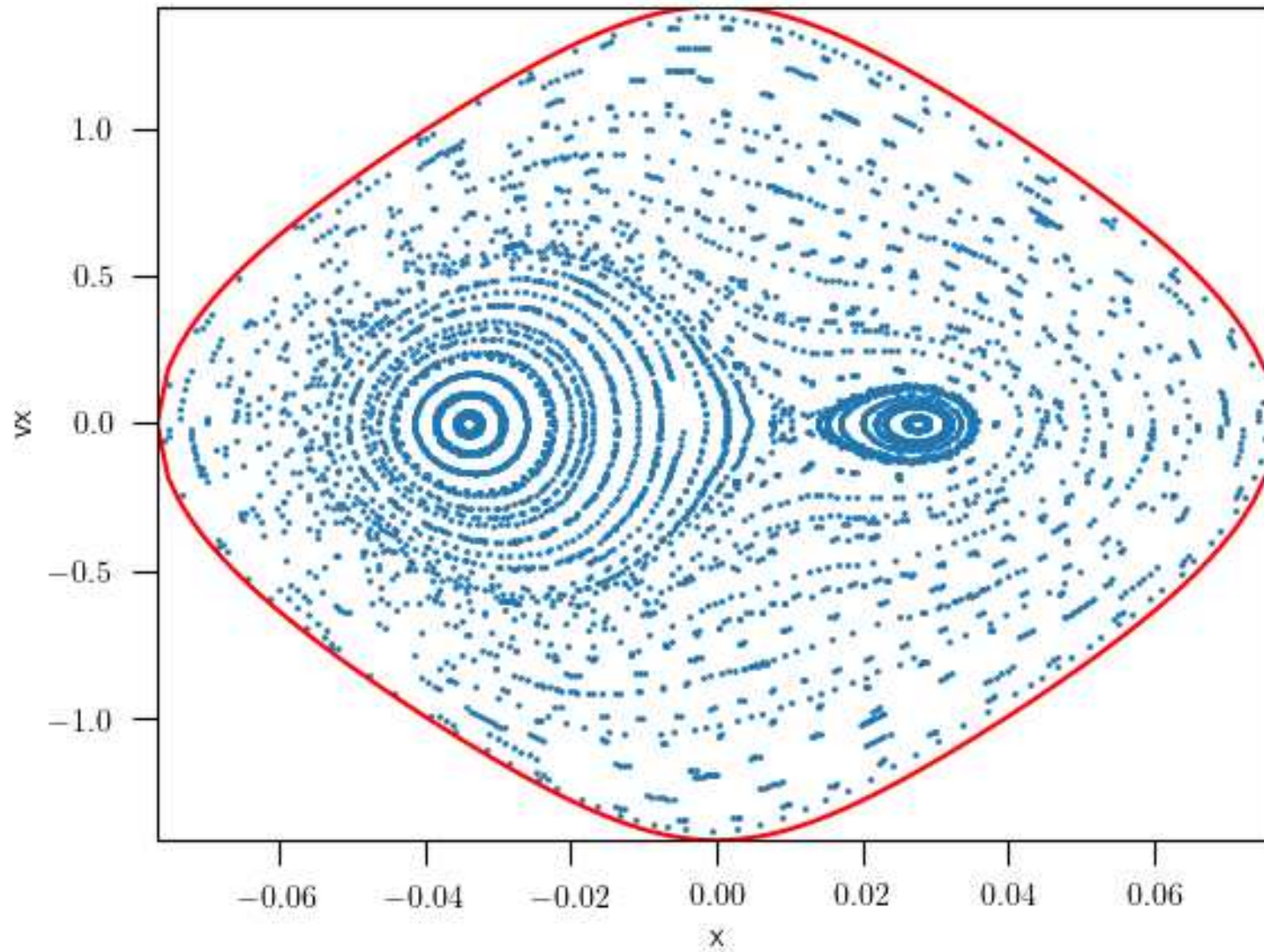
```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -2.7 --norbits 50
```

$$E = -2.6$$



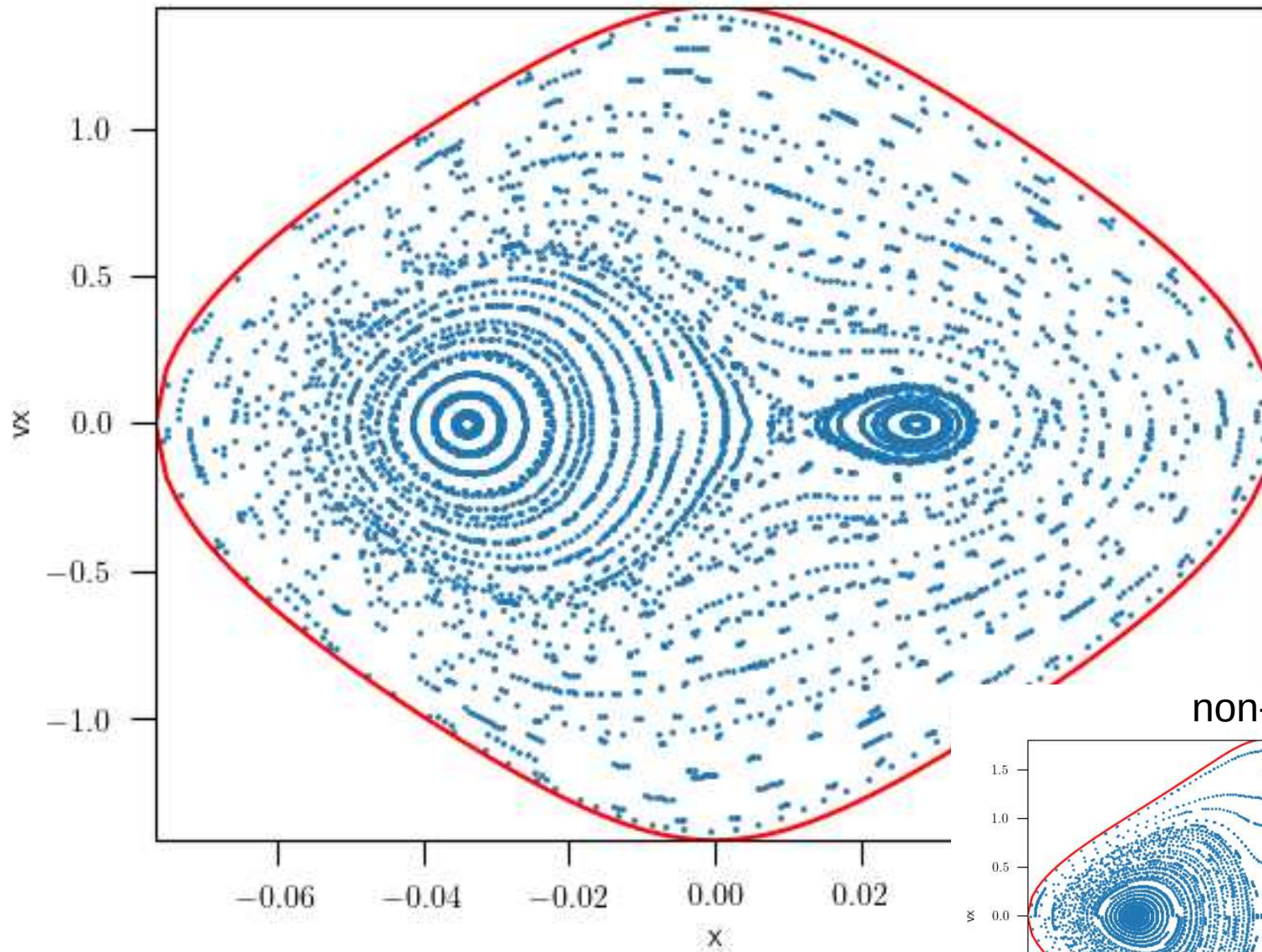
```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -2.6 --norbits 50
```

$$E = -2.5$$

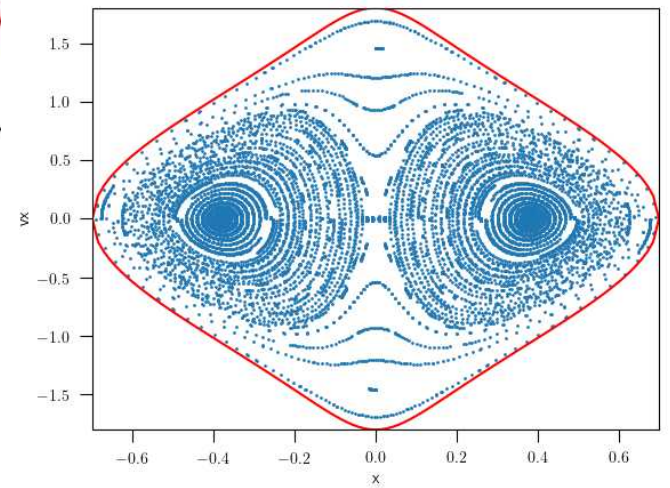


```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -2.5 --norbits 50
```

$$E = -2.5$$

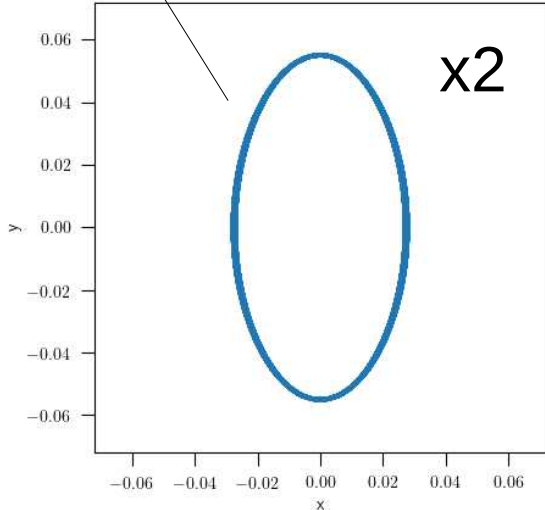
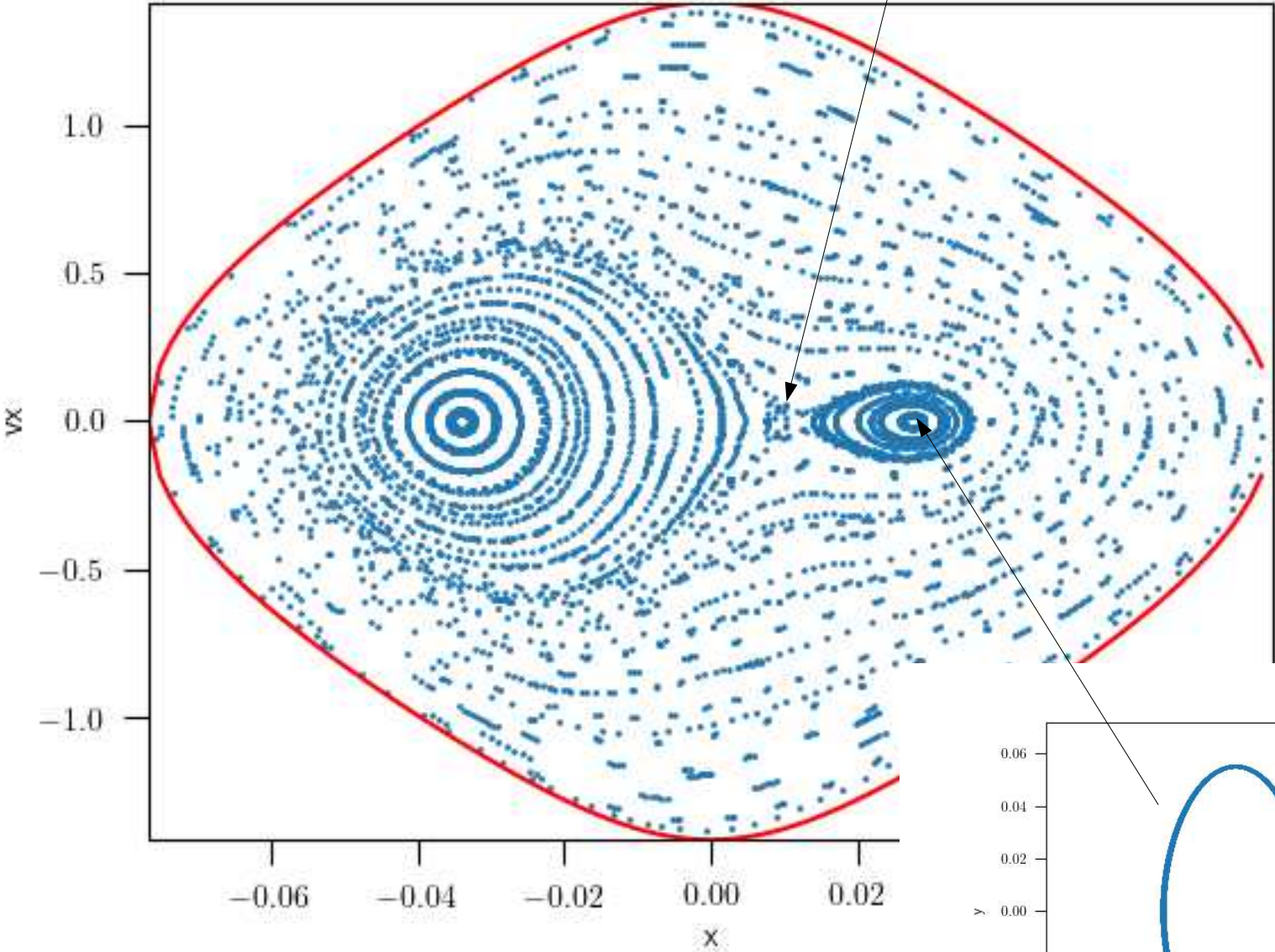


non-rotating case



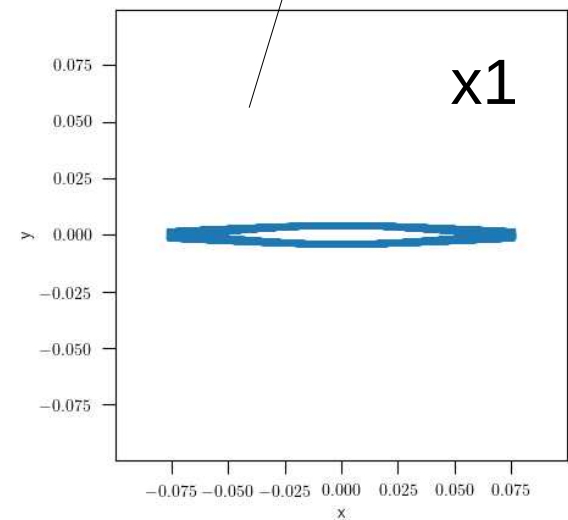
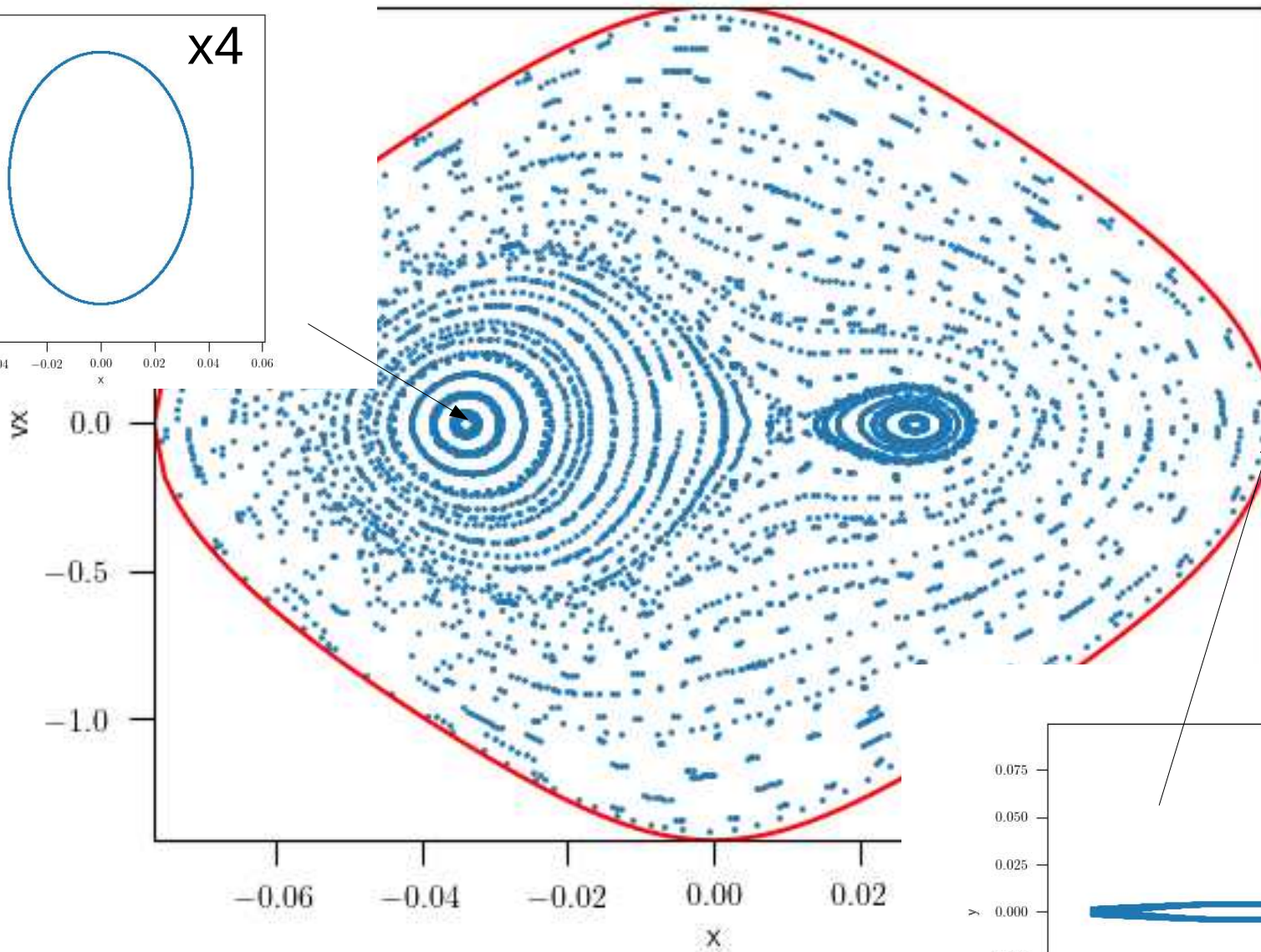
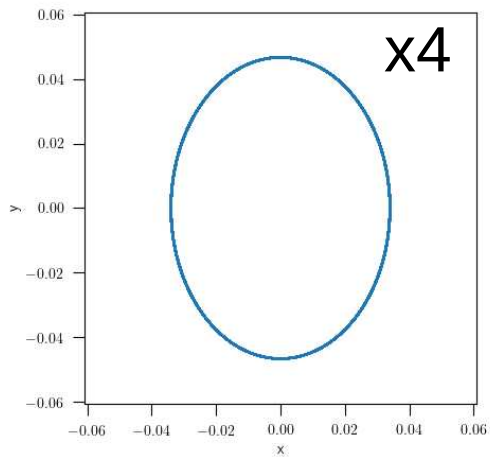
```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -2.5 --norbits 50
```

# Bifurcation : apparition of $x_2$ (stable)/ $x_3$ (unstable) orbits



```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -2.5 --x 0.0268
```

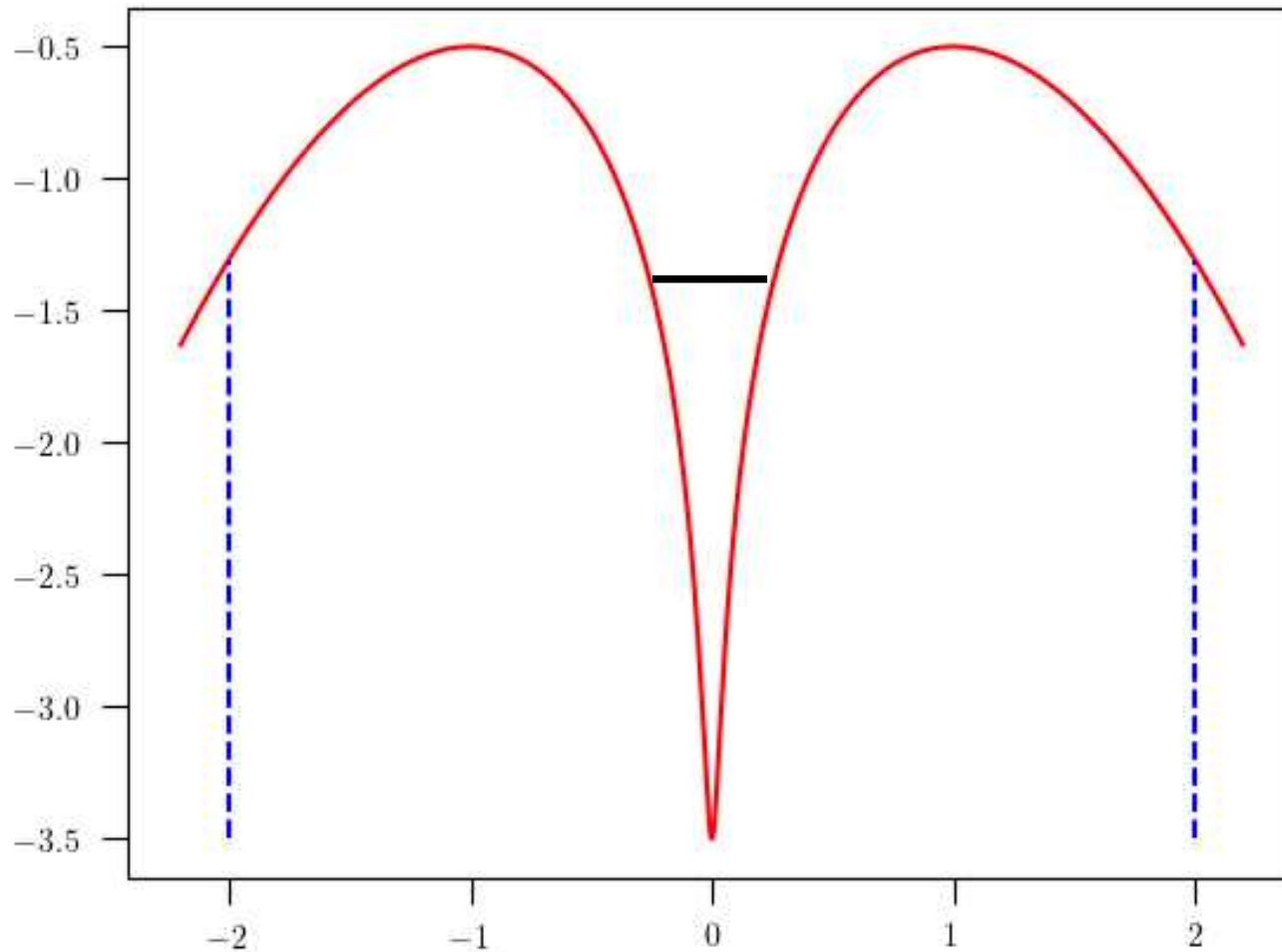
x1 : prograde x4 : retrograde



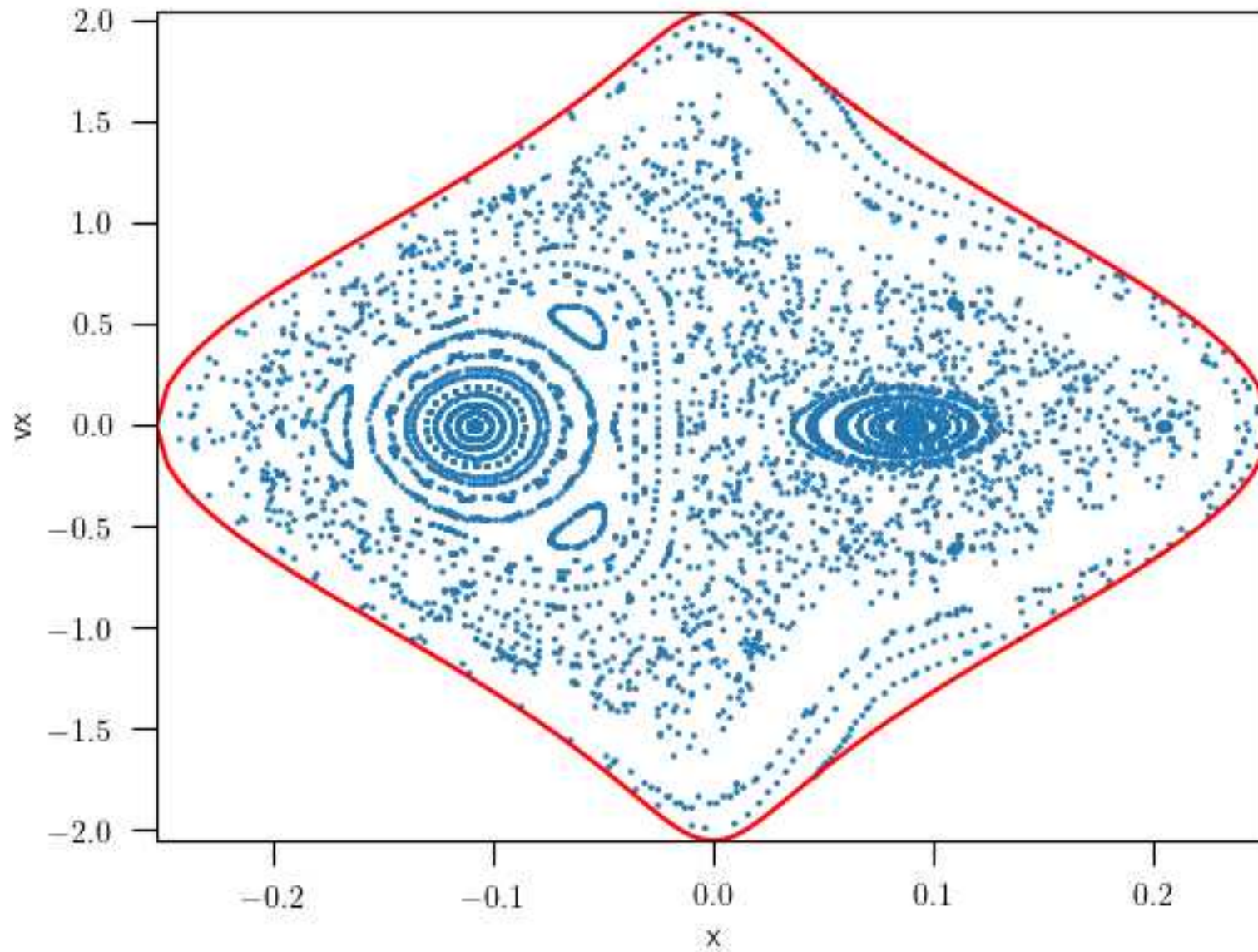
```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -2.5 --x 0.0766659  
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -2.5 --x -0.034
```

# Increasing the energy

$$E = -1.4$$

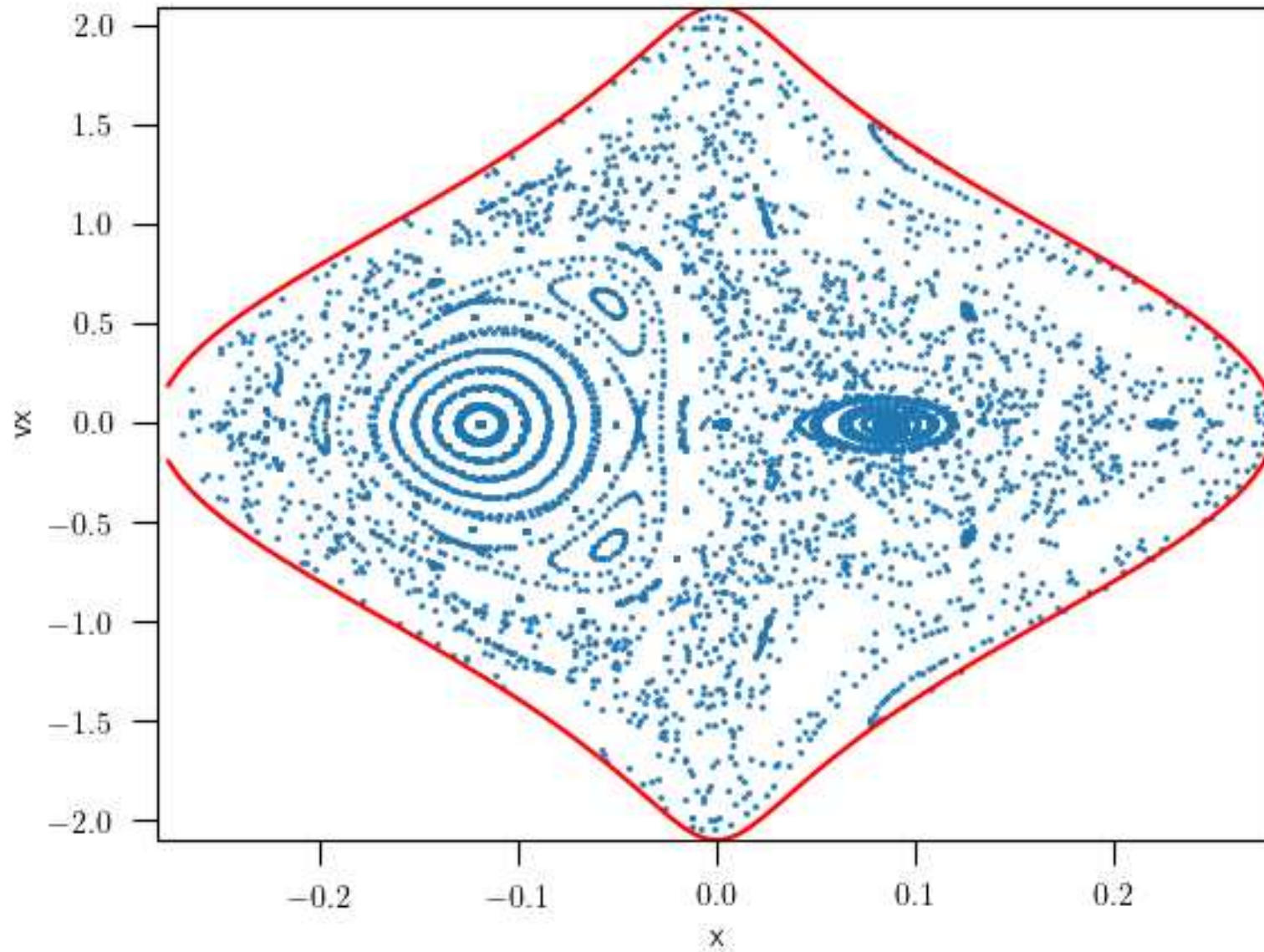


$$E = -1.4$$



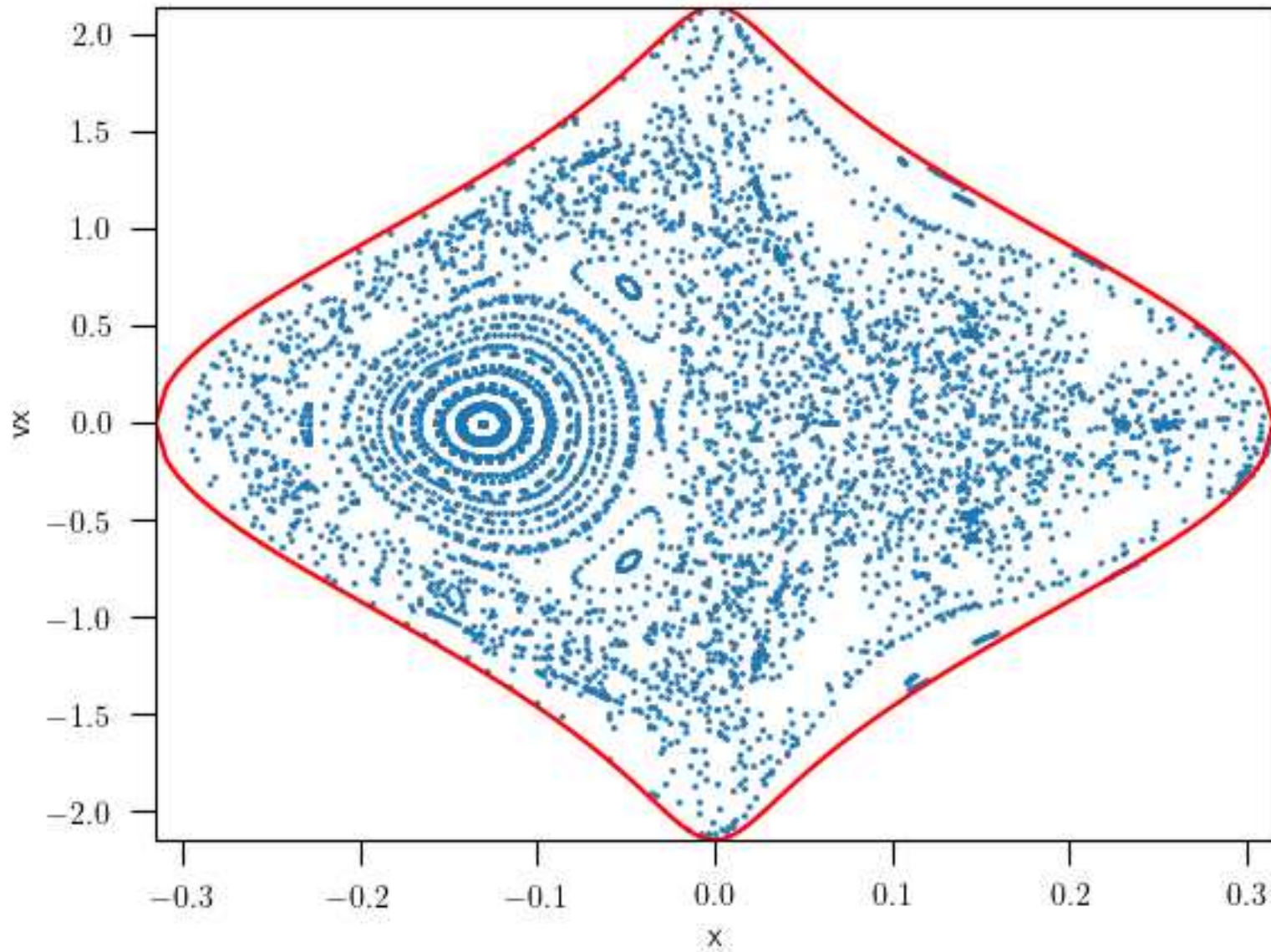
```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -1.4 --norbits 50
```

$$E = -1.3$$



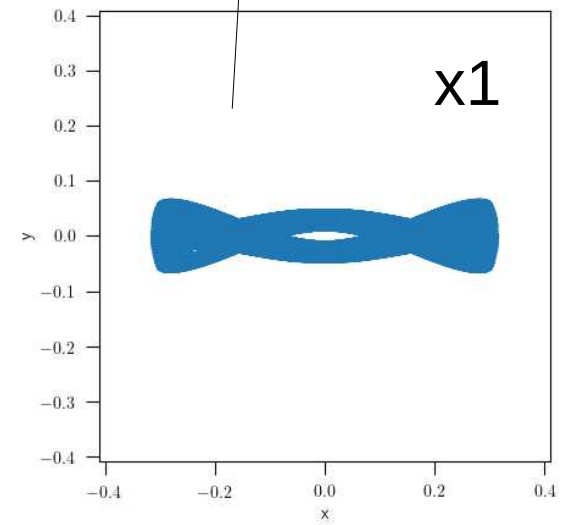
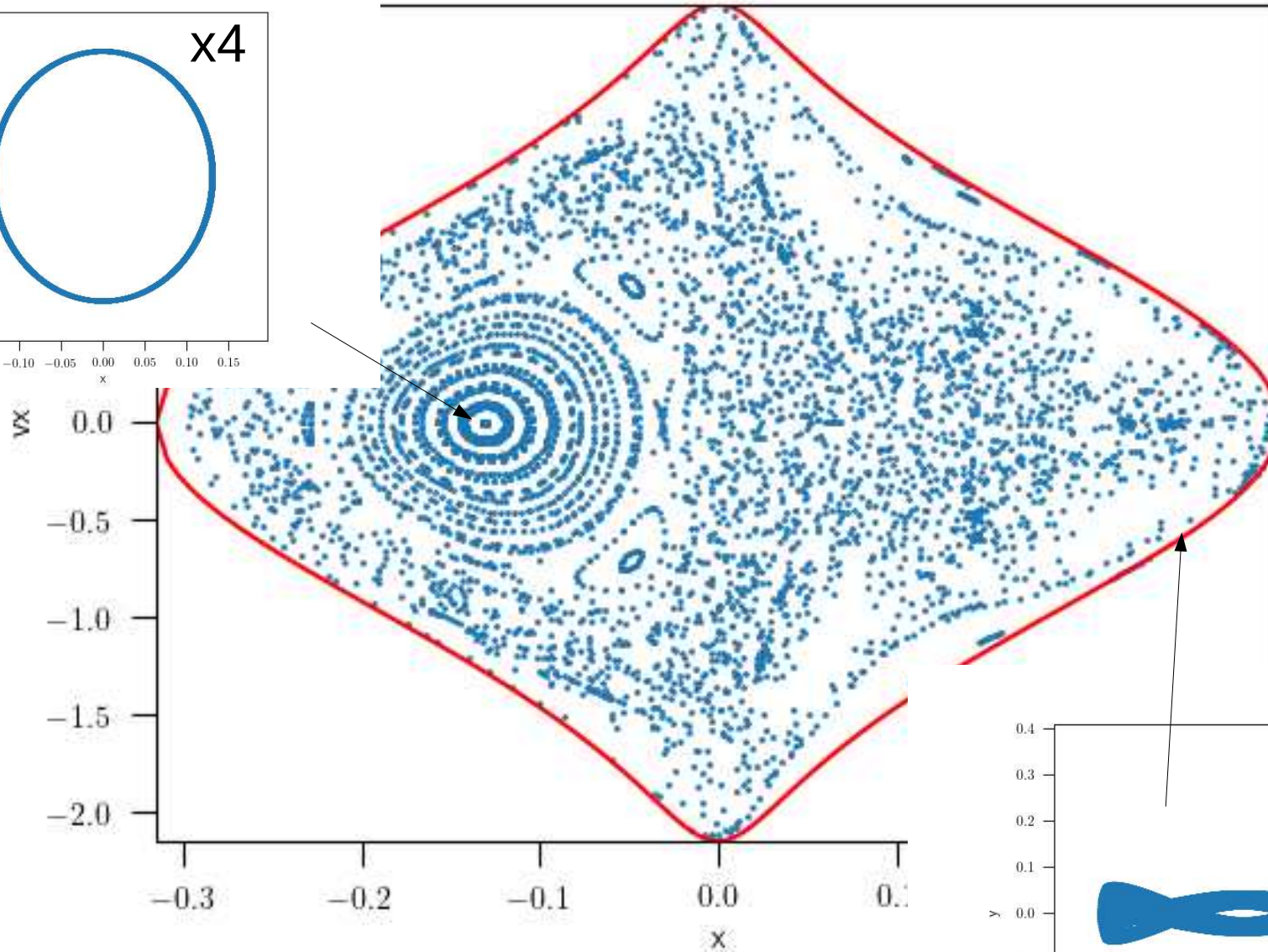
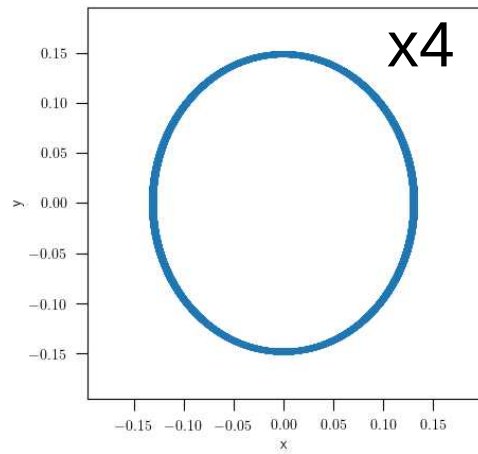
```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -1.3 --norbits 50
```

$E = -1.2$   
Bifurcation :  $x_2/x_3$  disappeared



```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -1.2 --norbits 50
```

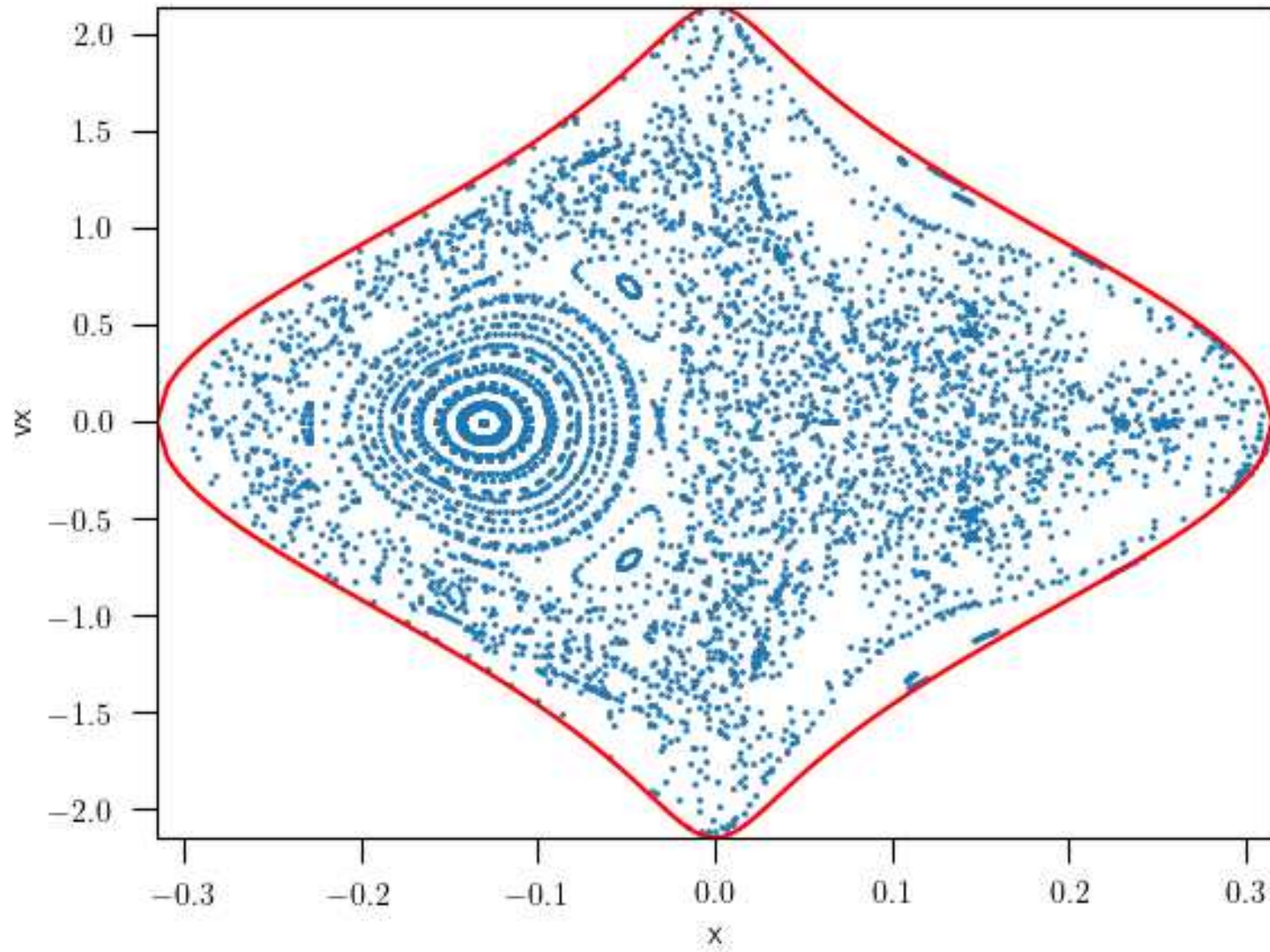
$$E = -1.2$$



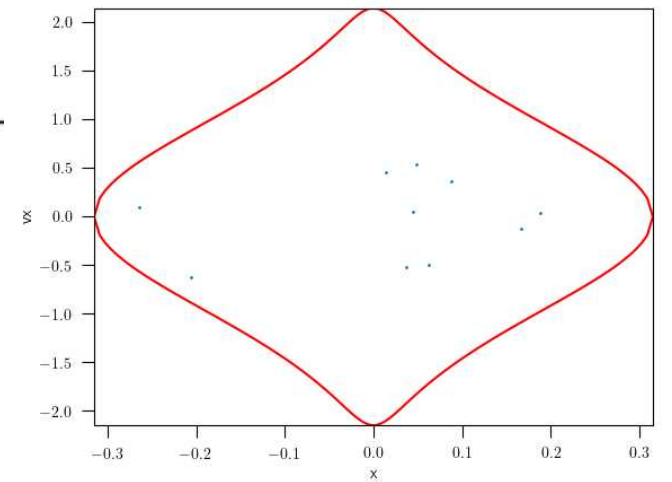
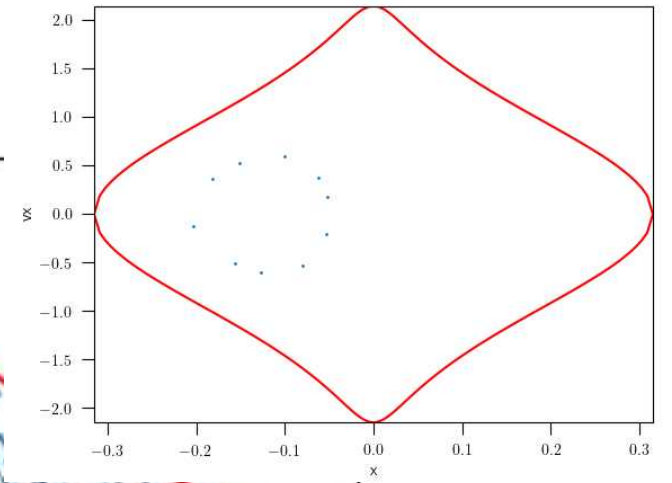
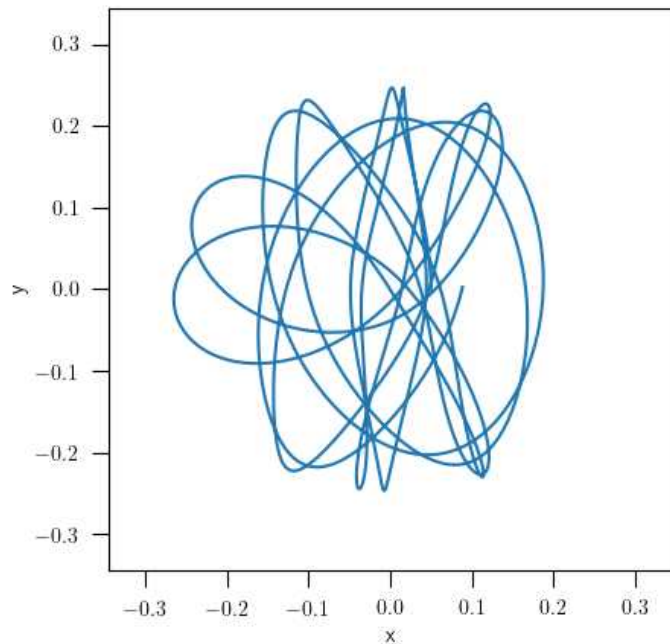
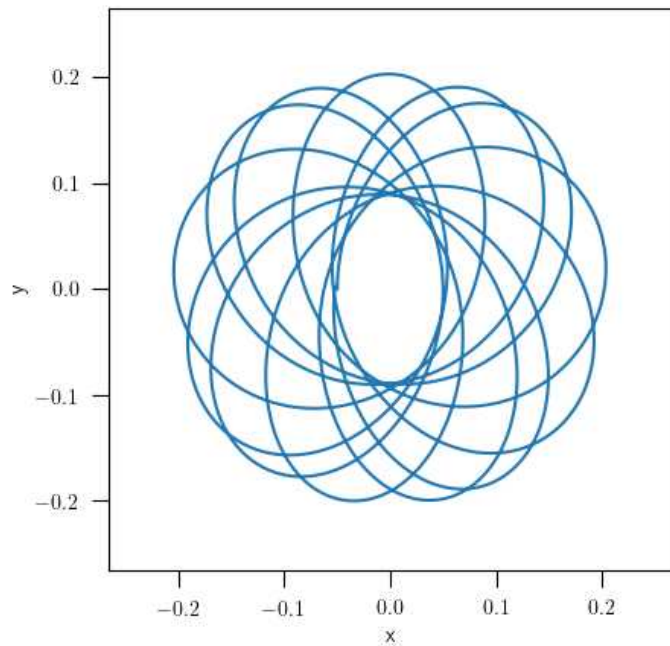
```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -1.2 --x 0.315099  
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -1.2 --x -0.1283
```

# Chaotic orbits

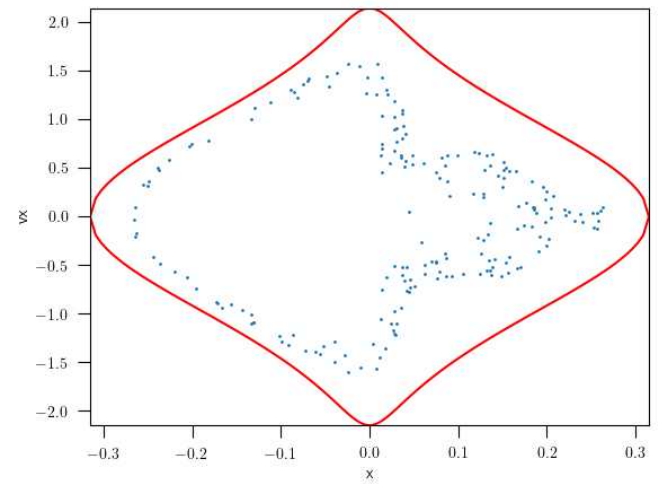
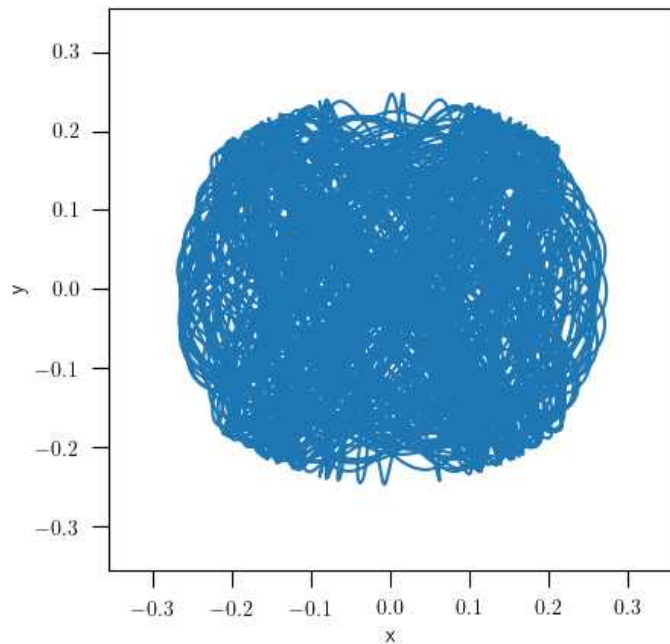
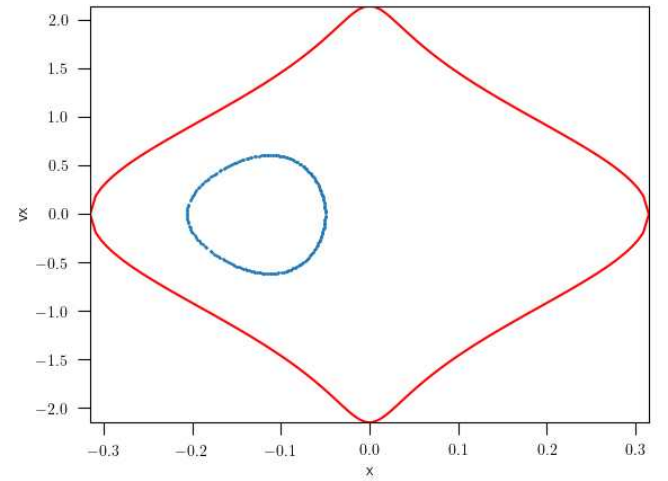
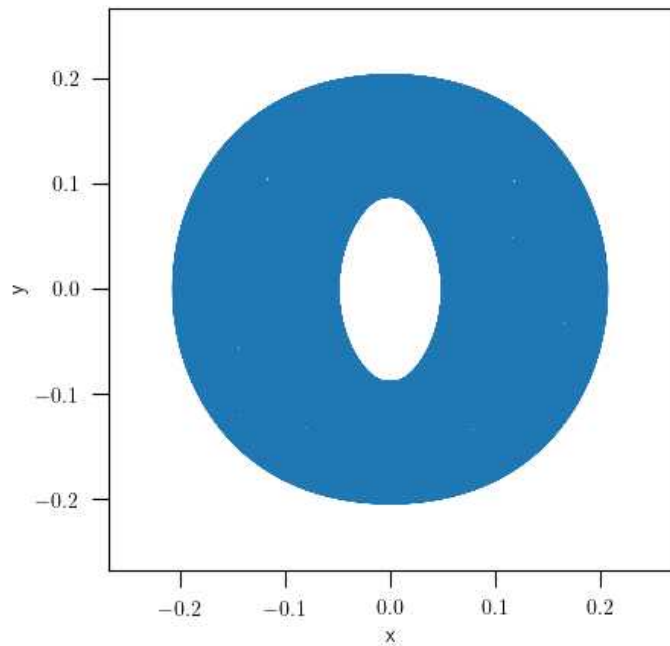
# Chaotic orbits



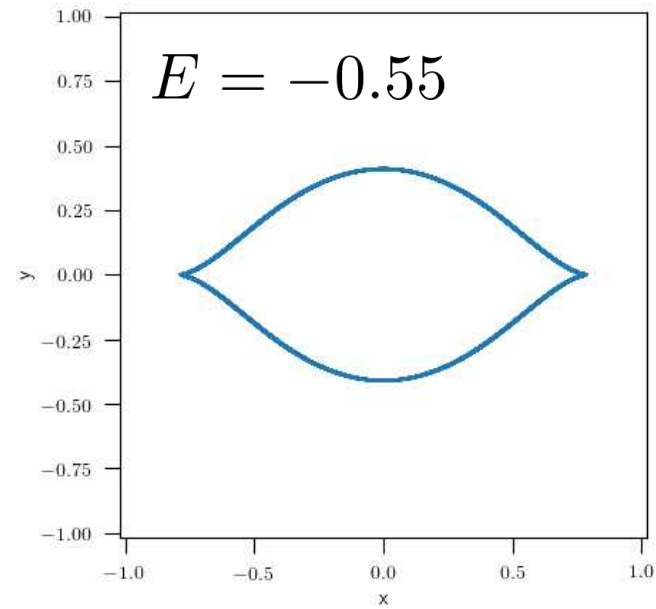
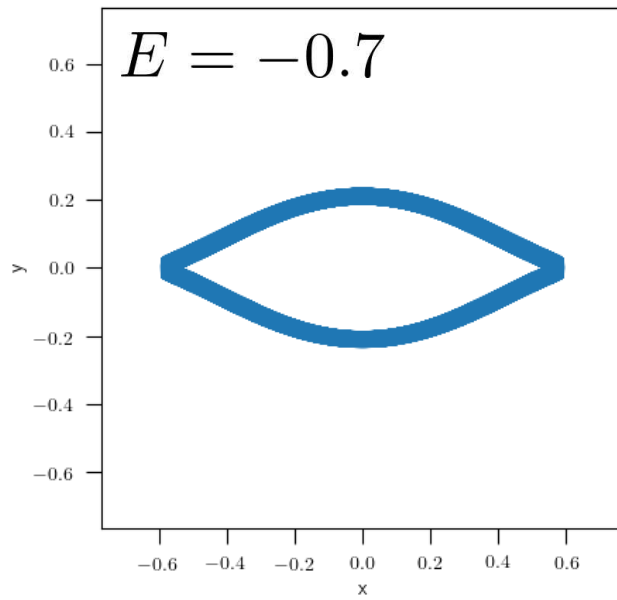
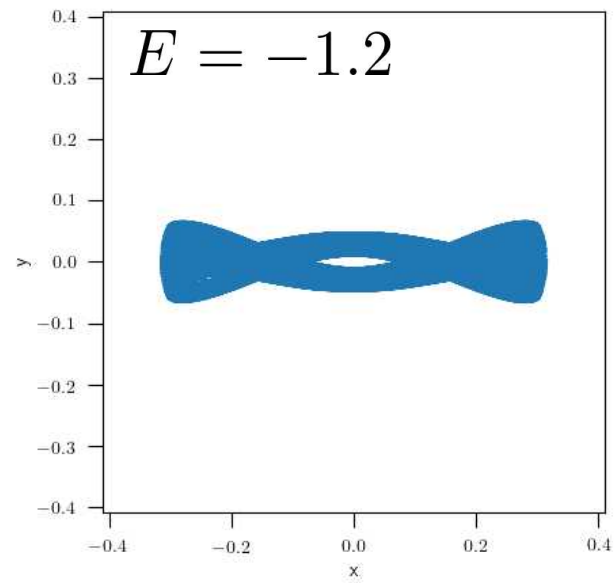
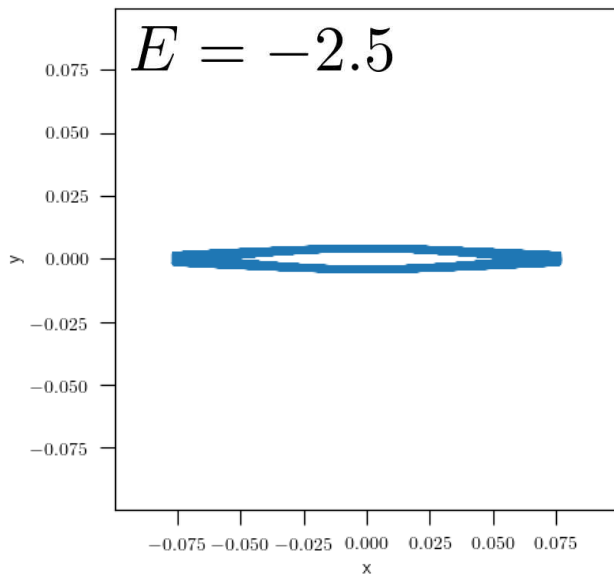
# chaotic orbits



# chaotic orbits



# Evolution of the $x_1$ orbit with increasing energy

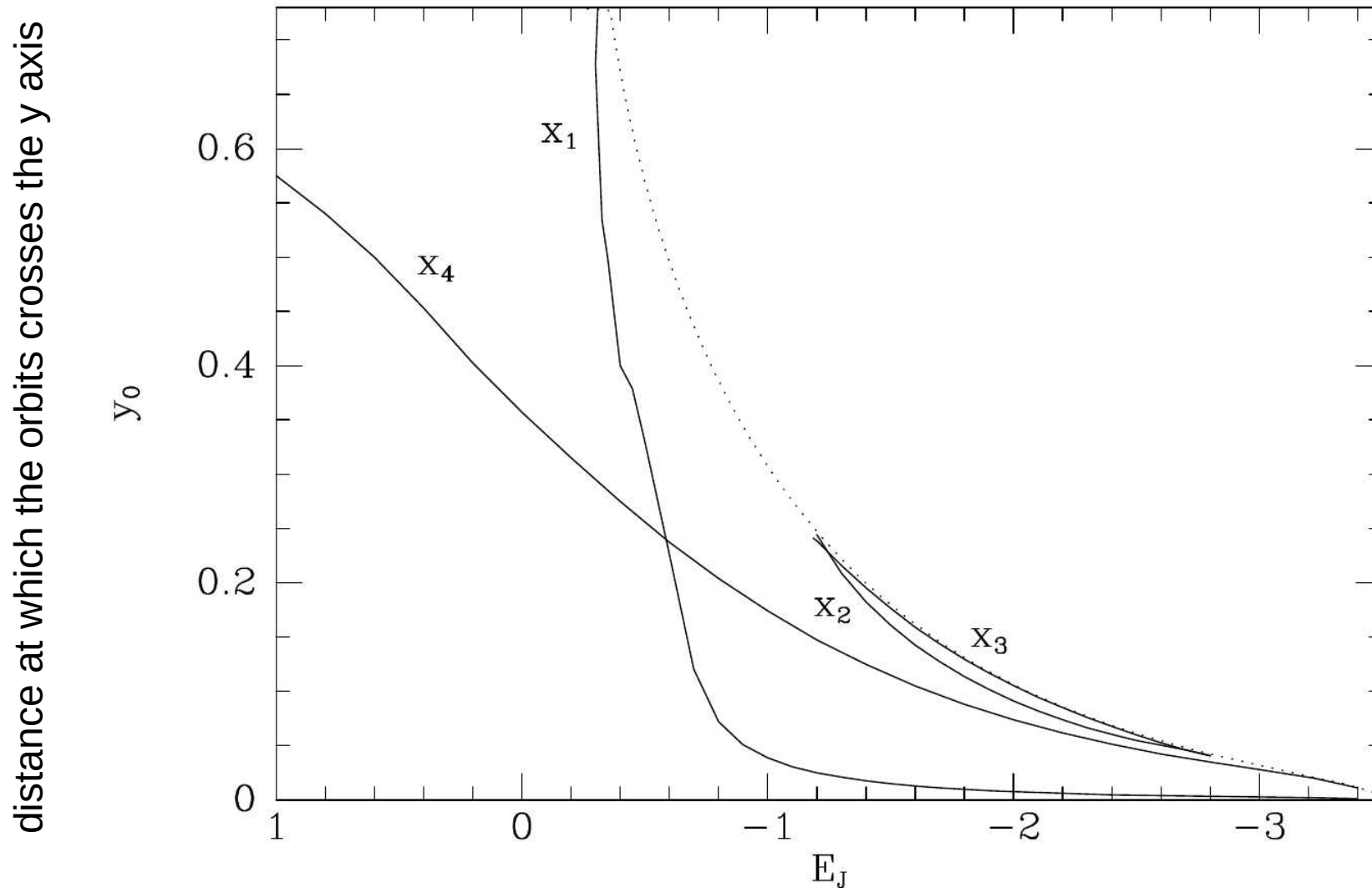


```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -2.5 --x 0.0766659
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -1.2 --x 0.315099
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -0.7 --x 0.590356
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -0.55 --x 0.783882
```

NGC 1300 SBb



### The X-orbit families (characteristics curves)



**Figure 3.18** A plot of the Jacobi constant  $E_J$  of closed orbits in  $\Phi_L(q = 0.8, R_c = 0.03, \Omega_b = 1)$  against the value of  $y$  at which the orbit cuts the potential's short axis. The dotted curve shows the relation  $\Phi_{\text{eff}}(0, y) = E_J$ . The families of orbits  $x_1$ – $x_4$  are marked.

# **Stellar Orbits**

## **Orbits in weak rotating bars**

# Objective

Analytical description of orbits in a rotating bar

- Split a loop orbit in two parts:
  - a circular motion of a guiding center
  - oscillations around the guiding center

## Orbits in weak rotating bars (planar potentials)

- the barred potential rotates with a pattern speed  $\Omega_b$

Lagrangian :

$$\mathcal{L}(\vec{x}, \dot{\vec{x}}) = \frac{1}{2} (\dot{\vec{x}} + \vec{\Omega}_b \times \vec{x})^2 - \phi(\vec{x})$$

In 2-D, with  $\vec{\Omega}_b = \Omega_b \vec{e}_z$

$$\mathcal{L}(x, y, \dot{x}, \dot{y}) = \frac{1}{2} (\dot{x} - y \Omega_b)^2 + \frac{1}{2} (\dot{y} + x \Omega_b)^2 - \phi(x, y)$$

In cylindrical coordinates

$$\mathcal{L}(R, \varphi, \dot{R}, \dot{\varphi}) = \frac{1}{2} \dot{R}^2 + \frac{1}{2} (R(\dot{\varphi} + \Omega_b))^2 - \phi(R, \varphi)$$

# Equations of motion in cylindrical coordinates (Euler-Lagrange)

$$\begin{cases} \ddot{R} &= R (\dot{\varphi} + \Omega_b)^2 - \frac{\partial \phi}{\partial R} \\ \frac{d}{dt} (R^2 (\dot{\varphi} + \Omega_b)) &= - \frac{\partial \phi}{\partial \varphi} \end{cases}$$

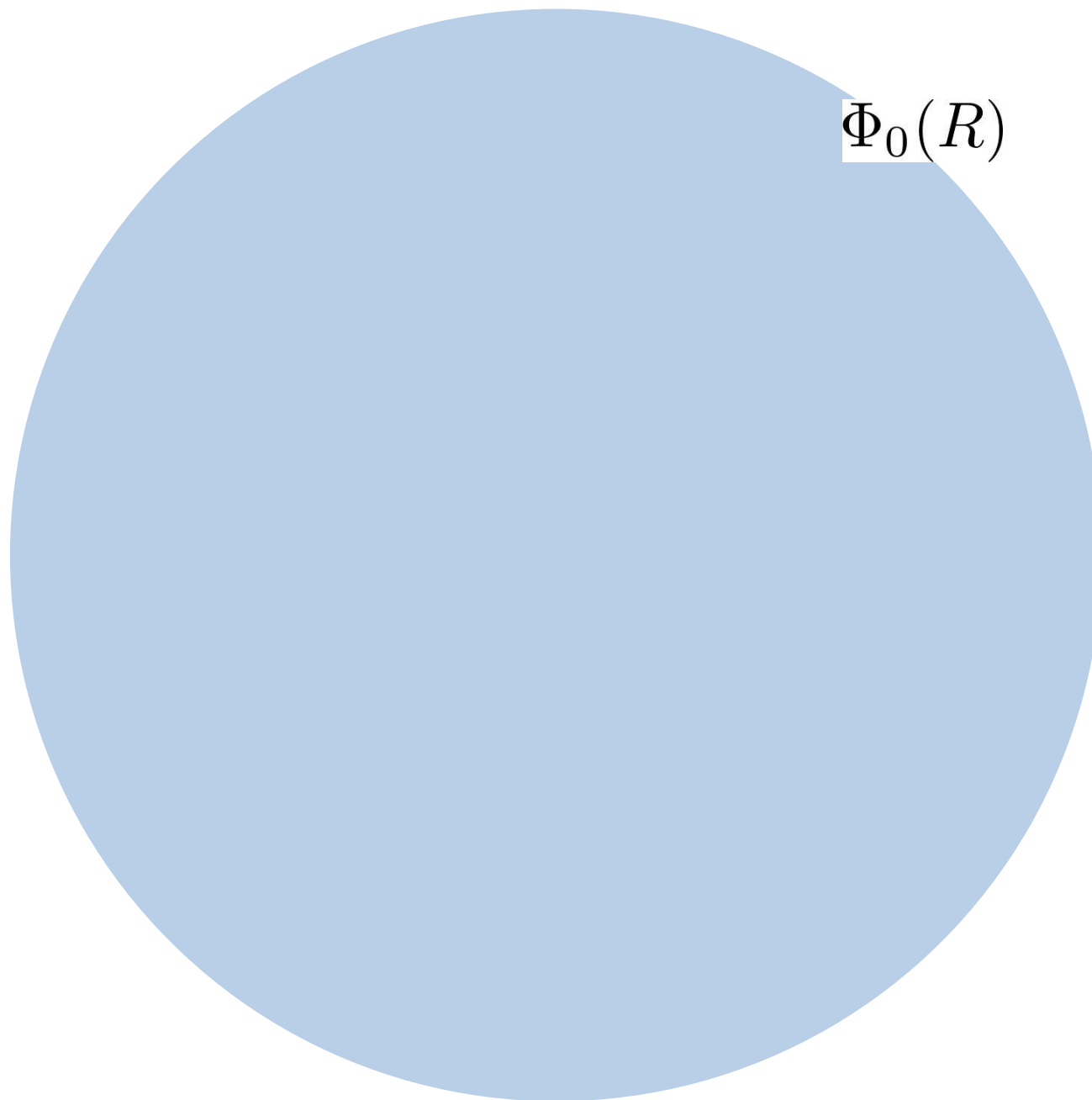
## Assumptions

① A weak perturbation :  $\phi(R, \varphi) = \underbrace{\phi_0(R)}_{\text{cyl. symmetry}} + \underbrace{\phi_1(R, \varphi)}_{\text{perturbation}} \quad \frac{|\phi_1|}{|\phi_0|} \ll 1$

$$\phi_1(R, \varphi) = \underbrace{\phi_b(R)}_{\text{radial dependency}} \underbrace{\cos(m\varphi)}_{\text{azimuthal dependency}}$$

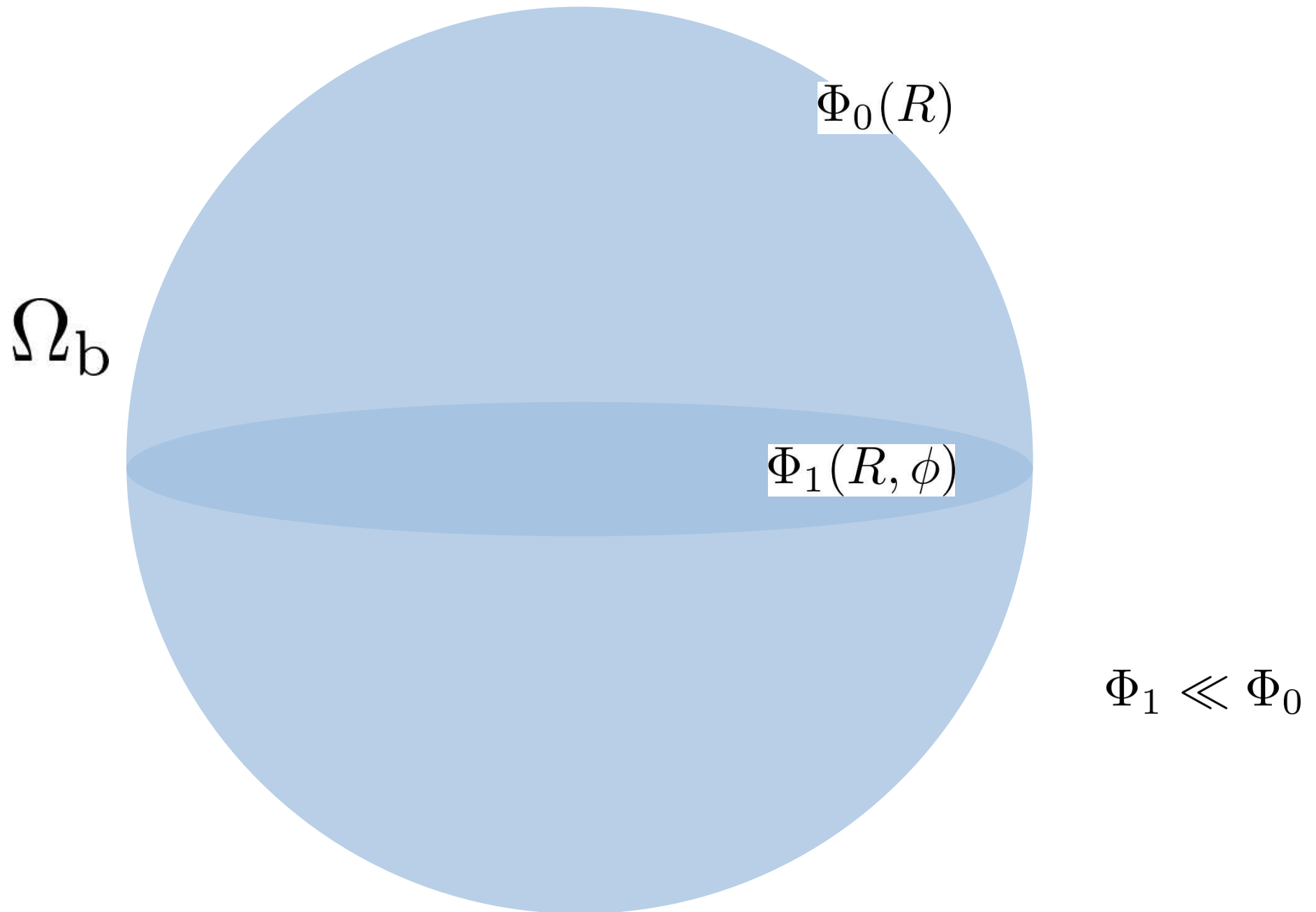
$m$  : perturbation mode

# The weakly-bared galaxy model

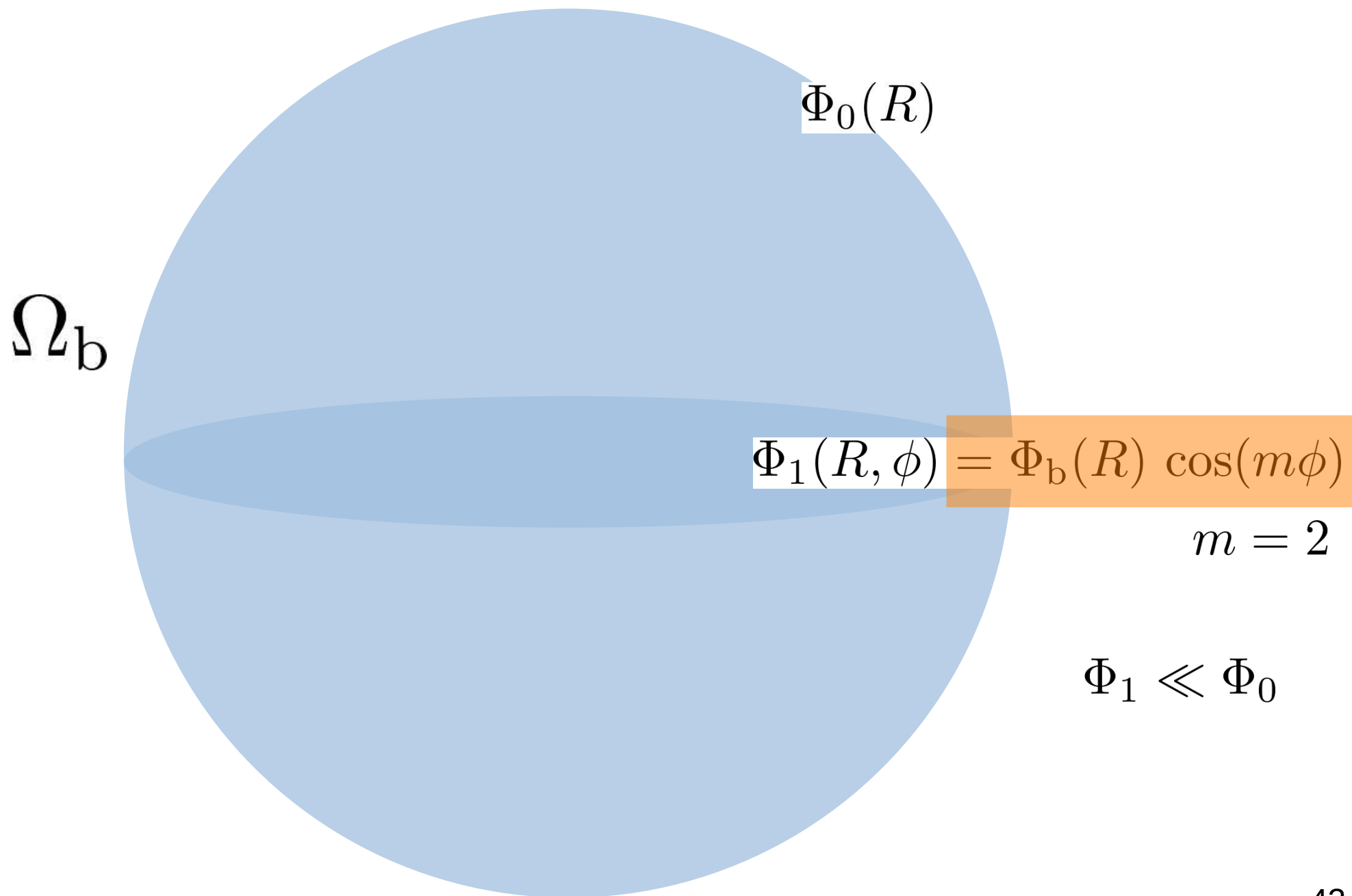


$$\Phi_0(R)$$

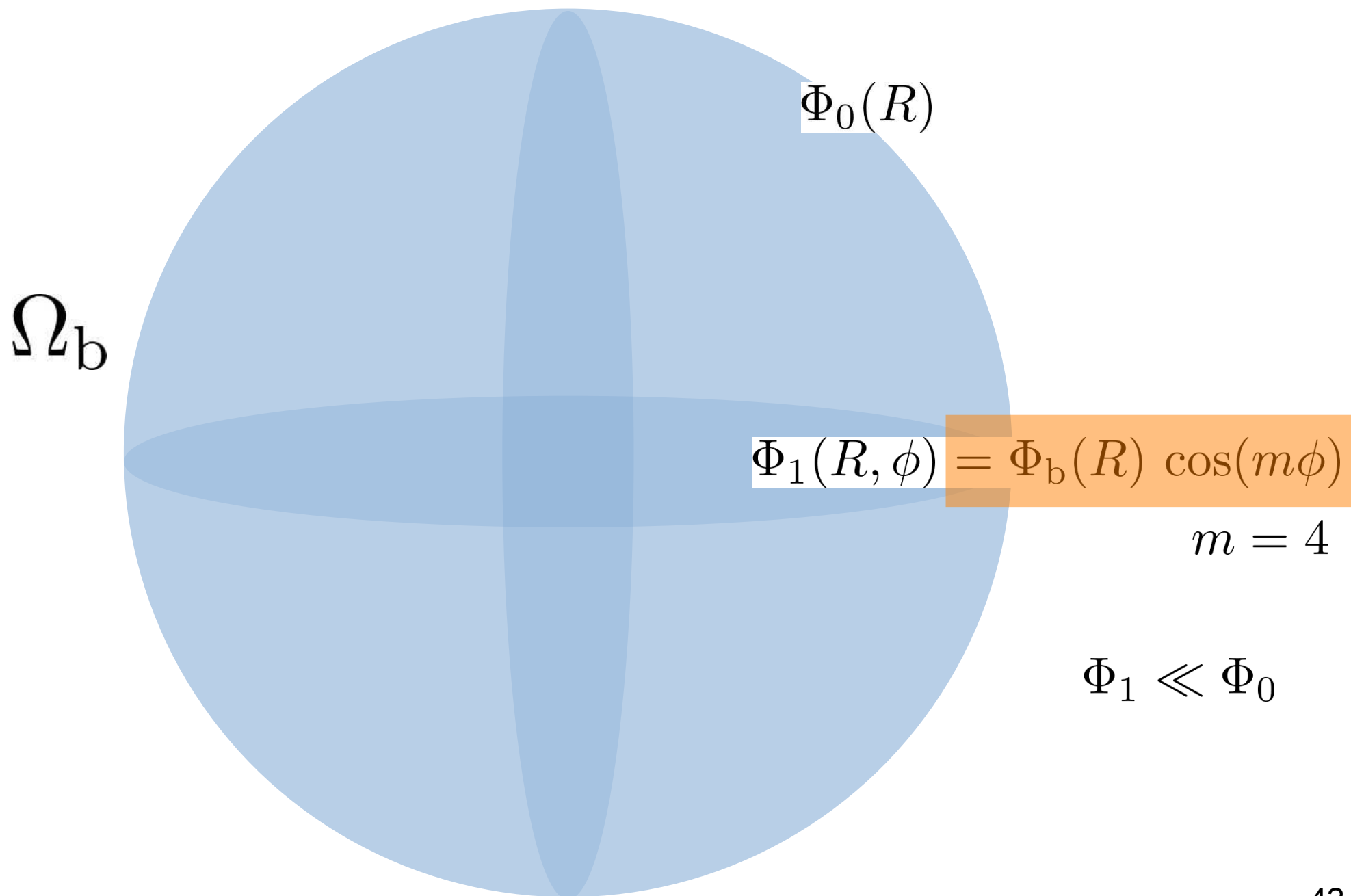
# The weakly-bared galaxy model



# The weakly-bared galaxy model



# The weakly-bared galaxy model



## Assumptions

② The motion may be decomposed into two parts

1) circular motion

2) perturbation

$$\begin{cases} R(t) = R_0(t) + R_1(t) & R_1 \ll R_0 \\ \varphi(t) = \varphi_0(t) + \varphi_1(t) & \varphi_1 \ll \varphi_0 \end{cases}$$

### Note

$$\begin{cases} R_0(t) = R_0 & (R_0 = \text{radius of the guiding center}) \\ \varphi_0(t) = (\Omega_0 - \Omega_b) t & (\Omega_0 = \text{circular frequency}) \end{cases}$$

# Solution of the EOM (2<sup>nd</sup> order terms)

EXERCICE

## Radial motion

$$R_n(\varphi_0) = C_1 \cos\left(\frac{\omega_e \varphi_0}{\Omega_0 - \omega_b} + d\right) - \left[ \frac{d\phi_b}{dR} + \frac{2\Omega R \phi_b}{R(\Omega - \Omega_b)} \right]_{R_0} \frac{\cos(m \varphi_0)}{\omega_e^2 - m^2(\Omega_0 - \Omega_b)^2}$$

$C_1, d$  : arbitrary constants  
 $\omega_b$  : radial epicycle frequency

## Azimuthal motion

$$\dot{\varphi}_n(t) = -2\Omega_0 \frac{R_n}{R_0} - \frac{\phi_b(R_0)}{R_0^2 (\Omega_0 - \Omega_b)} \cos\left(m(\Omega_0 - \Omega_b)t\right) + cte$$

Discussion

$$R_1(\varphi_0) = C_1 \cos\left(\frac{\omega_0 \varphi_0}{\Omega_0 - \Omega_1} + \alpha\right) - \left[ \frac{d\phi_1}{dR} + \frac{2\Omega_1 \phi_1}{R(\Omega - \Omega_1)} \right]_{R_0} \frac{\cos(m \varphi_0)}{\omega_0^2 - m^2(\Omega_0 - \Omega_1)^2}$$

$$\varphi_0 = (\Omega_0 - \Omega_1) t$$

Epicycles motions

① if  $\phi_1(R) = 0$  (no perturbation)

$$R_1(t) = C_1 \cos(\omega_0 t + \alpha)$$

$\equiv x(t)$  radial oscillations

$$\dot{\varphi}_1(t) = -2\Omega_0 \frac{R_1(t)}{R_0}$$

$\Rightarrow y(t)$  oscillations along the orbit

② if  $C_1 = 0$   $\phi_1 \neq 0$

$$R_1(\varphi_0) = - \left[ \frac{d\phi_1}{dR} + \frac{2\Omega_1 \phi_1}{R(\Omega - \Omega_1)} \right]_{R_0} \frac{\cos(m \varphi_0)}{\omega_0^2 - m^2(\Omega_0 - \Omega_1)^2}$$

etc

periodic in  $\varphi_0$  ( $\frac{2\pi}{m}$ )

$\Rightarrow$  closed orbit



③ if  $C_1 \neq 0$  oscillations around the closed orbit (same family)

The orbit is not necessary closed

# Resonances



two problematic terms

$$\frac{1}{\Omega_0 - \Omega_b} \quad \text{and} \quad \frac{1}{\kappa^2 - m^2(\Omega_0 - \Omega_b)^2}$$

$\Rightarrow R_1$  may diverge !

1)

$$\Omega_0 = \Omega_b$$

Corotation

we are at a radius where the circular frequency is similar to the pattern speed of the bar

as  $\dot{\varphi}_0 = \Omega_0 - \Omega_b \Rightarrow \underline{\dot{\varphi}_0 = 0}$

$\rightarrow$  stable in the rotating frame

2)

$$m(\Omega_0 - \Omega_b) = \pm \kappa$$

Lindblad resonances

freq. at which the star encounter the potential minimum

$\rightarrow$  the frequency at which a star encounter a potential minimum is similar to its radial frequency

$$\equiv \Omega_b = \Omega \pm \frac{\kappa}{m}$$

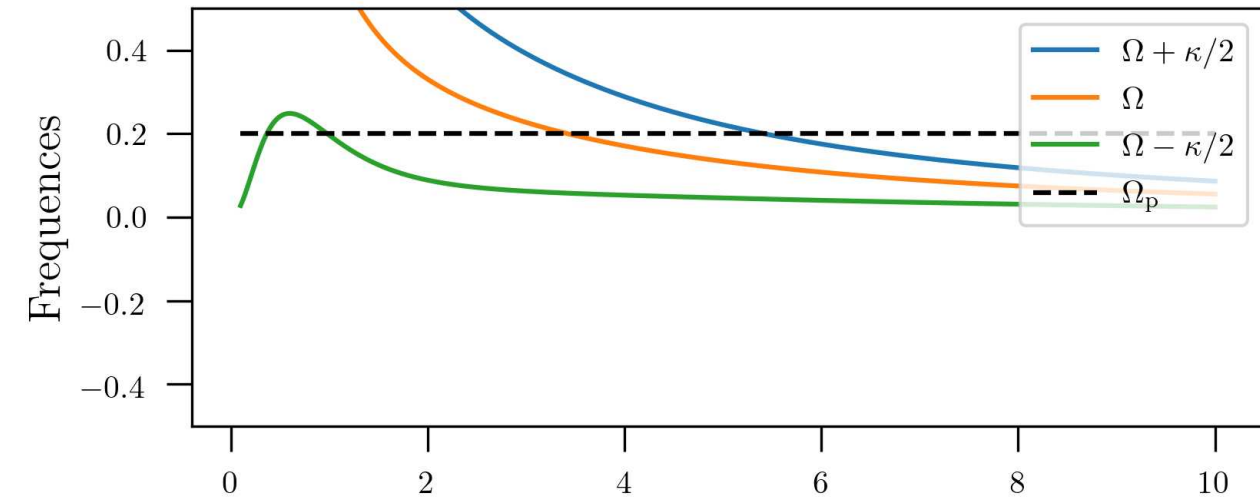
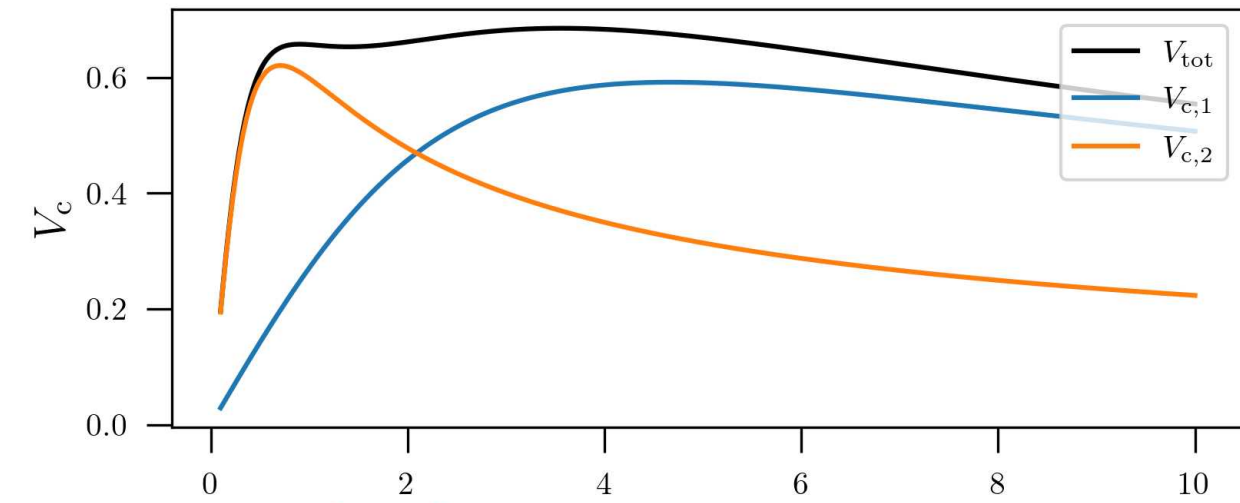
$\Rightarrow$  excitation

A circular orbit has two natural frequencies

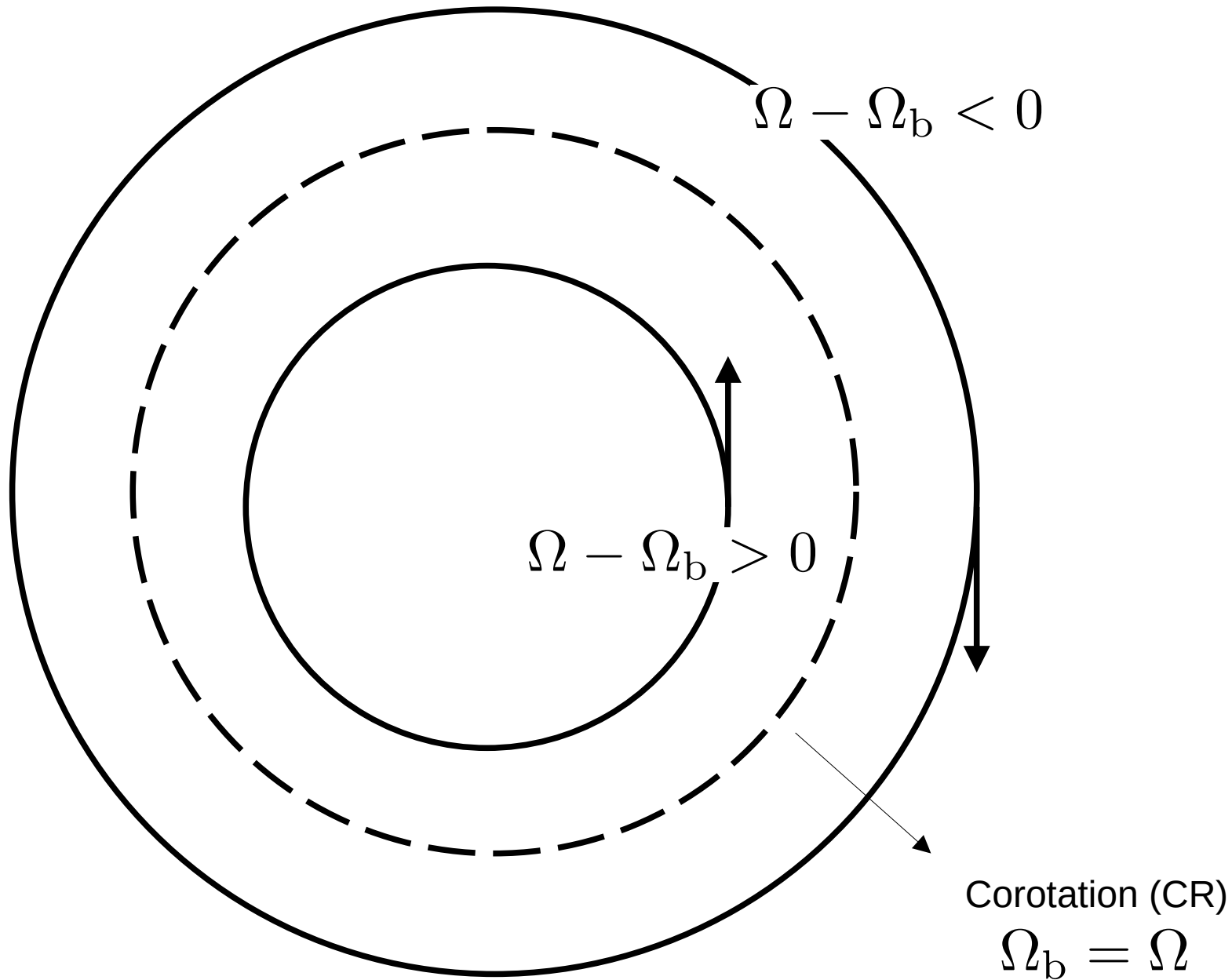
- ①  $\kappa$  : radial frequ.  $\rightarrow$
- ②  $0$  : azimuthal frequ.  $\rightarrow$   
(no change  $\Rightarrow$  frequ. = 0)

Resonances occur when the forcing frequency  $m(\Omega_0 - \Omega_b)$  is equal to one of these frequencies.

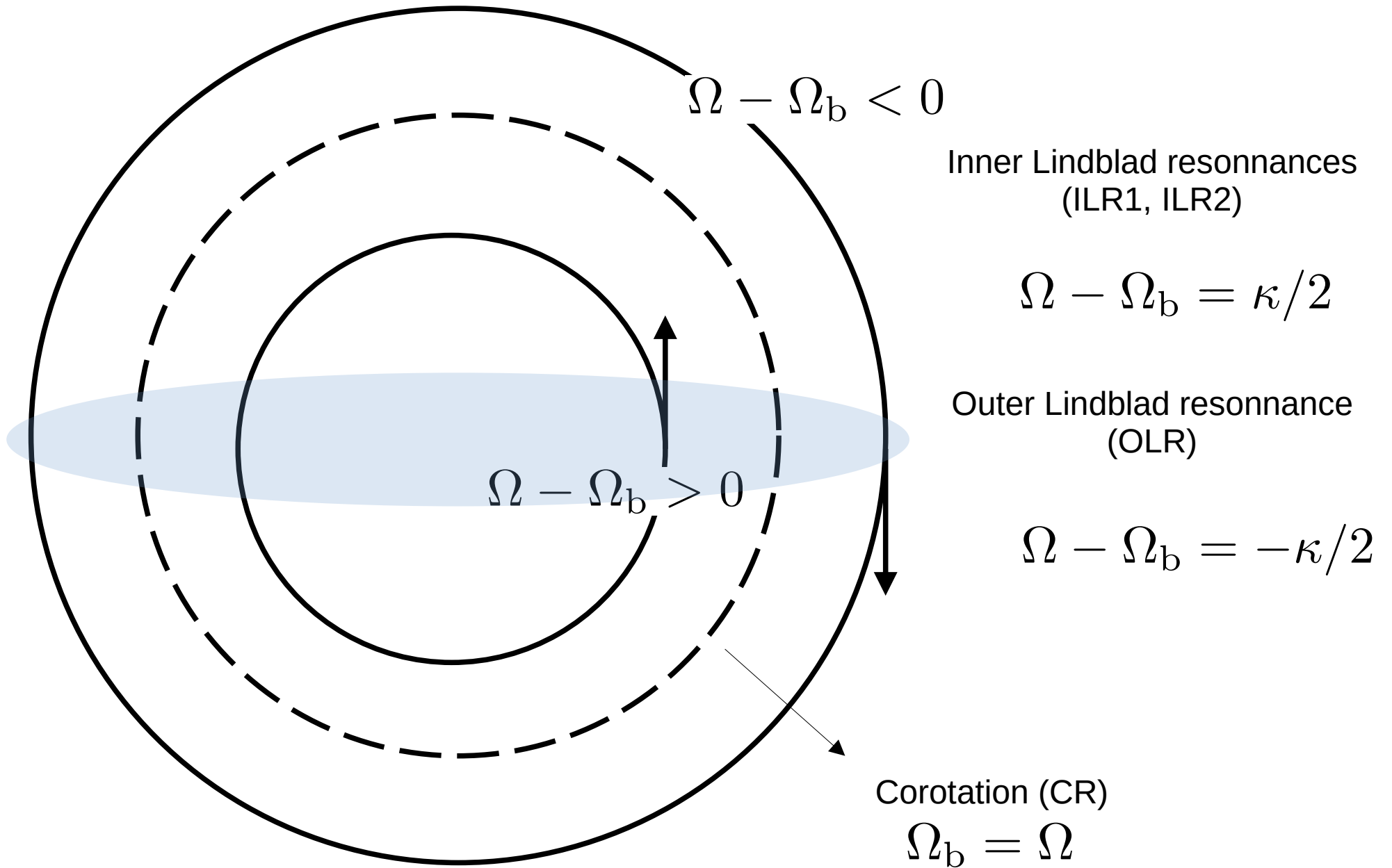
Disk : Miyamoto-Nagai  
Bulge : Plummer



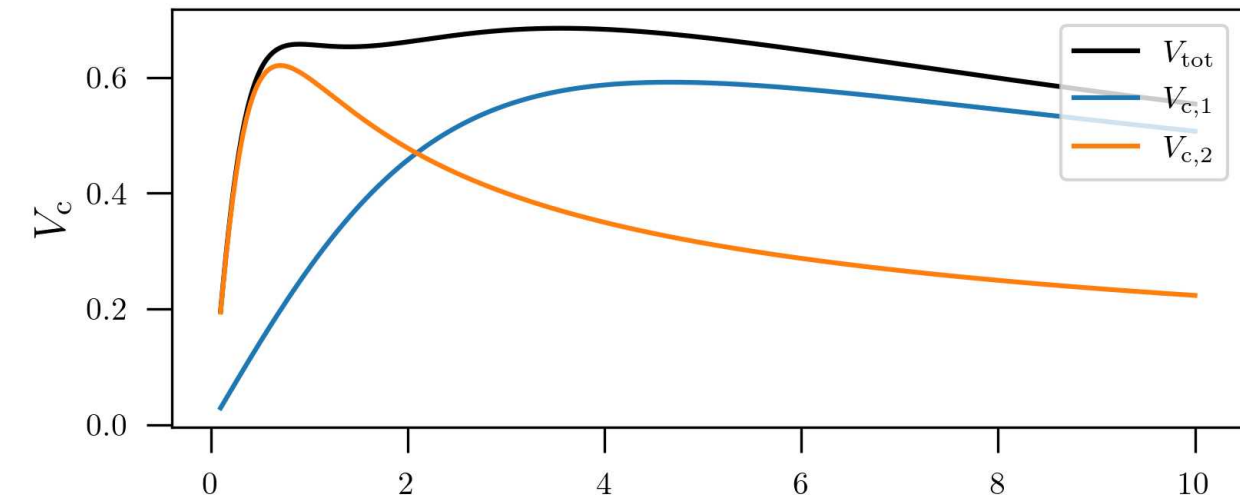
$\Omega = \Omega(R)$  is a decreasing function in a galaxy



$\Omega = \Omega(R)$  is a decreasing function in a galaxy

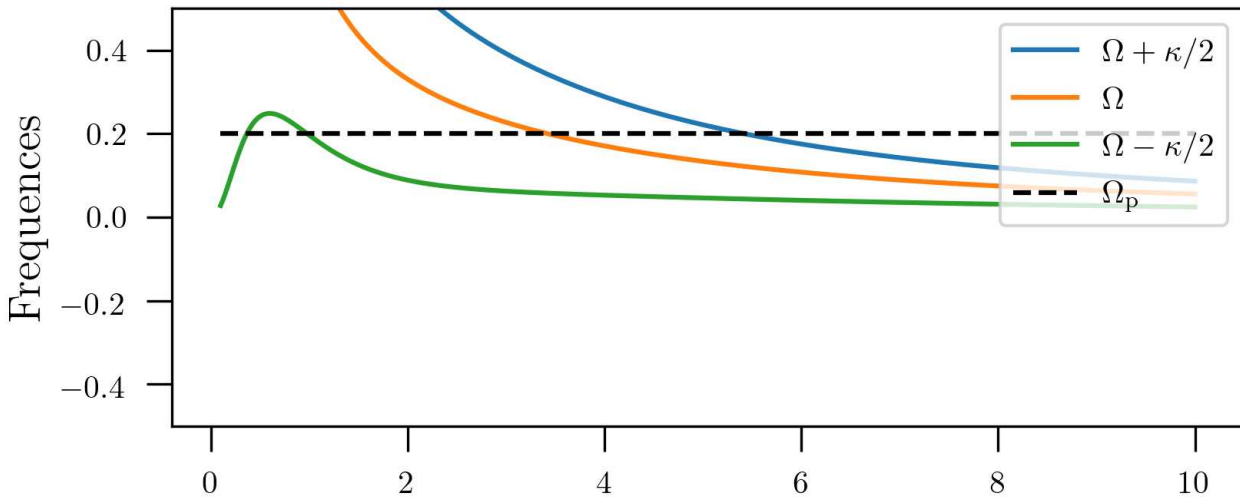


Disk : Miyamoto-Nagai  
 Bulge : Plummer



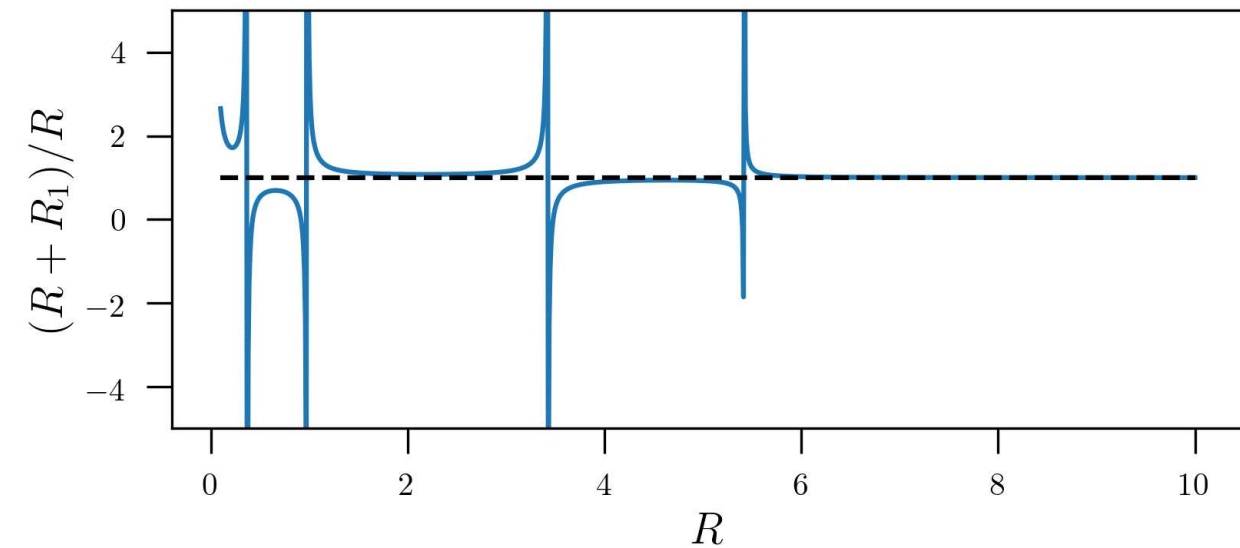
Inner Lindblad resonances  
 (ILR1, ILR2)

$$\Omega_b = \Omega - \kappa/2$$



Outer Lindblad resonance  
 (OLR)

$$\Omega_b = \Omega + \kappa/2$$

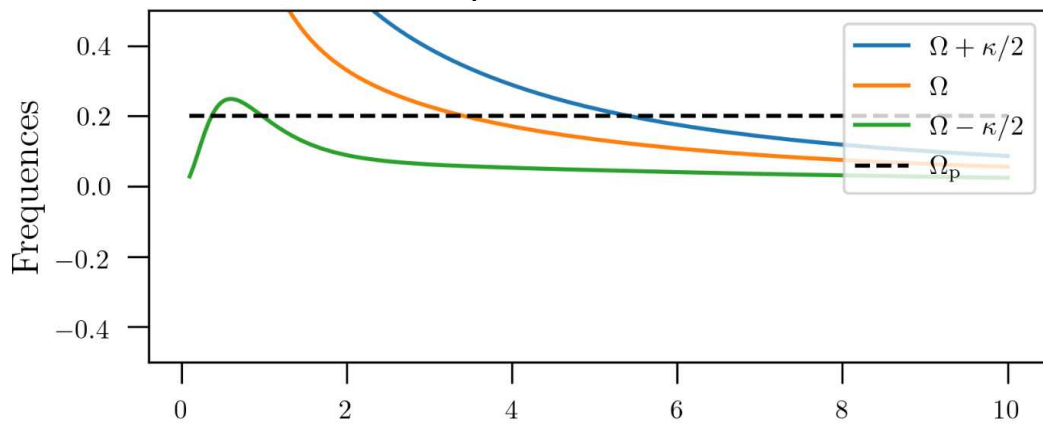
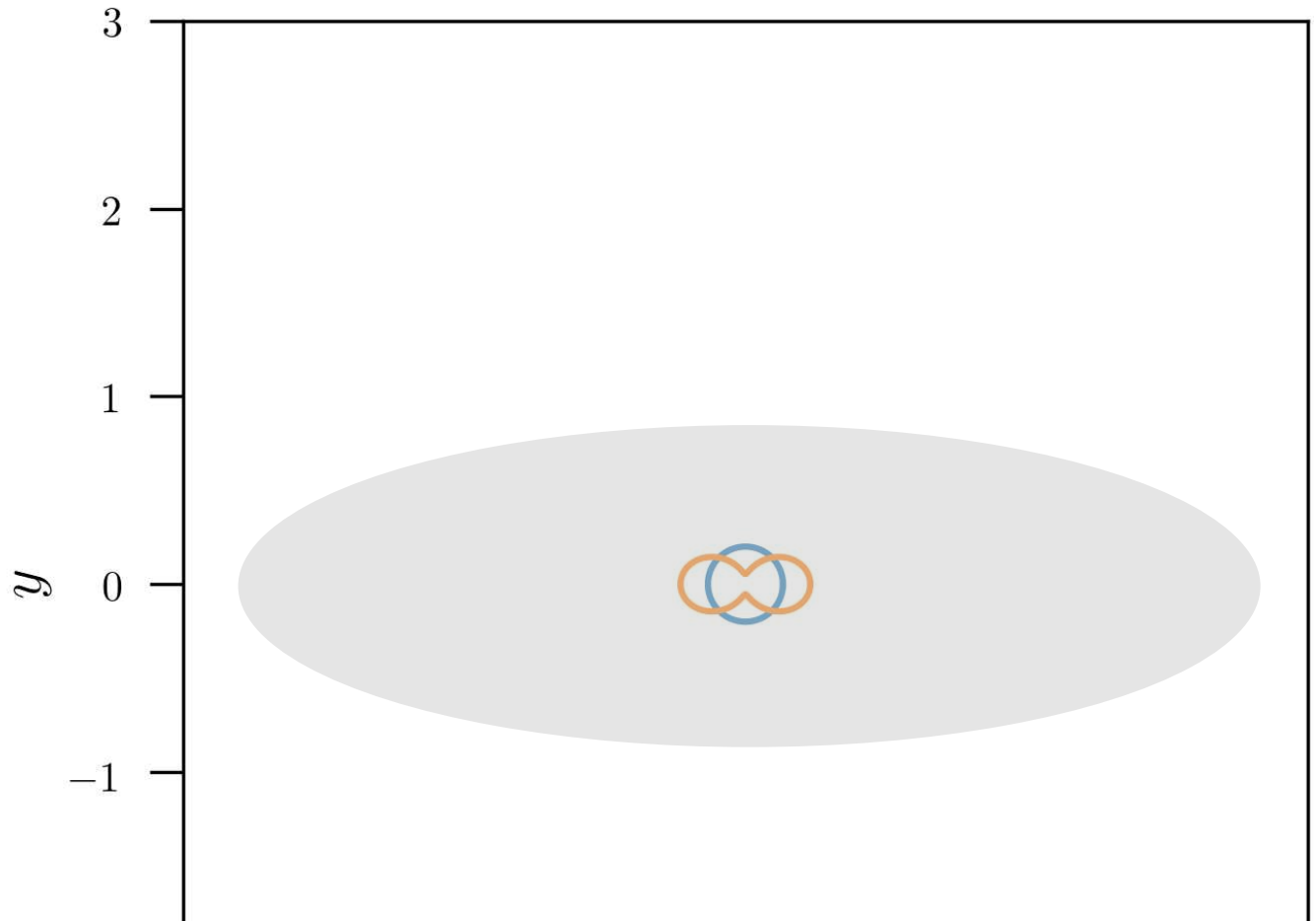


Corotation (CR)

$$\Omega_b = \Omega$$

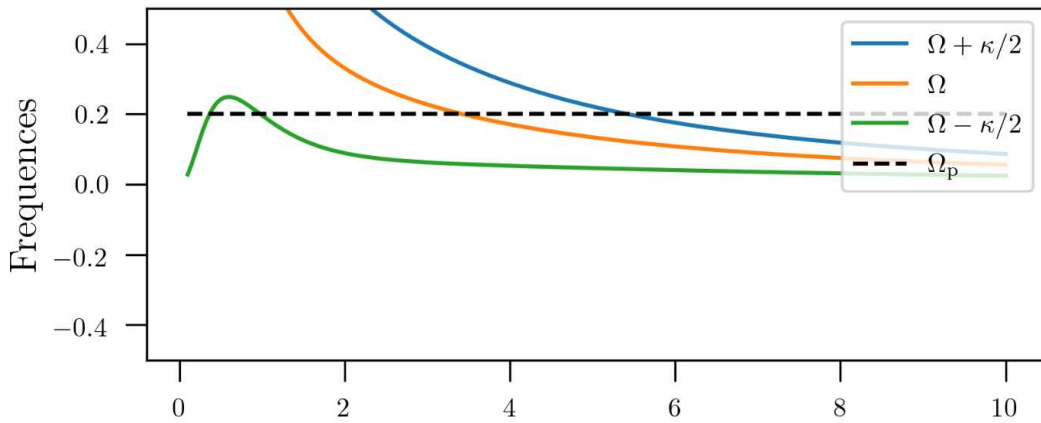
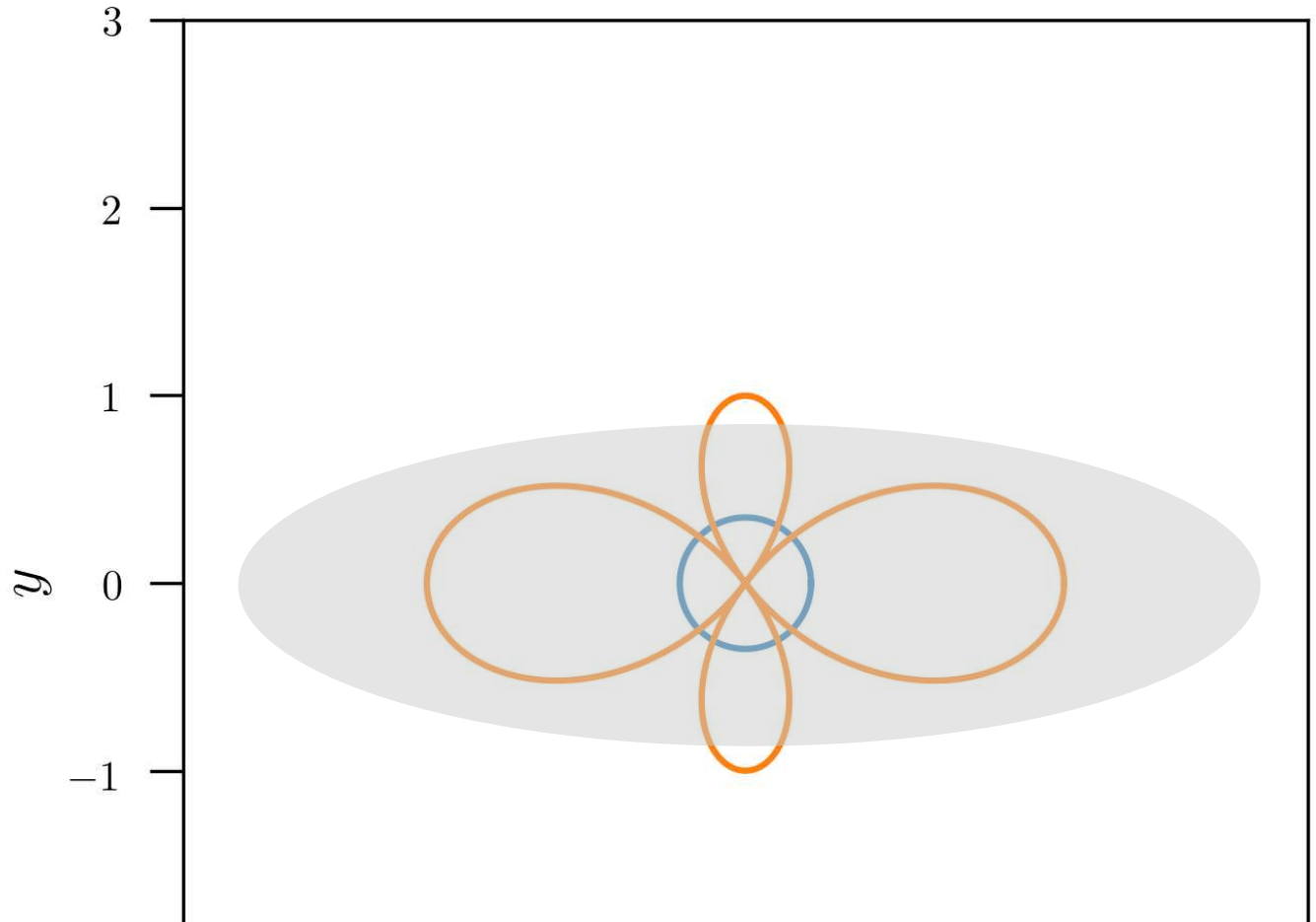
$R = 0.2$

$R < R_{\text{ILR1}}$



$R = 0.3$

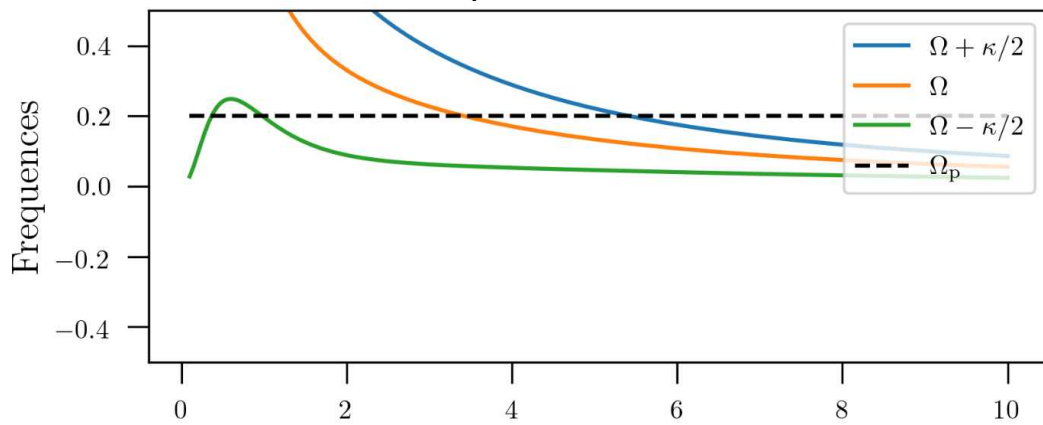
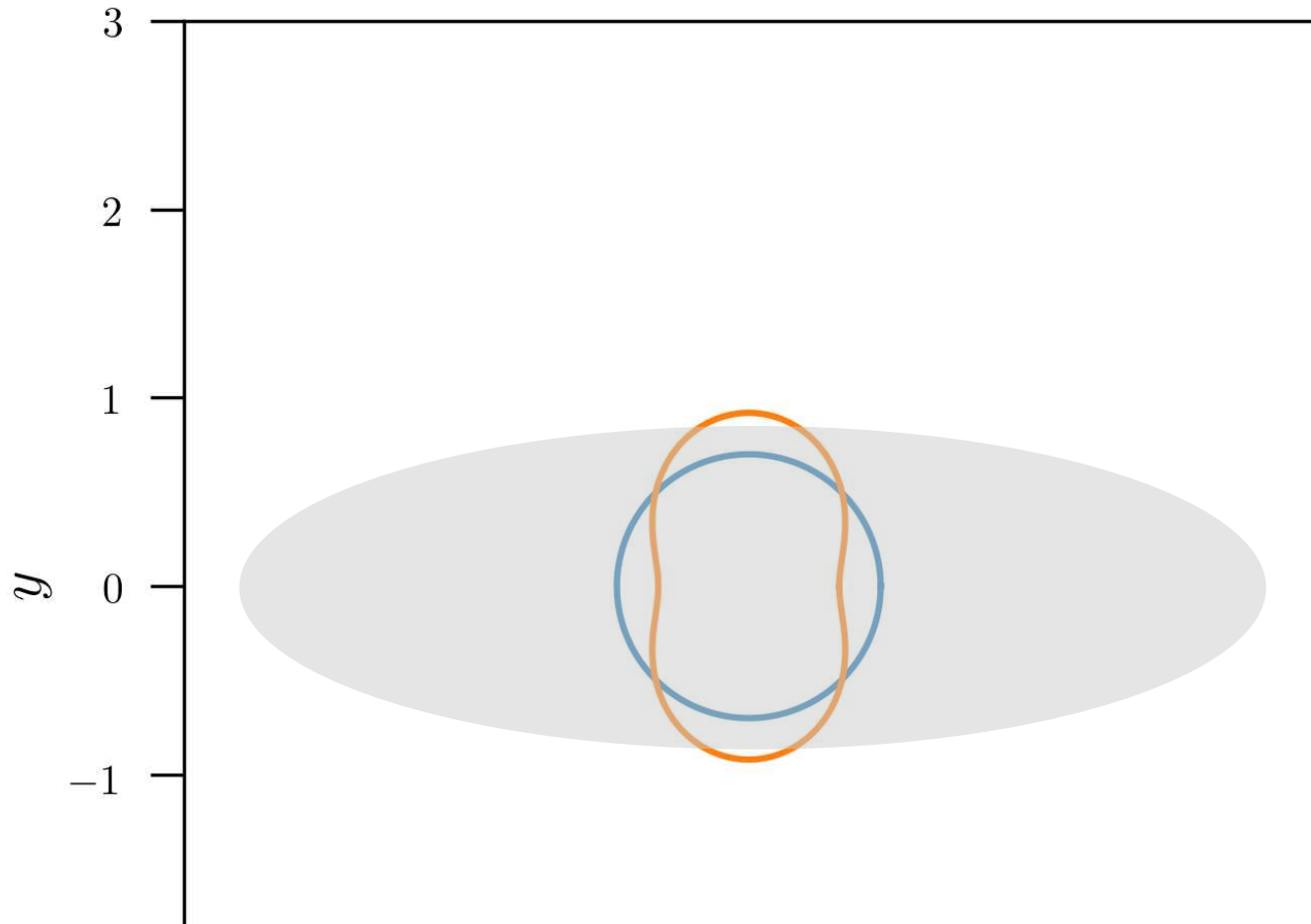
$R \cong R_{\text{ILR1}}$



1 2 3

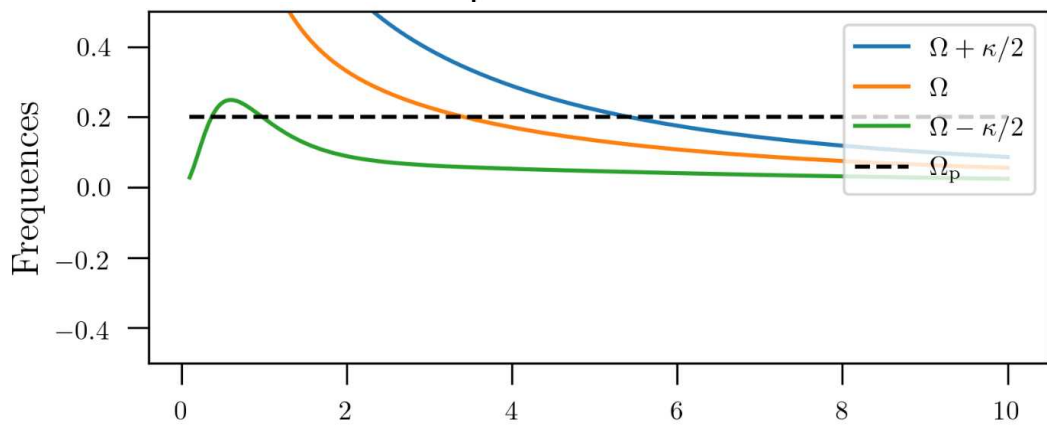
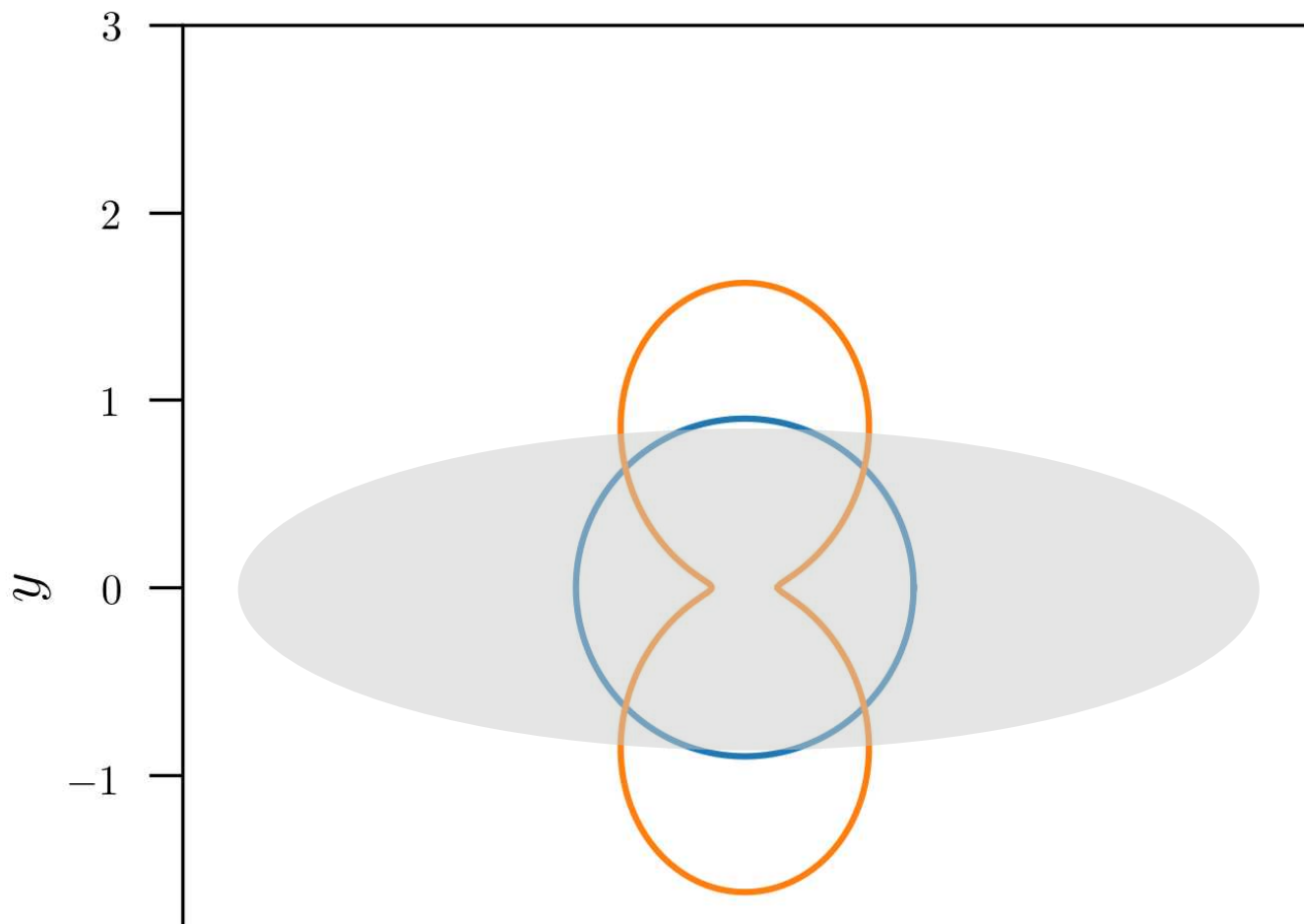
$$R = 0.7$$

$$R_{\text{ILR1}} < R < R_{\text{ILR2}}$$



$R = 0.9$

$R \cong R_{\text{ILR2}}$



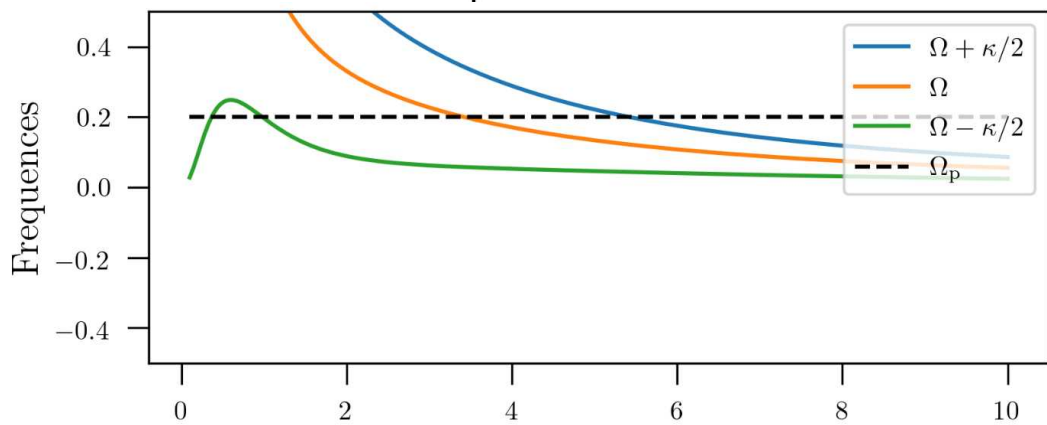
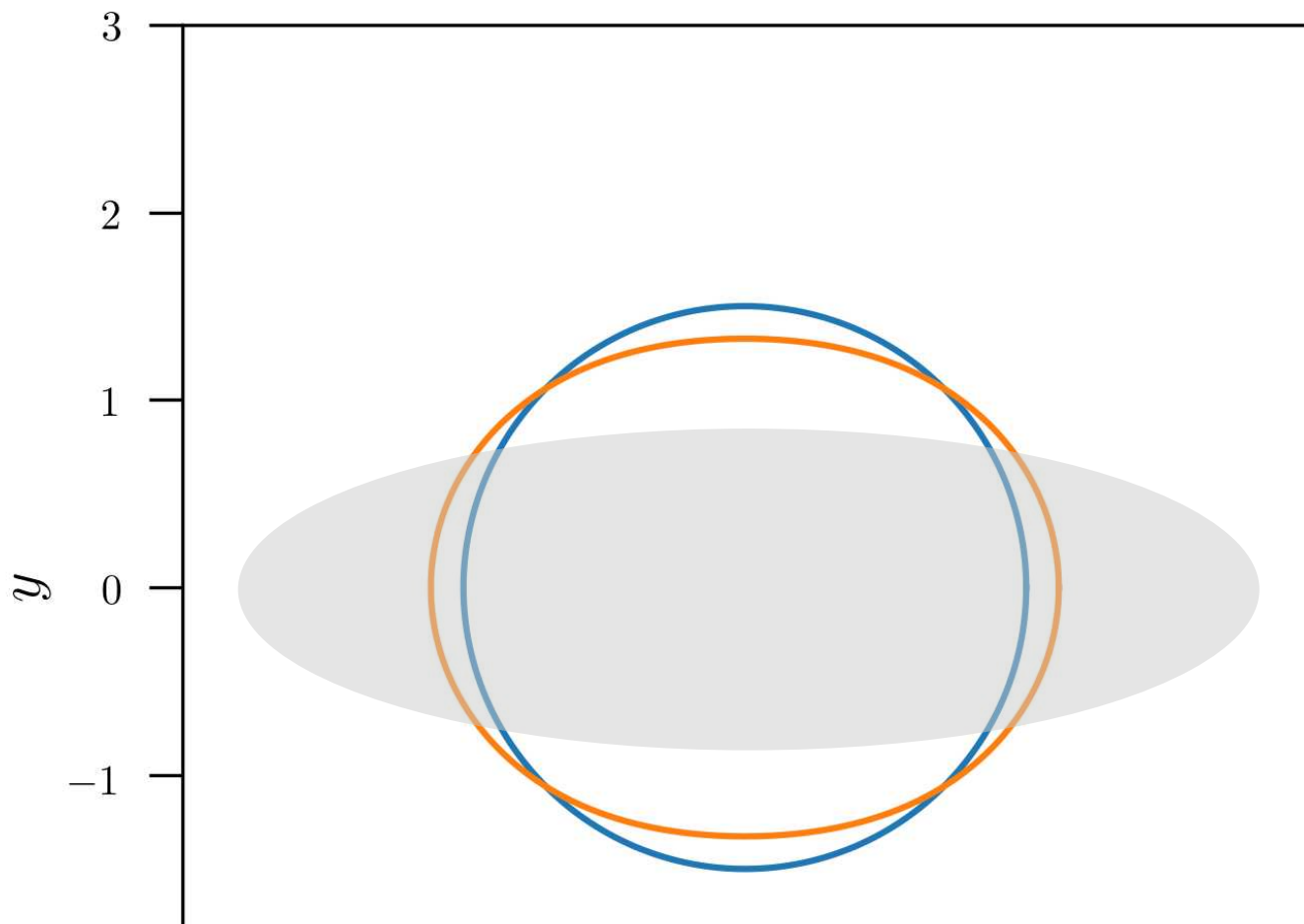
1

2

3

$R = 1.5$

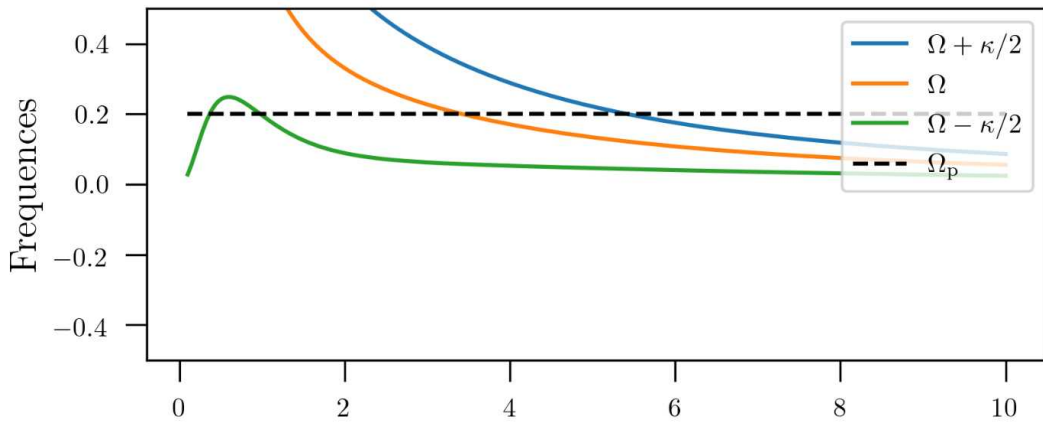
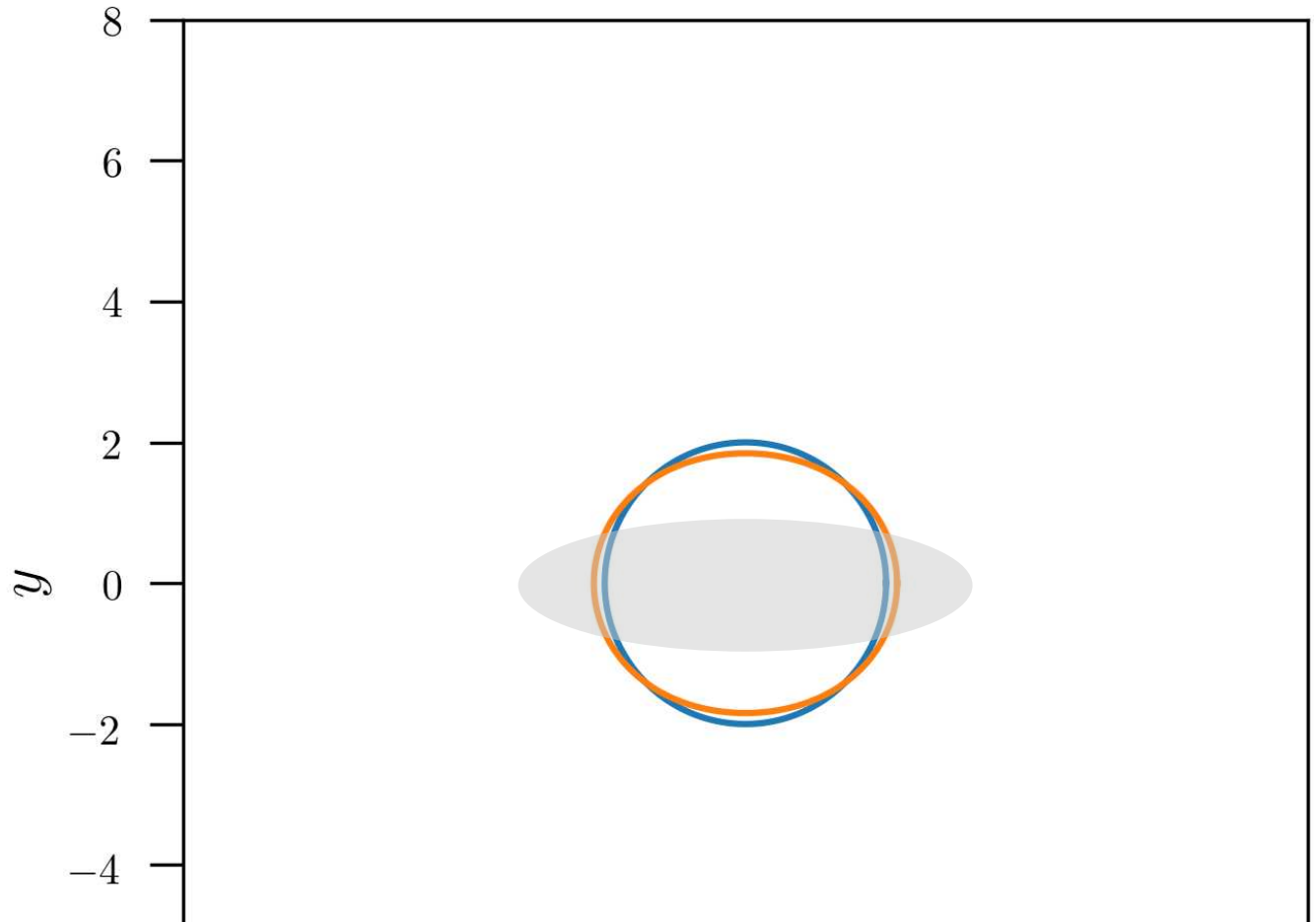
$R_{\text{ILR2}} < R < R_{\text{CR}}$



1 2 3

$R = 2.0$

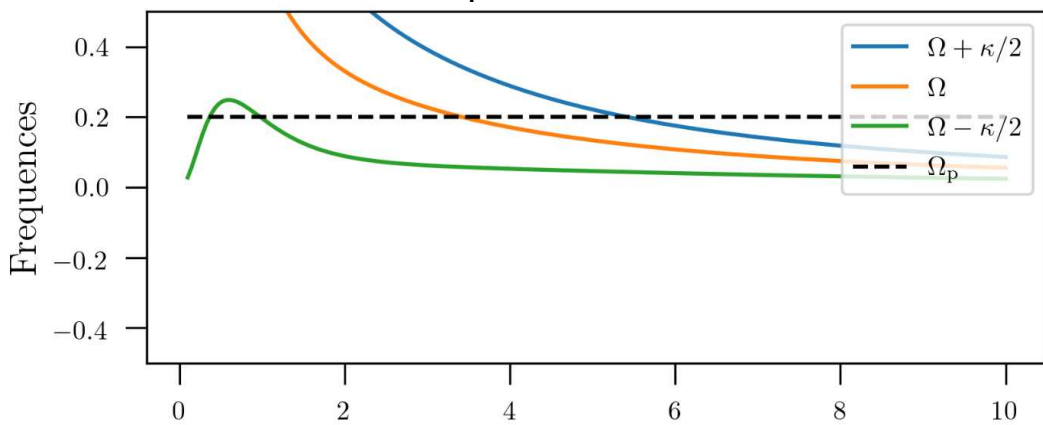
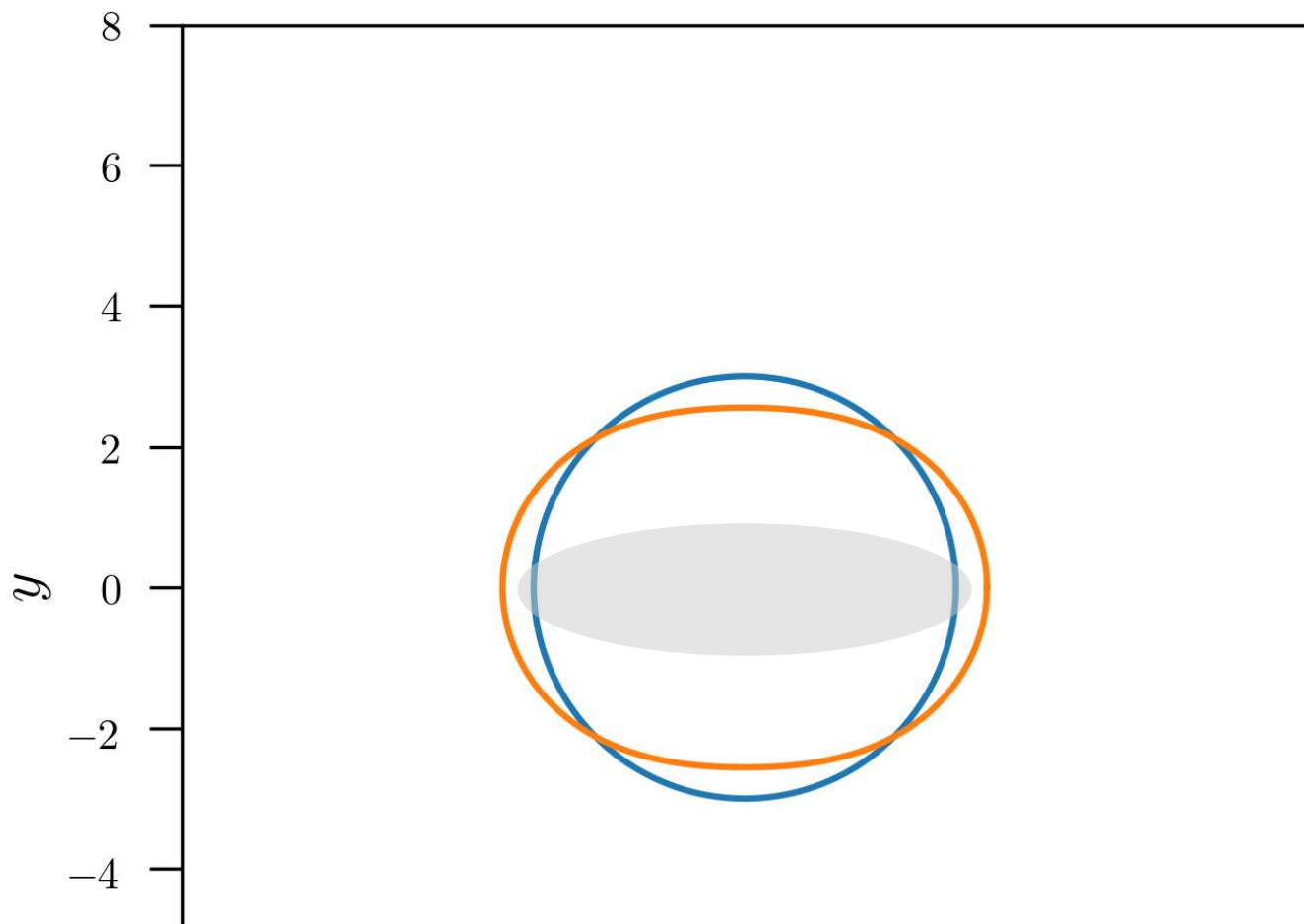
$R_{\text{ILR2}} < R < R_{\text{CR}}$



2 4 6 8

$R = 3.0$

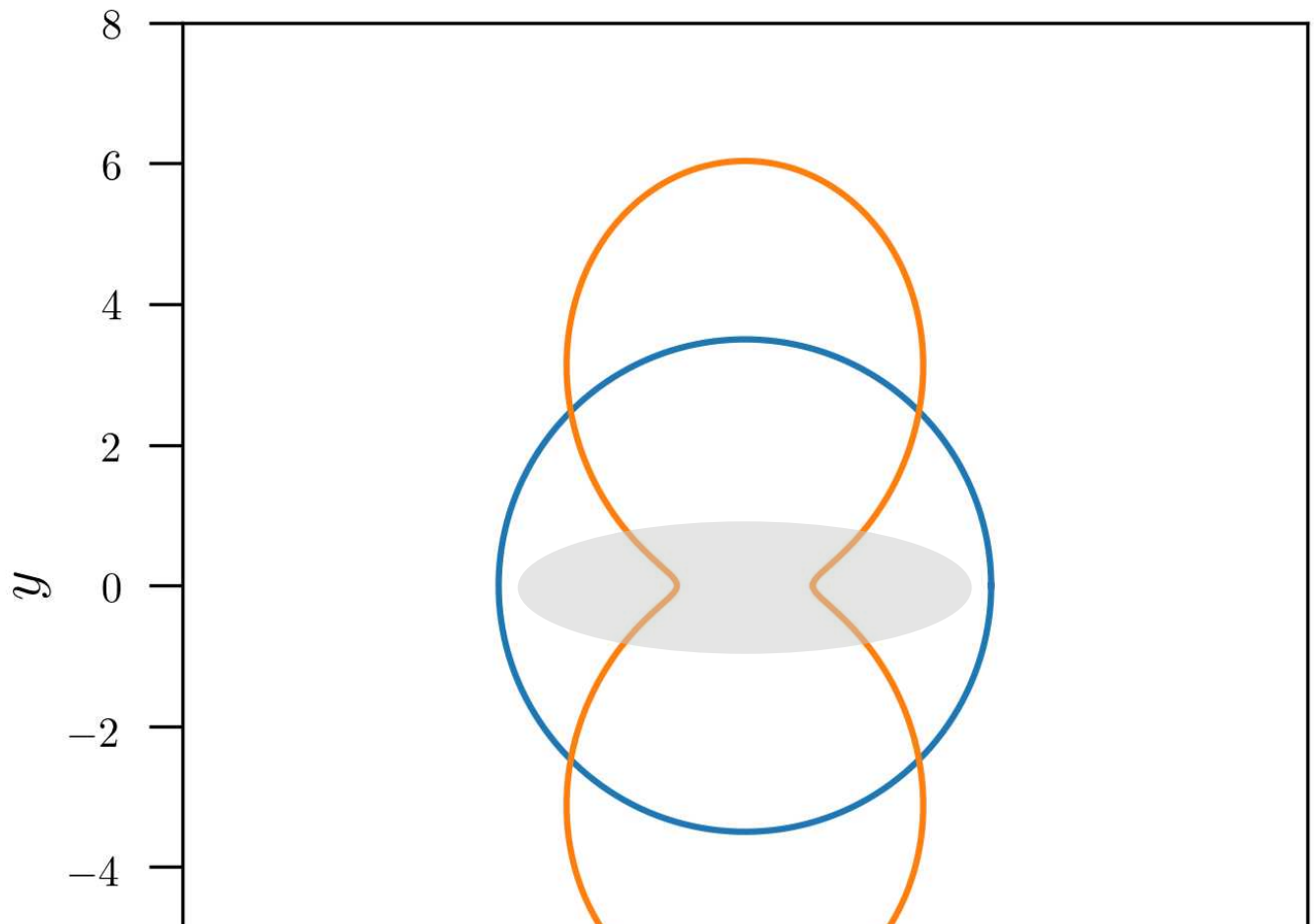
$R_{\text{ILR2}} < R < R_{\text{CR}}$



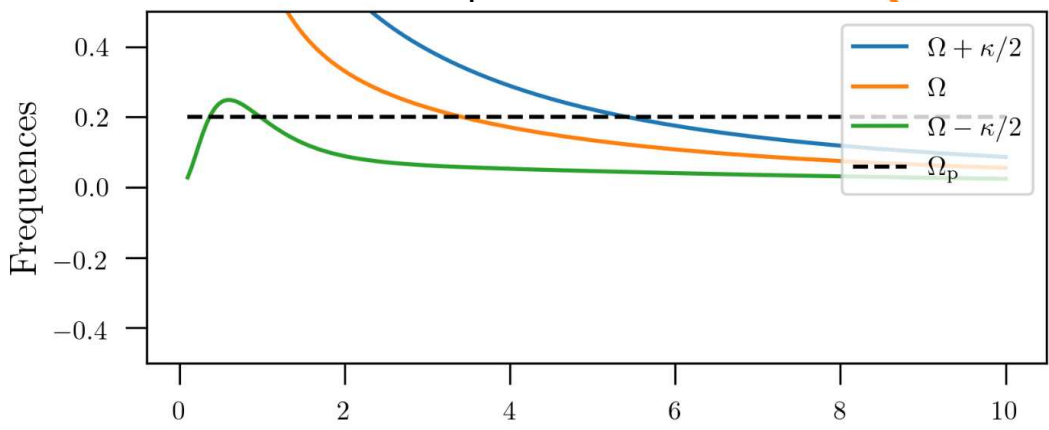
2 4 6 8

$R = 3.5$

$R \cong R_{CR}$

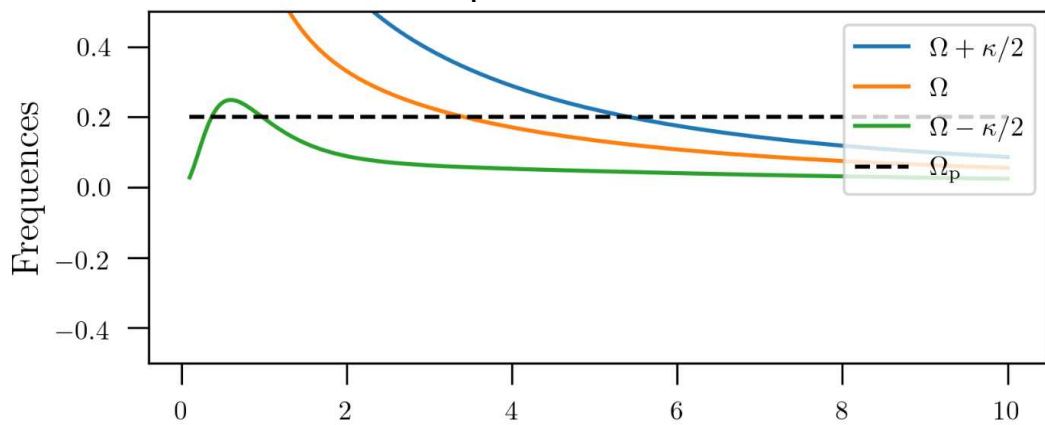
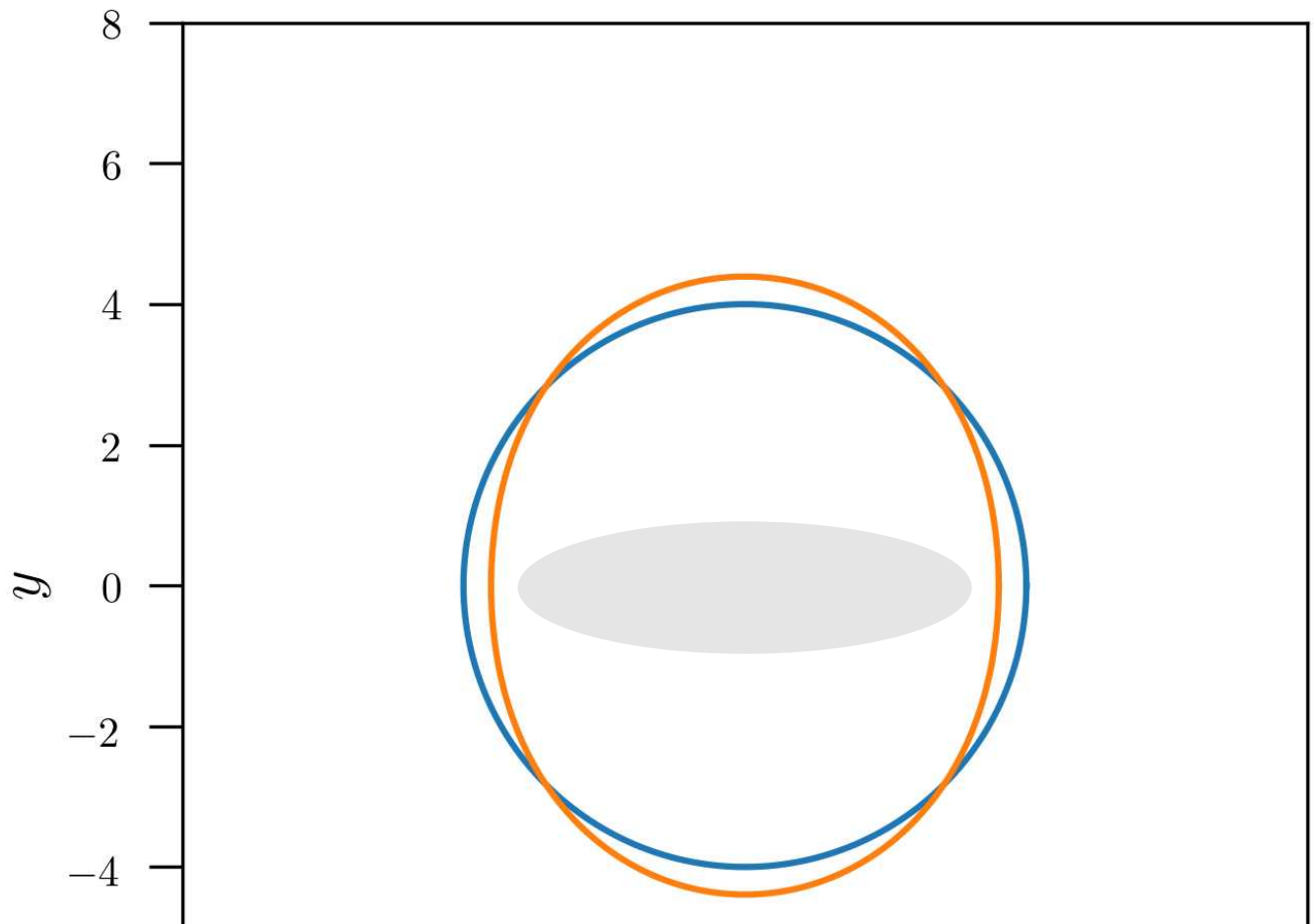


+ closed orbits  
around  $L_4, L_5$



$R = 4.0$

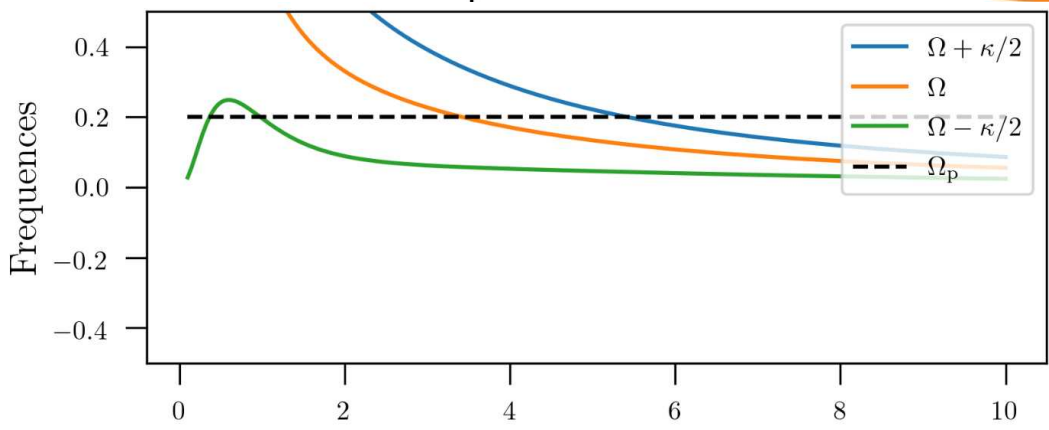
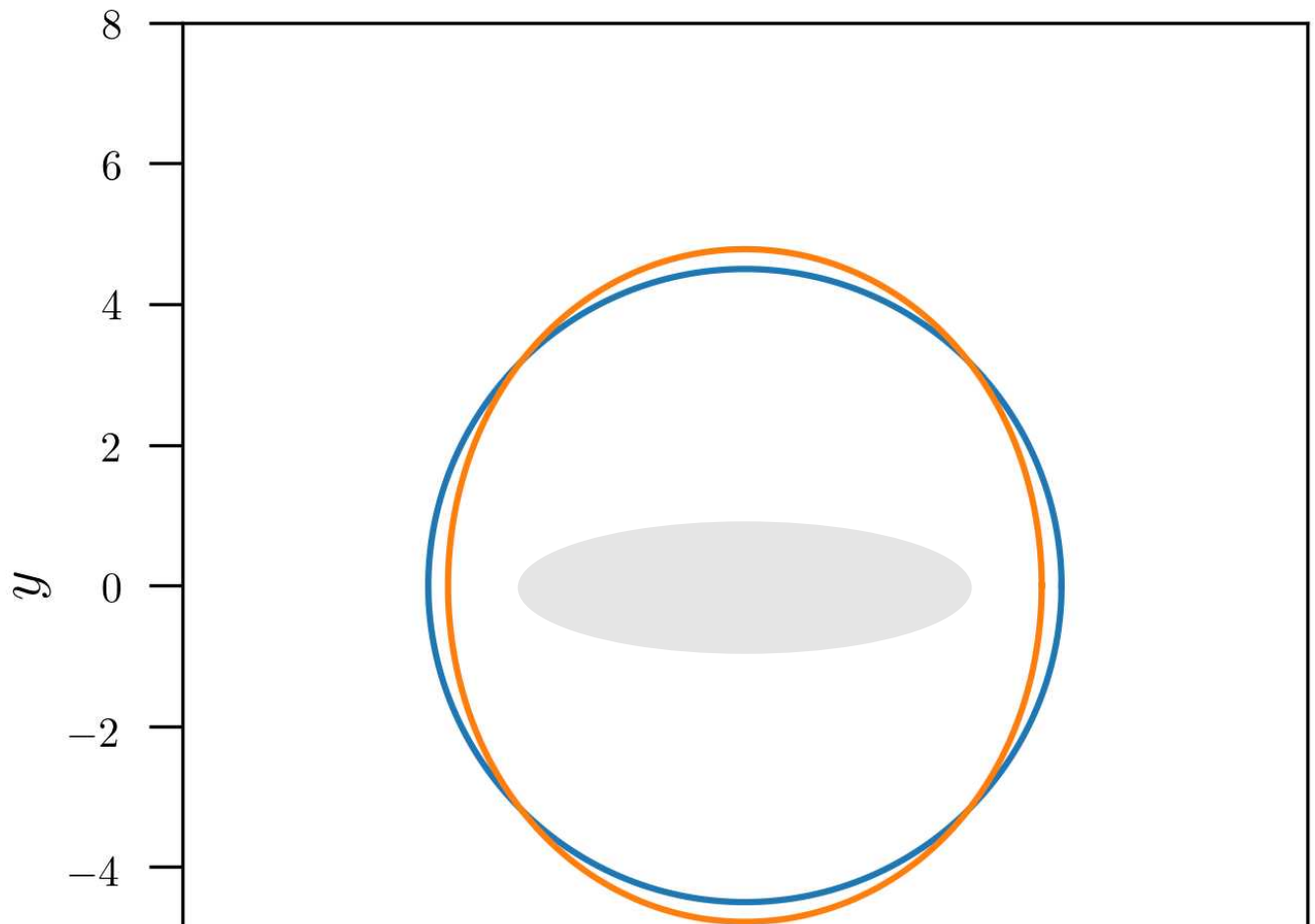
$R_{\text{CR}} < R < R_{\text{OLR}}$



2 4 6 8

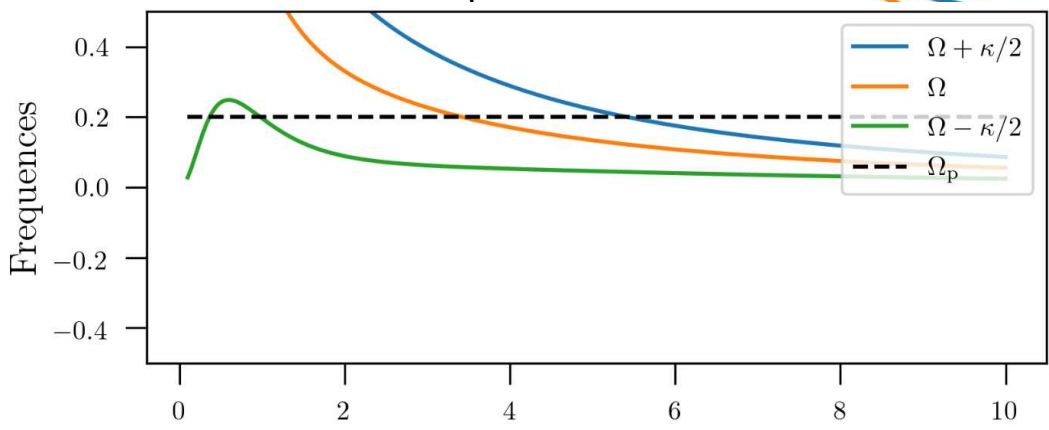
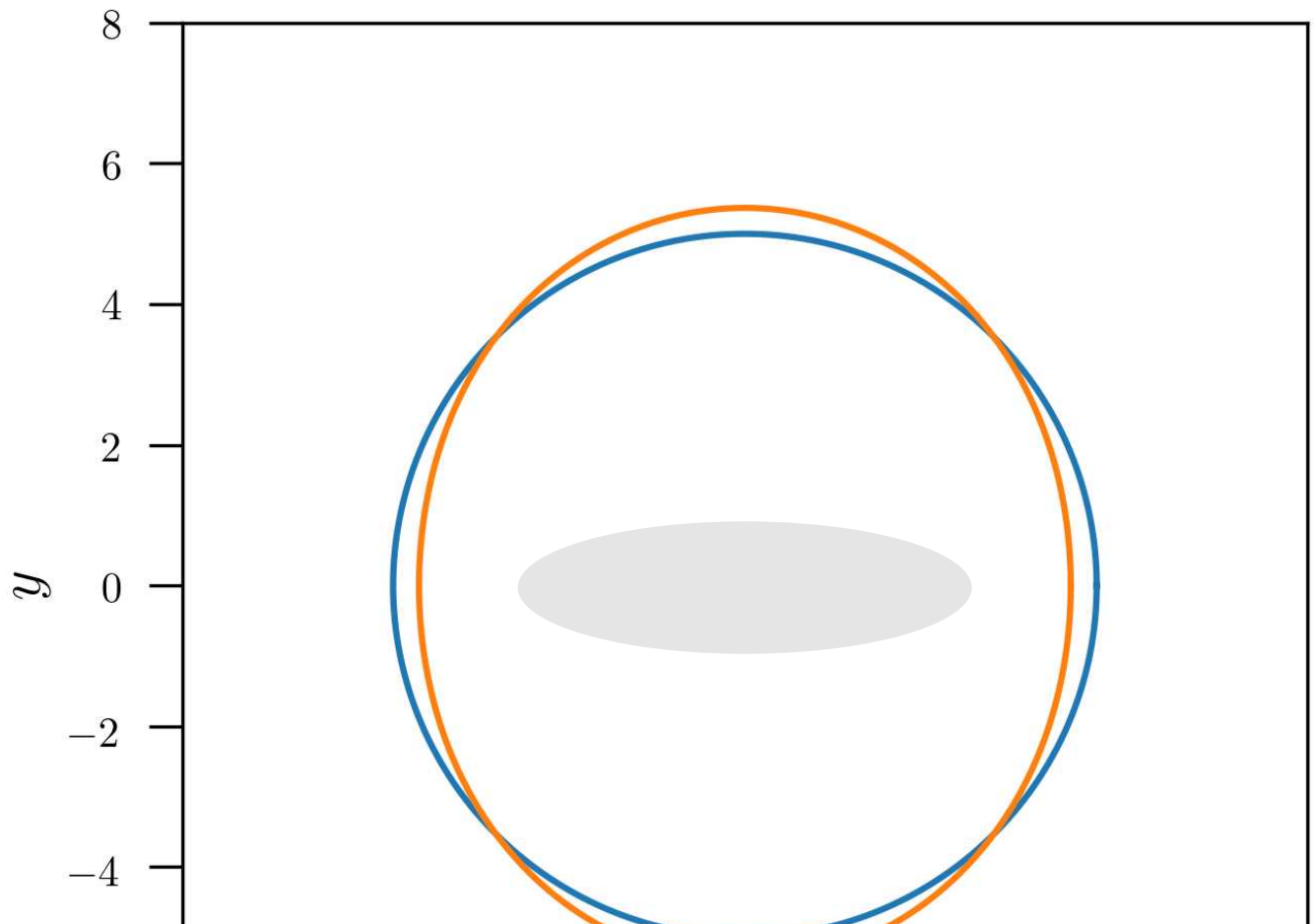
$R = 4.5$

$R_{\text{CR}} < R < R_{\text{OLR}}$



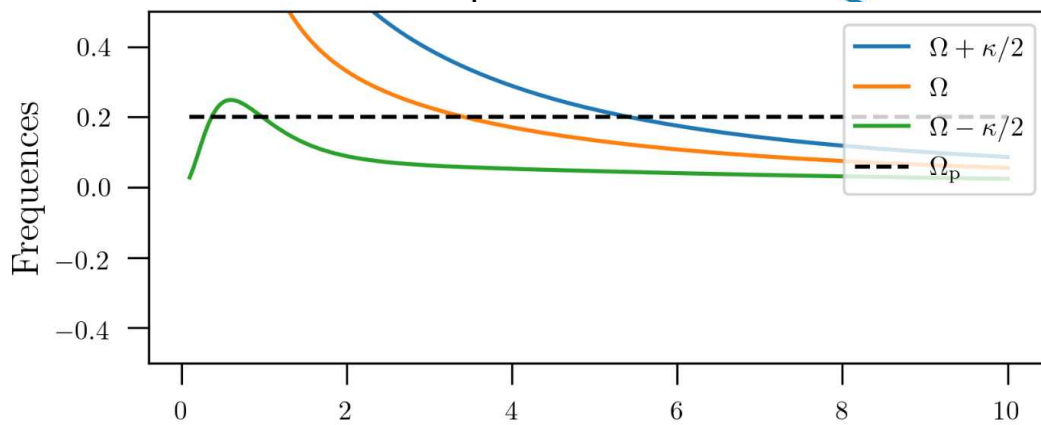
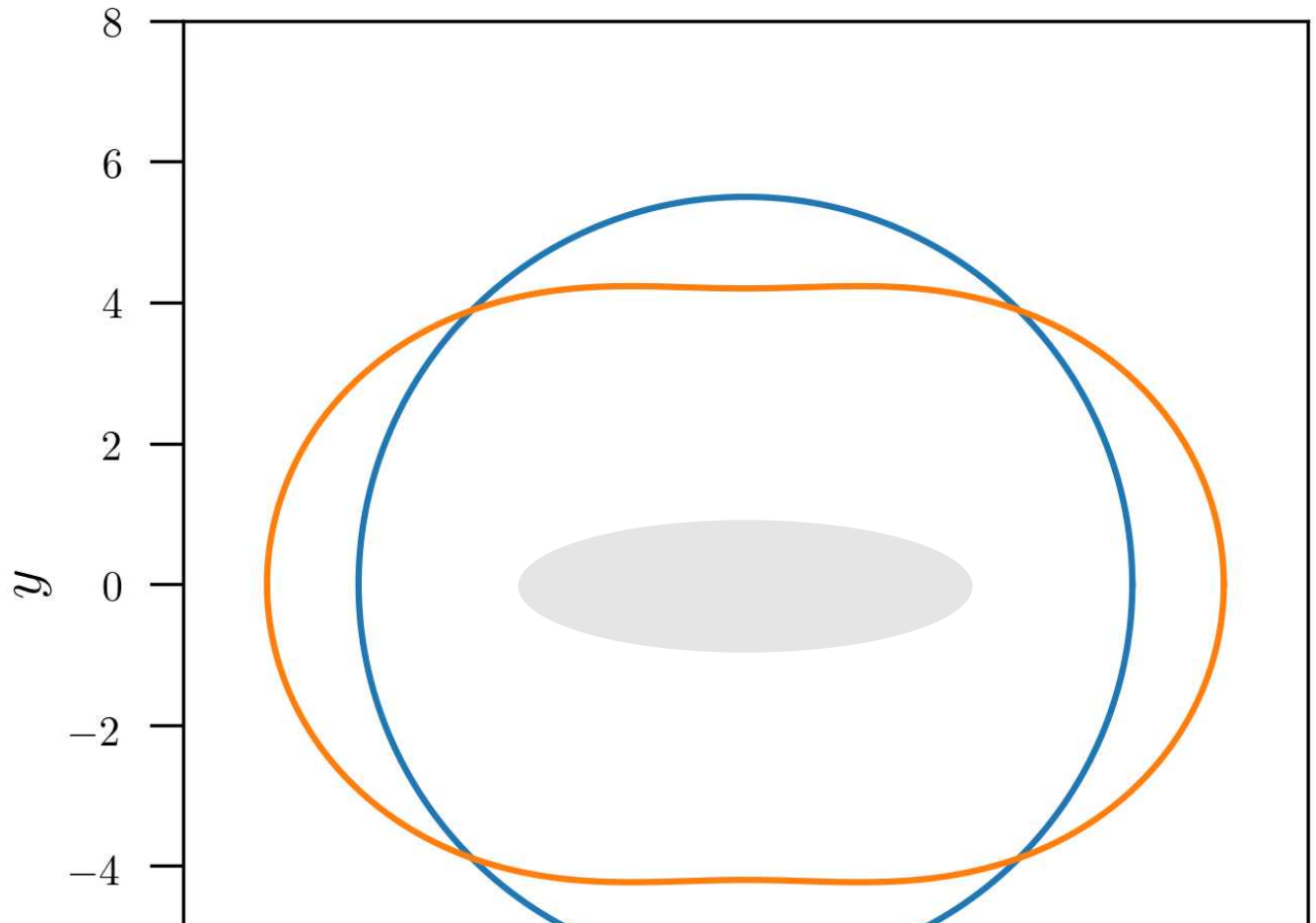
$R = 5.0$

$R_{\text{CR}} < R < R_{\text{OLR}}$



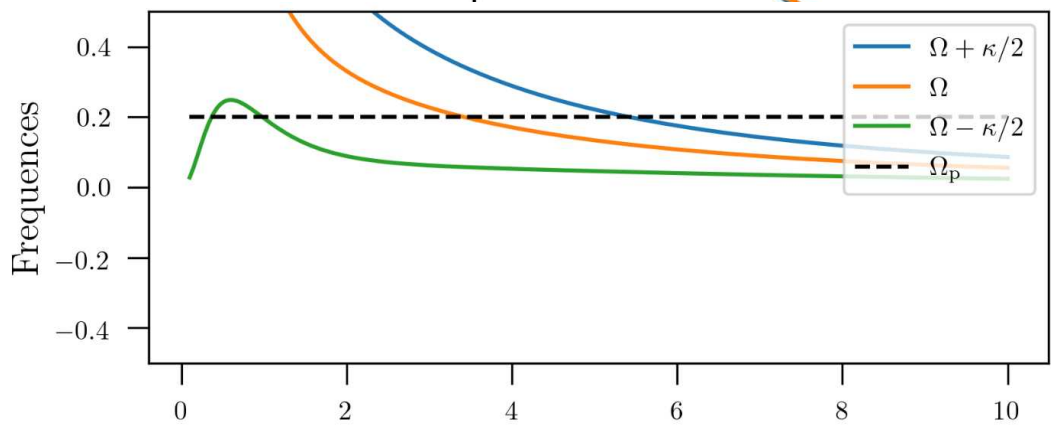
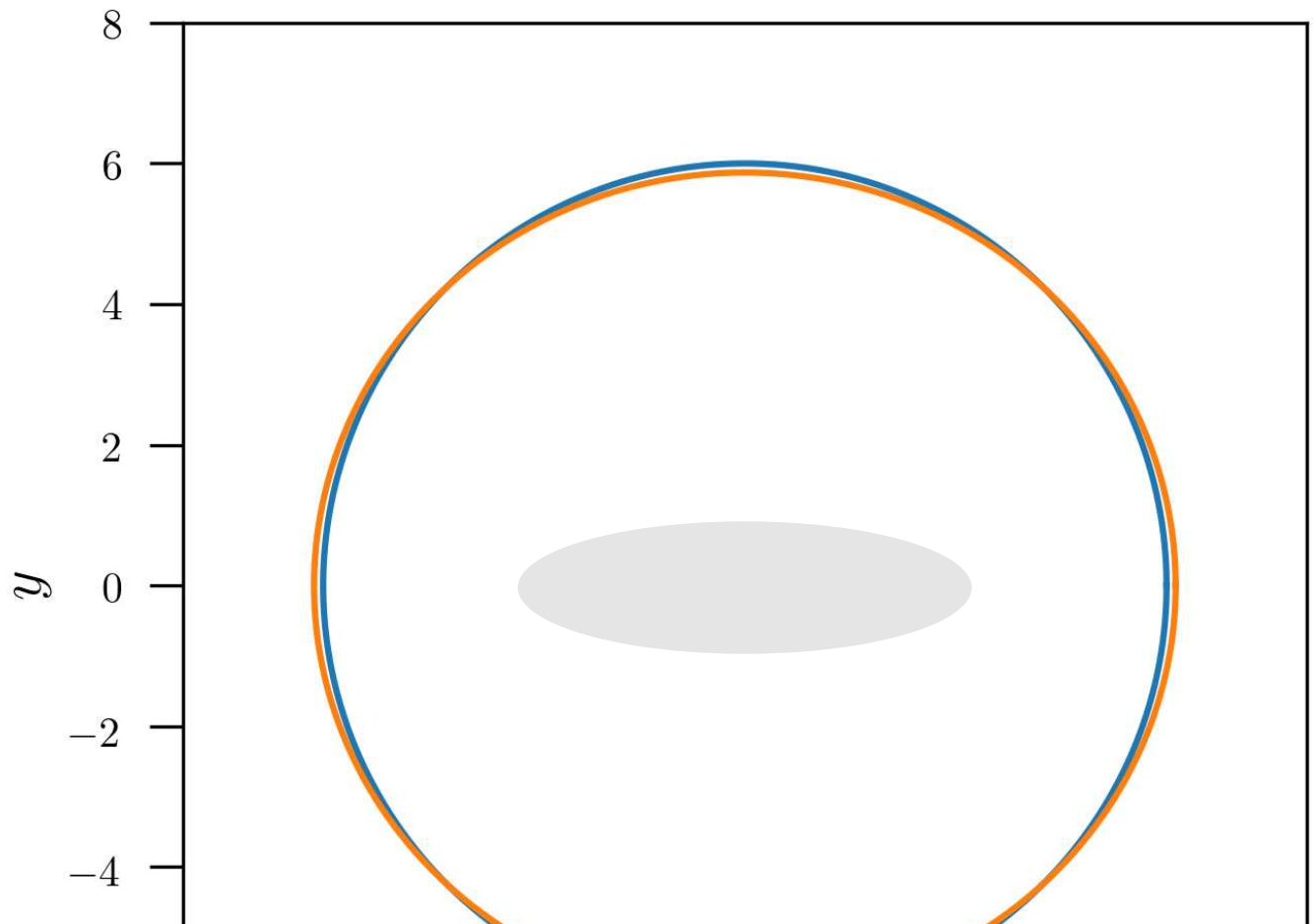
$R = 5.5$

$R \cong R_{\text{OLR}}$



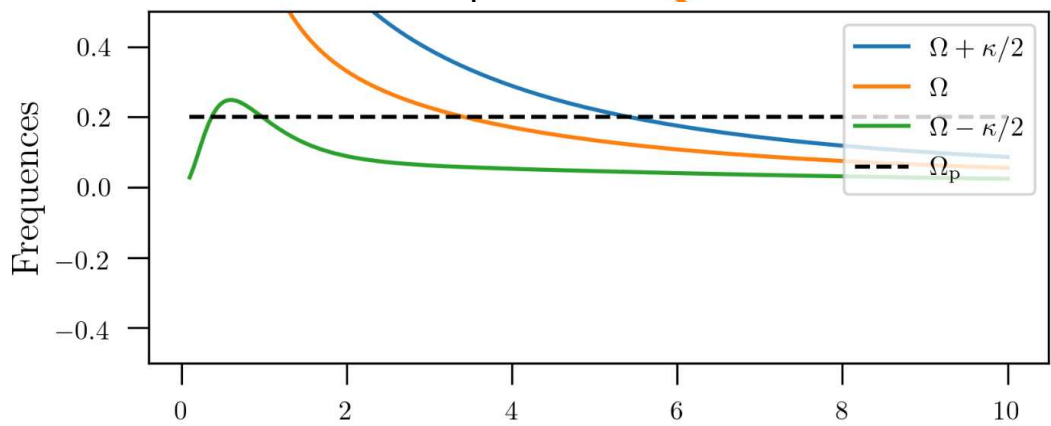
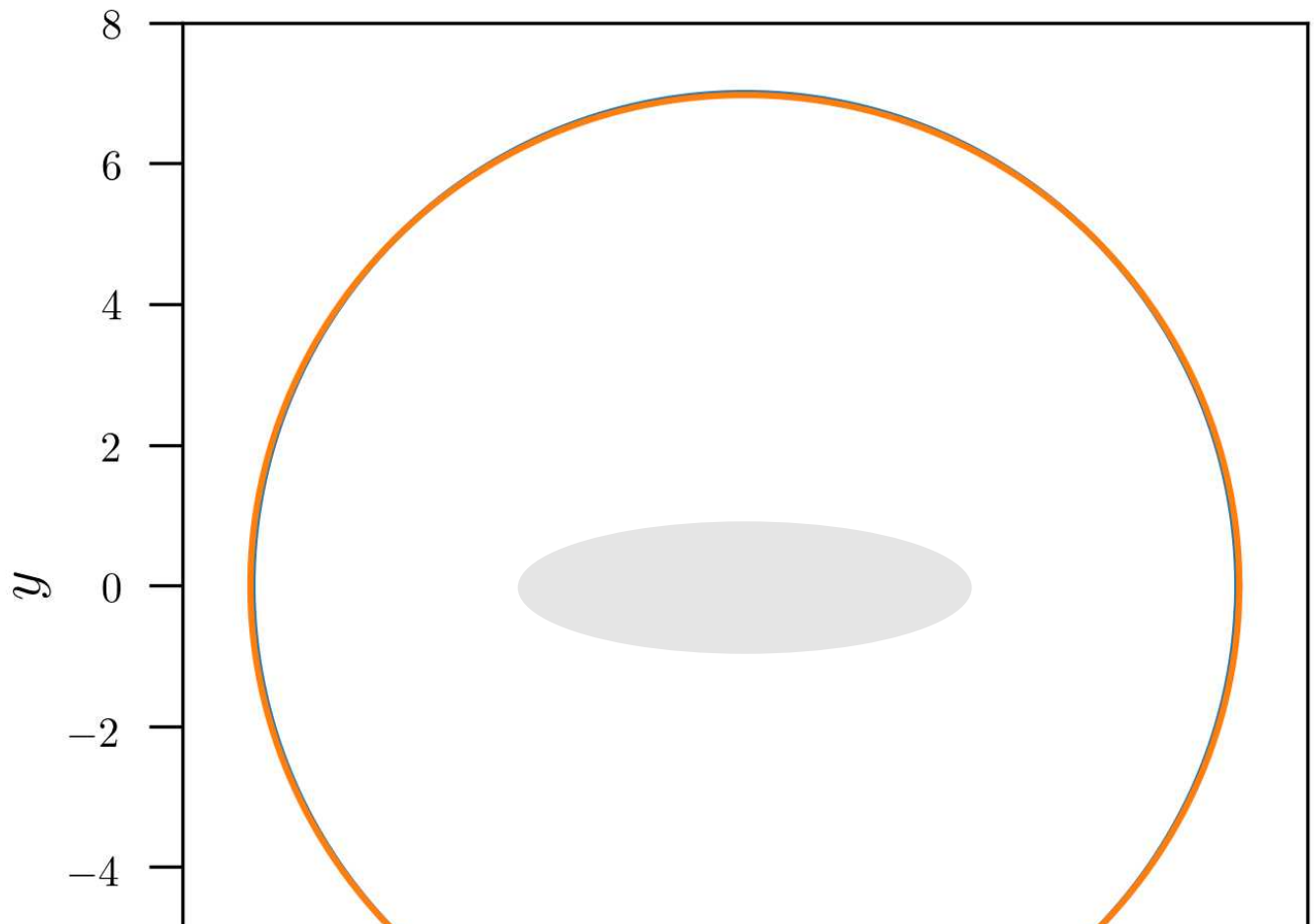
$R = 6.0$

$R_{\text{OLR}} < R$



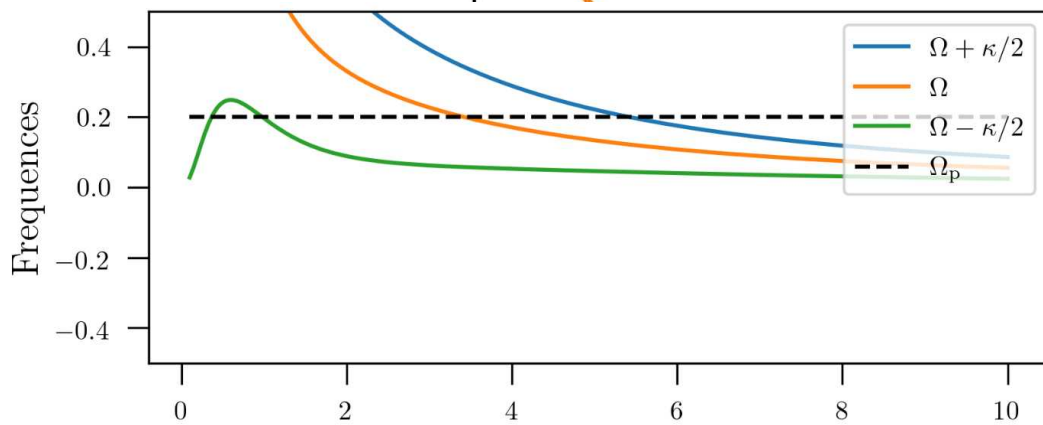
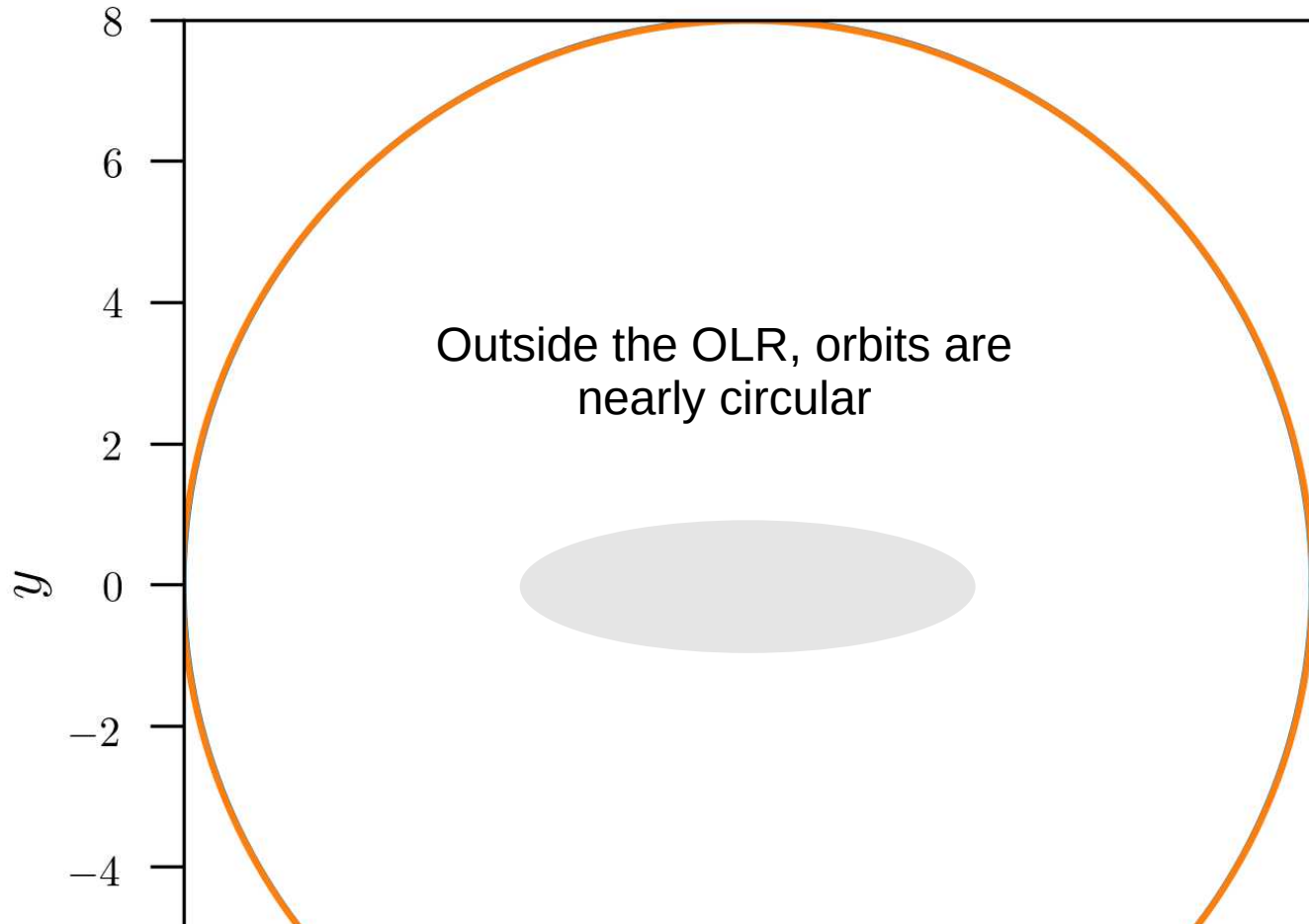
$R = 7.0$

$R_{\text{OLR}} < R$

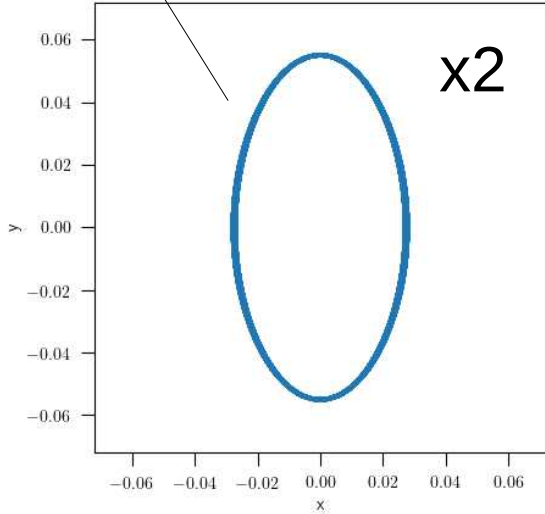
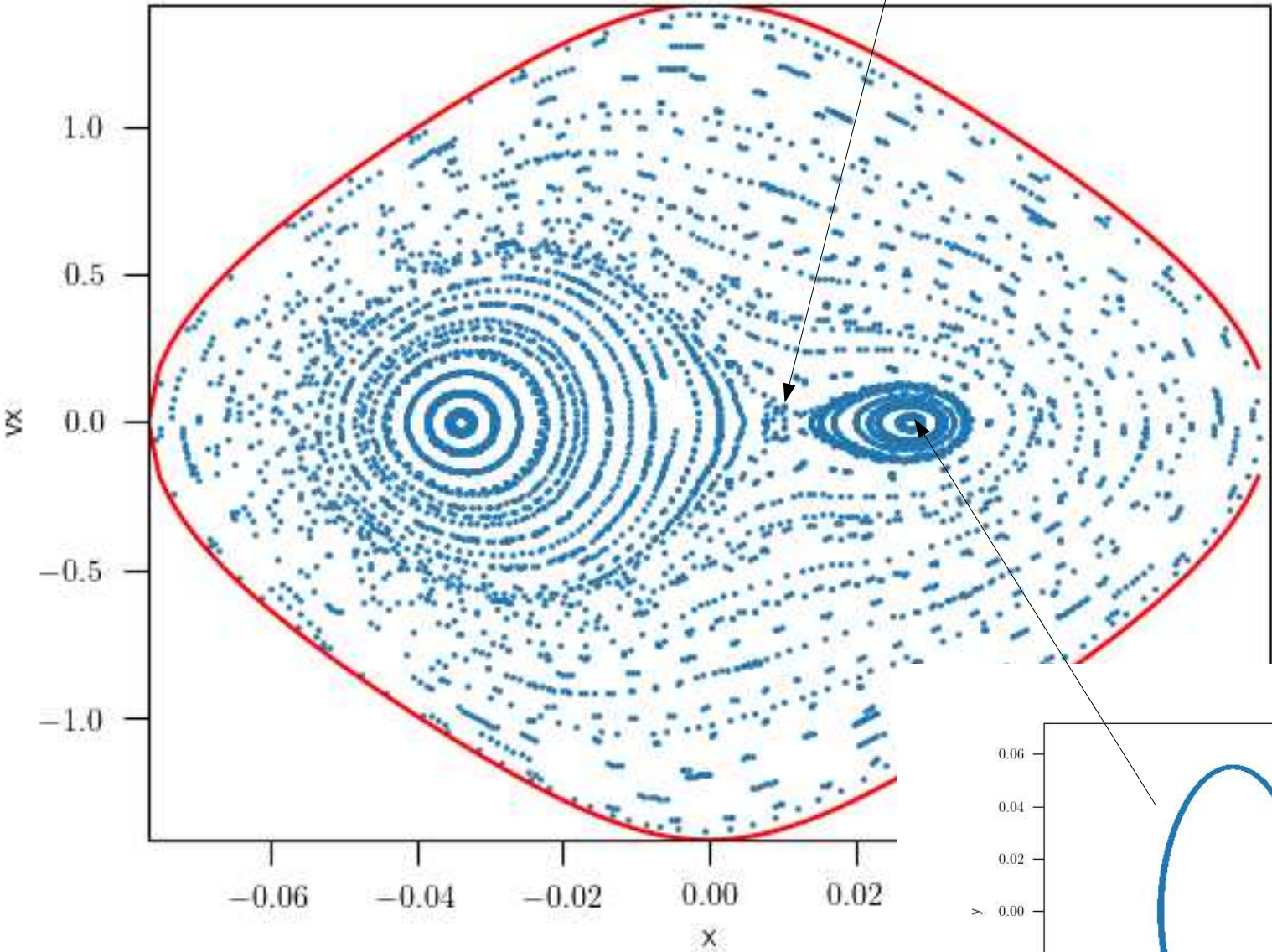


$$R = 8.0$$

$$R_{\text{OLR}} < R$$

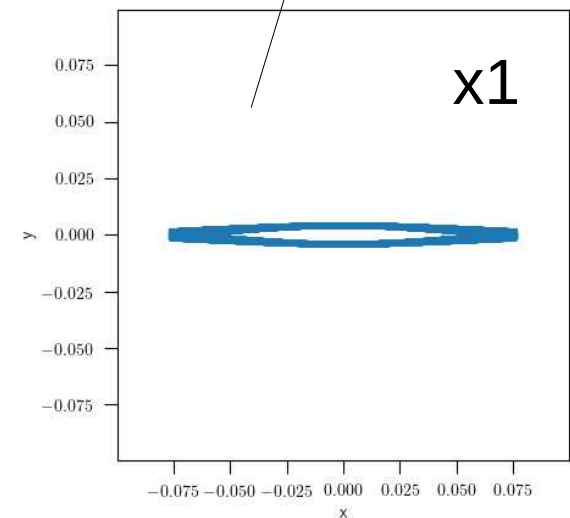
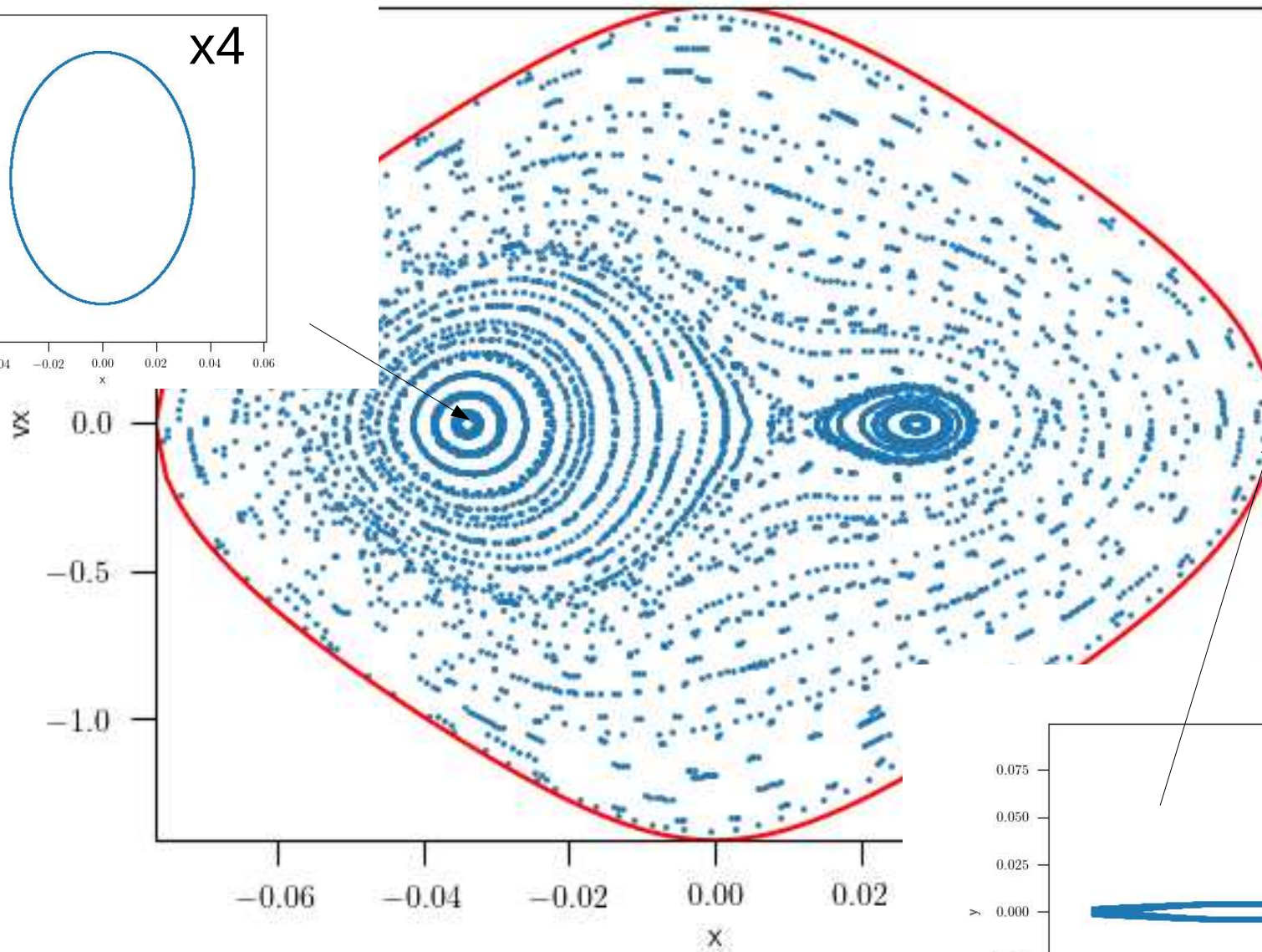
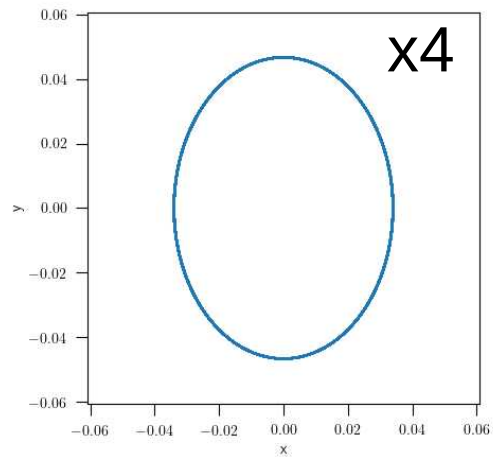


# Bifurcation : apparition of $x_2$ (stable)/ $x_3$ (unstable) orbits



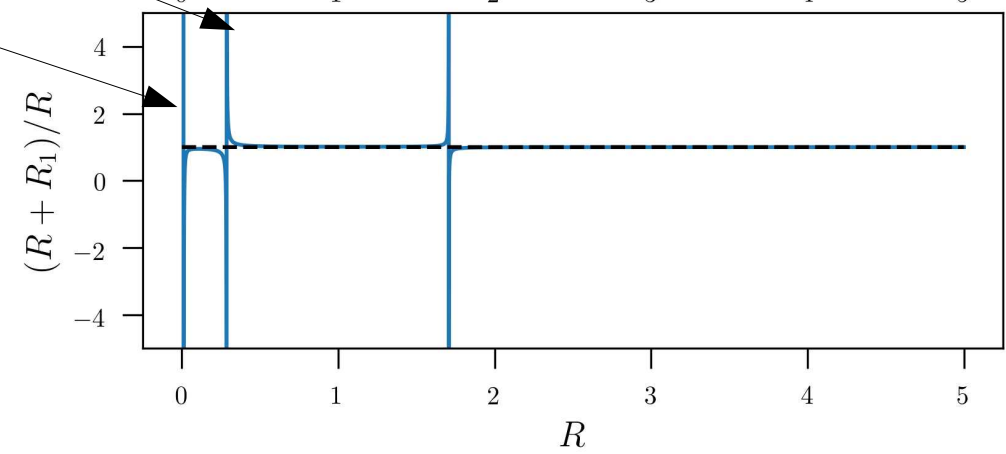
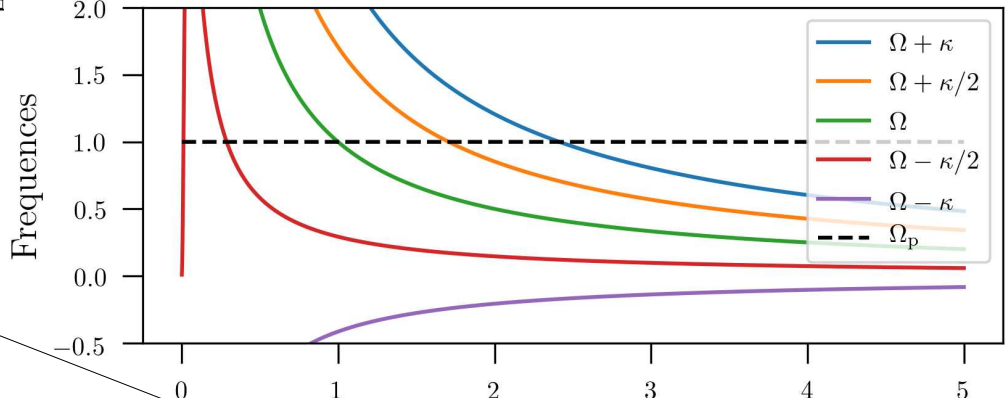
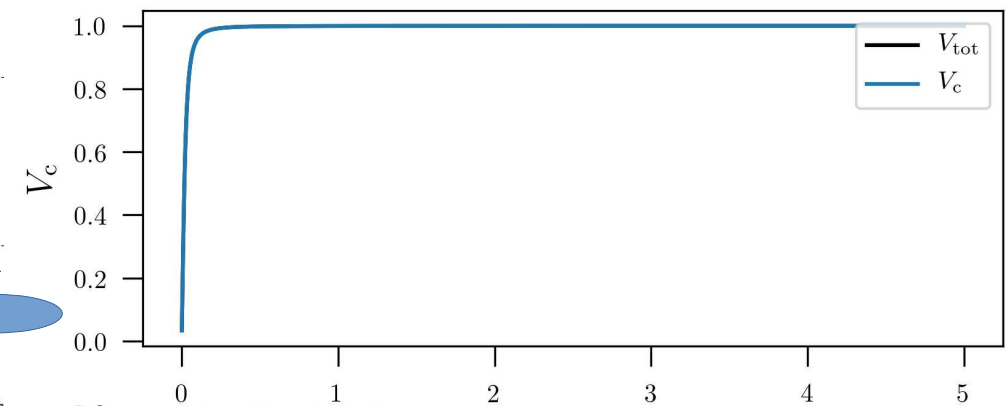
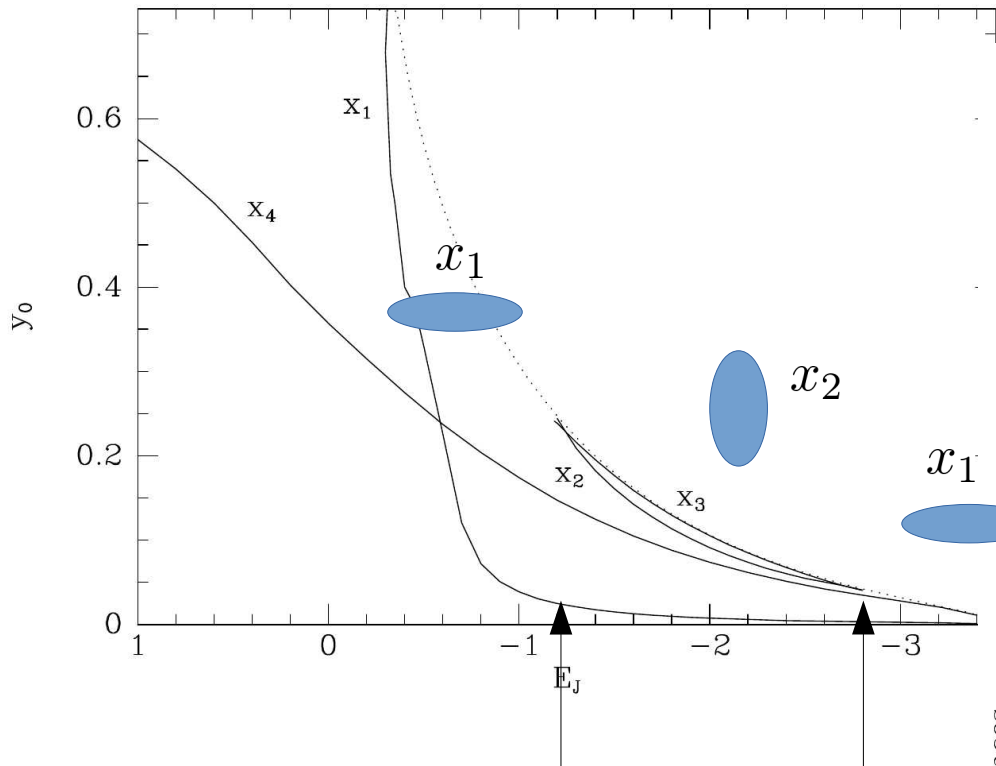
```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -2.5 --x 0.0268
```

x1 : prograde x4 : retrograde

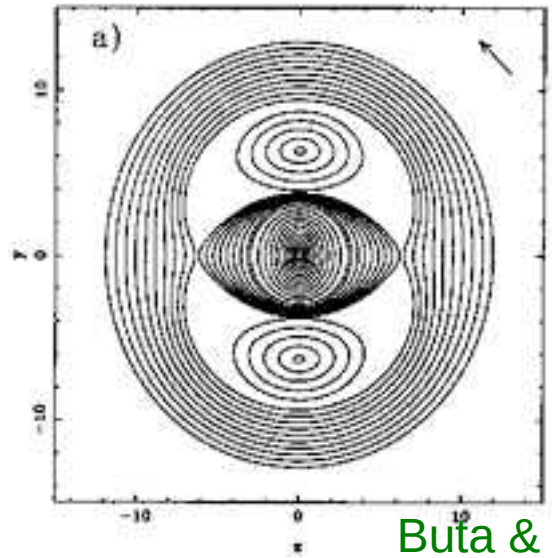


```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -2.5 --x 0.0766659  
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -2.5 --x -0.034
```

# Lindblad frequencies for the Logarithmic potential



$R_{ILR2}$        $R_{ILR1}$



Buta & Combes 1998

# **Equilibria of collisionless systems**

## **The collisionless Boltzmann equation**

# Introduction / Motivations

So far, we :

1. we modelled static potentials from a mass distribution (Poisson equation)
2. from the potential, we obtained forces and derived equations of motion leading us study orbits in different idealized potentials :
  - spherical potentials
  - axi-symmetric potentials (epicycles motions)
  - orbits in bared rotating potentials (motions around Lagrange points)

But :

1. We did not used any velocity constraints. We only used the positions of stars through the emission of light.
2. Nothing tells us that the models we used are at the equilibrium.  
This is not guarantee, if, for e.g., all velocities are zero...
3. We did do not include the self-gravity of the model or perturbations on it due to the orbits of stars.

# Introduction / Motivations

## Goal :

Build a self-consistent way galaxies, ensuring that they are at the equilibrium, i.e., if we compute the evolution of the galaxy under its own gravity, the evolution will be stationary.

→ requires the description of the **density** but also the **velocity field**

$$\rho(\vec{x}) \qquad \vec{v}(\vec{x})$$

## Assumptions :

1. We will consider systems with a very large number of “particles” (stars, DM)

→ the collisionless approximation is valid

→ real orbits deviate not too much from the one predicted from the model  
(very large relaxation time)

We will seek for solution corresponding to  $t_{\text{relax}} = \infty$

2. We will consider systems composed of N identical particles, i.e., with all the same mass.

All particles will be equivalent

# Introduction / Motivations

Goal :

Build a self-consistent way galaxies, ensuring that they are at the equilibrium, i.e., if we compute the evolution of the galaxy under its own gravity, the evolution will be stationary.

→ requires the description of the **density** but also the **velocity field**

$$\rho(\vec{x})$$

$$\vec{v}(\vec{x})$$

But :

It is impossible to describe analytically the orbits of billions of stars :

→ **we need a probabilistic approach**

# Distribution function (DF)

Definition ①  $f(\vec{x}, \vec{v}, t)$  or  $f(\vec{w}, t)$  such that  
 $f(\vec{x}, \vec{v}, t) d^3\vec{x} d^3\vec{v}$  or  $f(\vec{w}, t) d^6\vec{w}$   
is the probability that at the time  $t$ ,  
a randomly chosen star "i" has its position  $\vec{x}_i$ ;  
an velocity  $\vec{v}_i$ , or phase space coordinates  $\vec{w}_i$ ;  
in the ranges  $\vec{x}_i \in [\vec{x}, \vec{x} + d\vec{x}]$   
 $\vec{v}_i \in [\vec{v}, \vec{v} + d\vec{v}]$   
 $\equiv \vec{w}_i \in [\vec{w}, \vec{w} + d\vec{w}]$

obviously:  
(normalisation)

$$\int f(\vec{x}, \vec{v}, t) d^3\vec{x} d^3\vec{v} = 1$$
$$\equiv \int f(\vec{w}, t) d^6\vec{w} = 1$$

the particle  
is for sure  
somewhere in  
the phase space

$f(\vec{x}, \vec{v}, t)$  is the probability density of the phase space.

## Distribution function (DF)

Definition (2)  $\tilde{f}(\vec{x}, \vec{v}, t)$  or  $\tilde{f}(\vec{w}, t)$  such that

$$\tilde{f}(\vec{x}, \vec{v}, t) d^3\vec{x} d^3\vec{v} \quad \text{or} \quad \tilde{f}(\vec{w}, t) d^6\vec{w}$$

is the number of stars having position  $\vec{x}$  and velocities  $\vec{v}$  ( $\vec{w}$ ) in the intervals at time  $t$ :

$$\vec{x}_i \in [\vec{x}, \vec{x} + d\vec{x}]$$

$$\vec{v}_i \in [\vec{v}, \vec{v} + d\vec{v}]$$

$$\vec{w}_i \in [\vec{w}, \vec{w} + d\vec{w}]$$

obviously:  
(normalisation)

$$\int \tilde{f}(\vec{x}, \vec{v}, t) d^3\vec{x} d^3\vec{v} = N$$
$$\equiv \int \tilde{f}(\vec{w}, t) d^6\vec{w} = N$$

There are exactly  $N$  particles in the phase space

$\tilde{f}(\vec{x}, \vec{v}, t)$  is the number density of the phase space.

Combining Det. ① and Det ②

$$N f(\bar{x}, \bar{v}, t) = \tilde{f}(\bar{x}, \bar{v}, t)$$

Notes

- we will sometimes forget the " $\sim$ "
- the time dependence " $t$ " will not be systematically written

Using definition ①

The probability of finding a star "i" in the subvolume of the phase space  $\nu$  is:

$$P = \int_{\nu} f(\vec{w}) d^6\vec{w}$$

However, imagine that we are using another canonical coordinate system  $\vec{W}$  (in which the Hamilton equations are valid)

e.g.  $(x, y, p_x = \dot{x}, p_y = \dot{y}) \rightarrow (r, \theta, p_r = \dot{r}, p_\theta = r^2\dot{\theta})$

$$P^w = \int_{\nu} F(\vec{W}) d^6\vec{W} = P$$

The probability must not be affected by a coordinate change.

If  $\nu$  is taken small enough, we can assume  $g(\vec{w})$  and  $F(\vec{W})$  to be constant and hence

$$g(\vec{w}_\nu) \int_\nu d^6\vec{w} = F(\vec{W}_\nu) \int_\nu d^6\vec{w}$$

But, for canonical coordinates, the phase space volume element is the same:

$$\int_\nu d^6\vec{w} = \int_\nu d^6\vec{W}$$

Thus

$$g(\vec{w}) = F(\vec{W})$$

The density of the phase space is independent of the coordinate system

Corollary: We can use any system of canonical coordinates  $\vec{w} = (\vec{q}, \vec{p})$  to define the distribution function

# The collisionless Boltzmann equation


- What is the evolution of  $f(\vec{w})$  over time?

As the mass, the probability is a conserved quantity.  $\rho = N\bar{f}$

the number of stars is a conserved quantity.

in the phase space

Continuity equation (similar than for hydrodynamics)

$V$   the time variation of the mass in  $V$

$$\frac{dM}{dt} = \sum_{\text{faces}} \rho \vec{v} \cdot d\vec{S}$$

$\rightarrow$  mass flux

Mass conservation

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}_x \cdot (\rho \vec{v}) = 0$$

mass flux through the surface  
of the volume

# The collisionless Boltzmann equation

- What is the evolution of  $f(\vec{w})$  over time?

As the mass, the probability is a conserved quantity.  $\rho = N f$

the number of stars is a conserved quantity.

in the phase space

Continuity equation (similar than for hydrodynamics)



the time variation of the mass in  $V$

$$\frac{dM}{dt} = \sum_{\text{faces}} \underbrace{\rho \vec{v} \cdot d\vec{S}}_{\text{mass flux}}$$

Mass conservation

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}_x \cdot (\rho \vec{v}) = 0$$

Probability conservation

$$\frac{\partial f}{\partial t} + \vec{\nabla}_w \cdot (f \vec{w}) = 0$$

mass flux through the surface  
of the volume

probability flux through the surface  
of the volume

## Analogy with the continuity equation in hydrodynamics

$$\rho(\vec{x}, t) \quad \vec{v} = \frac{d}{dt} \vec{x}$$

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}_x \cdot (\rho \vec{v}) = 0$$

Lagrangian derivative

$$\frac{d}{dt} \rho(\vec{x}, t) = \frac{\partial \rho}{\partial t} + \vec{v} \cdot \vec{\nabla}_x \rho$$

Flux divergence

$$\vec{\nabla}_x \cdot (\rho \vec{v}) = \vec{v} \cdot \vec{\nabla}_x \rho + \rho \vec{\nabla}_x \cdot \vec{v}$$

$$\vec{v} \cdot \vec{\nabla}_x \rho = \vec{\nabla}_x \cdot (\rho \vec{v}) - \rho \vec{\nabla}_x \cdot \vec{v}$$

## Lagrangian derivative

$$\begin{aligned}\frac{d}{dt} f(\vec{x}, t) &= \frac{\partial f}{\partial t} + \vec{v} \cdot \vec{\nabla}_x f \\ &= \underbrace{\frac{\partial f}{\partial t} + \vec{\nabla}_x \cdot (f \vec{v})}_{= 0 \text{ continuity Eqn.}} - f \vec{\nabla}_x \cdot \vec{v}\end{aligned}$$

$$\frac{d}{dt} f(\vec{x}, t) = - f \vec{\nabla}_x \cdot \vec{v}$$

the increase of  
 $f$  along the flow  
is due to compression

incompressible fluid :

$$\vec{\nabla}_x \cdot \vec{v} = 0$$

# Analogy with the continuity equation in hydrodynamics

$$f(\vec{x}, t) \quad \vec{v} = \frac{d}{dt} \vec{x}$$

$$\frac{\partial f}{\partial t} + \vec{\nabla}_{\vec{x}} \cdot (f \vec{v}) = 0$$

Lagrangian derivative

$$\frac{d}{dt} f(\vec{x}, t) = \frac{\partial f}{\partial t} + \vec{v} \cdot \vec{\nabla}_{\vec{x}} f$$

Flux divergence

$$\vec{\nabla}_{\vec{x}} \cdot (f \vec{v}) = \vec{v} \cdot \vec{\nabla}_{\vec{x}} f + f \vec{\nabla}_{\vec{x}} \cdot \vec{v}$$

$$\vec{v} \cdot \vec{\nabla}_{\vec{x}} f = \vec{\nabla}_{\vec{x}} \cdot (f \vec{v}) - f \vec{\nabla}_{\vec{x}} \cdot \vec{v}$$

$$f(\vec{w}, t) \quad \dot{\vec{w}} = \frac{d}{dt} \vec{w}$$

$$\frac{\partial f}{\partial t} + \vec{\nabla}_{\vec{w}} \cdot (f \dot{\vec{w}}) = 0$$

Lagrangian derivative

$$\frac{d}{dt} f(\vec{w}, t) = \frac{\partial f}{\partial t} + \dot{\vec{w}} \cdot \vec{\nabla}_{\vec{w}} f$$

Flux divergence

$$\vec{\nabla}_{\vec{w}} \cdot (f \dot{\vec{w}}) = \dot{\vec{w}} \cdot \vec{\nabla}_{\vec{w}} f + f \vec{\nabla}_{\vec{w}} \cdot \dot{\vec{w}}$$

$$\dot{\vec{w}} \cdot \vec{\nabla}_{\vec{w}} f = \vec{\nabla}_{\vec{w}} \cdot (f \dot{\vec{w}}) - f \vec{\nabla}_{\vec{w}} \cdot \dot{\vec{w}}$$

## Lagrangian derivative

$$\begin{aligned}\frac{d}{dt} f(\vec{x}, t) &= \frac{\partial f}{\partial t} + \vec{v} \cdot \vec{\nabla}_x f \\ &= \underbrace{\frac{\partial f}{\partial t} + \vec{\nabla}_x (f \vec{v})}_{= 0 \text{ continuity Eqn.}} - f \vec{\nabla}_x \cdot \vec{v}\end{aligned}$$

$$\frac{d}{dt} f(\vec{x}, t) = - f \vec{\nabla}_x \cdot \vec{v}$$

the increase of  $f$  along the flow is due to compression

incompressible fluid:

$$\vec{\nabla}_x \cdot \vec{v} = 0$$

## Lagrangian derivative

$$\begin{aligned}\frac{d}{dt} f(\vec{w}, t) &= \frac{\partial f}{\partial t} + \dot{\vec{w}} \cdot \vec{\nabla}_w f \\ &= \underbrace{\frac{\partial f}{\partial t} + \vec{\nabla}_w (f \dot{\vec{w}})}_{= 0 \text{ continuity Eqn.}} - \underbrace{f \vec{\nabla}_w \cdot \dot{\vec{w}}}_{= 0 \text{ canonical coords.}}\end{aligned}$$

$$\frac{d}{dt} f(\vec{w}, t) = 0$$

(replace  $\dot{\vec{w}}$  with Hamilton equations)

$\Rightarrow$  behaves like an incompressible fluid

The flow through the phase space is incompressible

Seen from an observer that follows the flow in the phase space, i.e. an orbit:  $f$  is constant

# Liouville's theorem (corollary)

In the motion of a stellar system, any volume of phase space remains constant

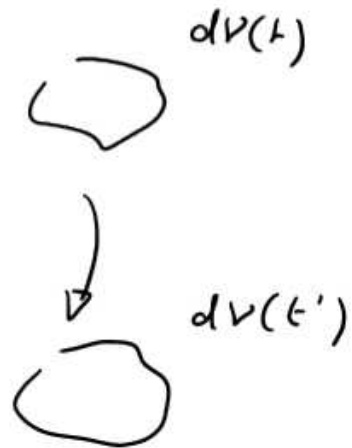
$dV$  : an infinitesimal volume of the phase space

$dN(t)$ : the number of stars in  $dV(t)$  at  $t$

$$dN(t) = \tilde{f}(\tilde{w}, t) dV(t)$$

$dN(t')$ : the number of stars in  $dV(t')$  at  $t'$

$$dN(t') = \tilde{f}(\tilde{w}, t') dV(t')$$



$$\text{But } dN(t) = dN(t')$$

$$\equiv \frac{dN}{dt} = 0$$

Because EoM are 1<sup>st</sup> order differential equations, only the points that were in  $dV$  at  $t$  are in  $dV'$  at  $t'$

Thus

$$\frac{dN}{dt} = \frac{d}{dt} \left( \tilde{f}(w, t) dV(t) \right)$$

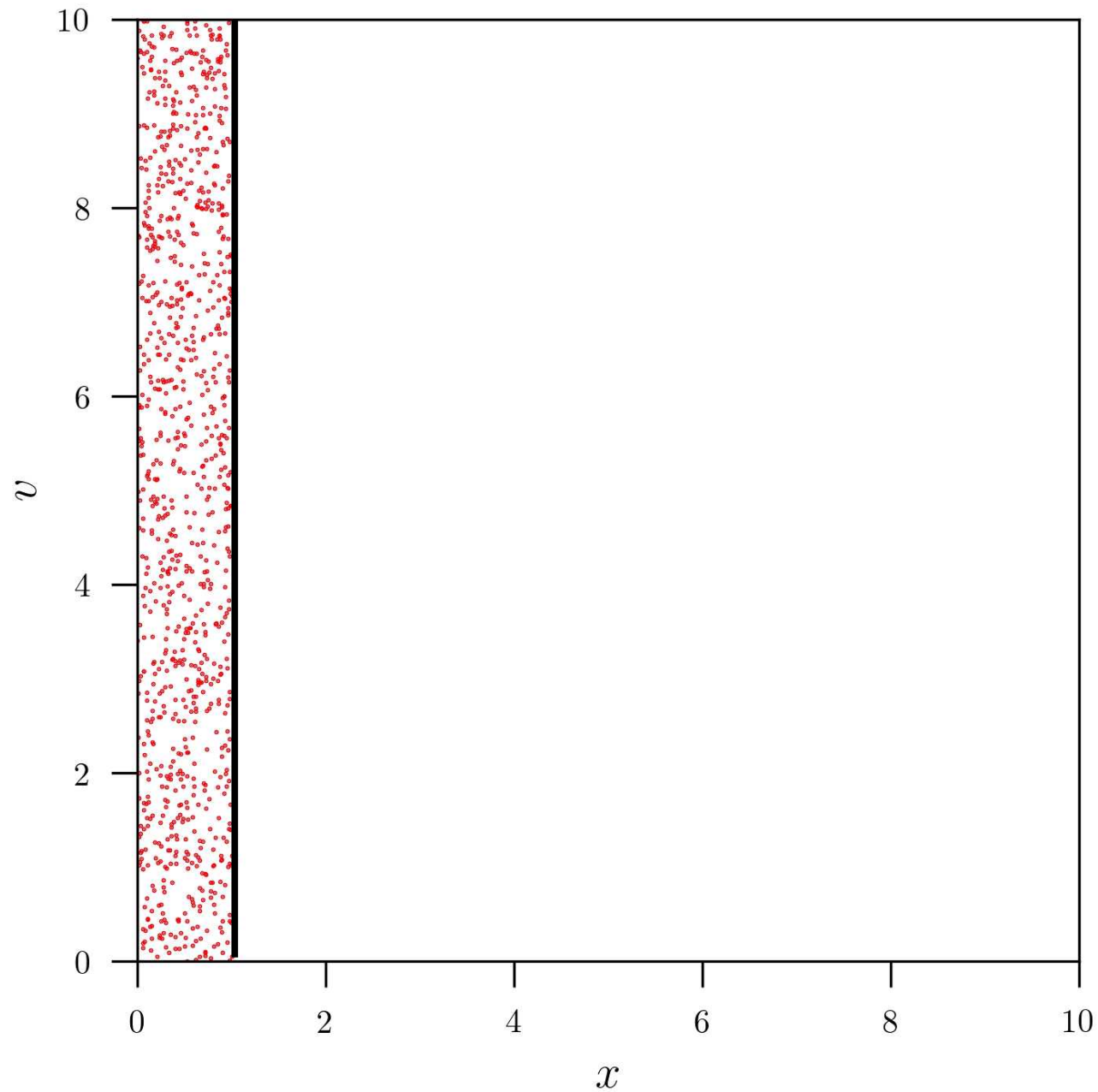
$$= \underbrace{\frac{d}{dt} \left( \tilde{f}(w, t) \right)}_{=0 \text{ (Boltzmann equation)}} dV(t) + \tilde{f}(w, t) \frac{d}{dt} (dV(t)) = 0$$

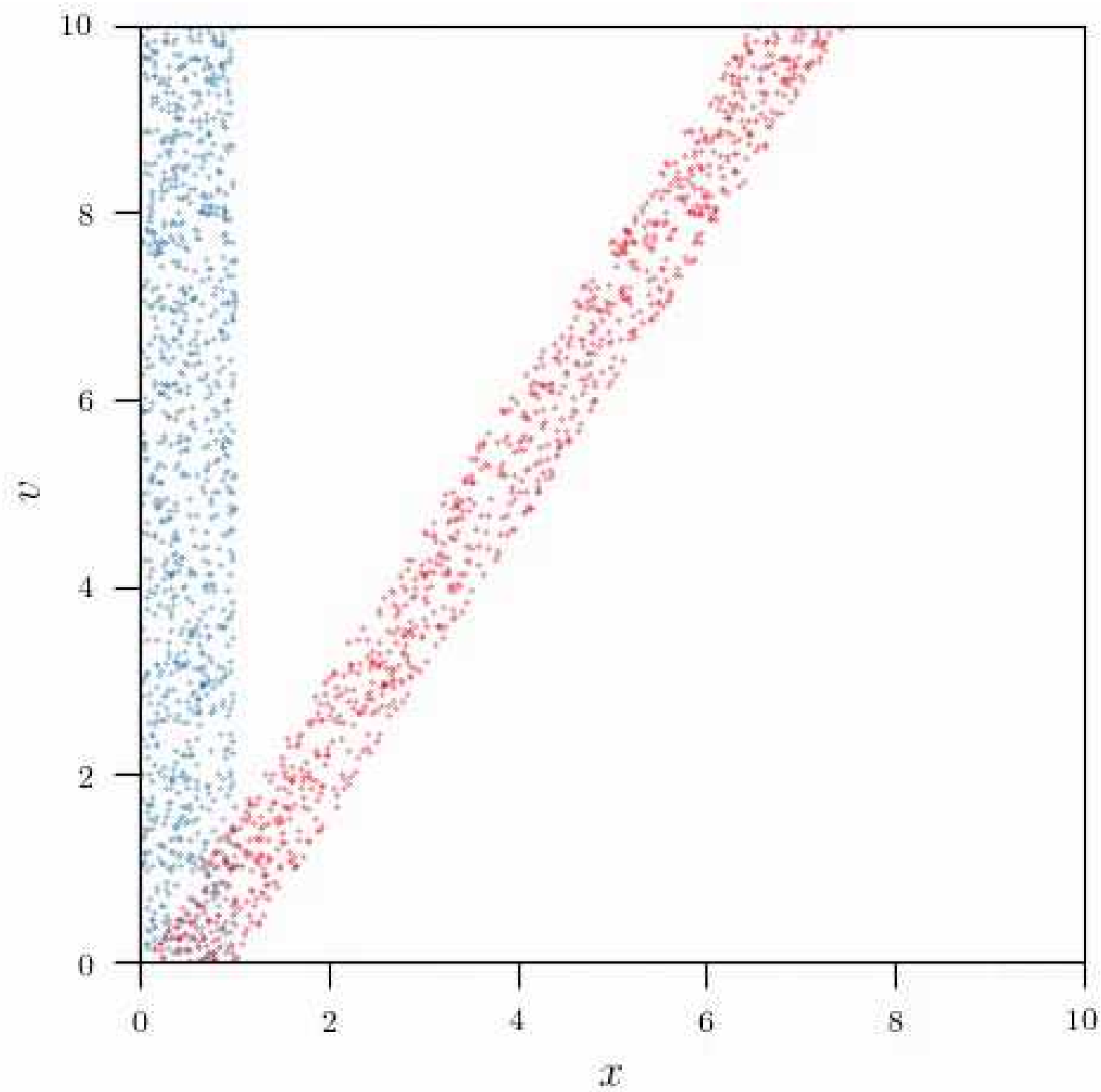
$$\Rightarrow \frac{d}{dt} (dV(t))$$

$$dV(t) = \text{cte}$$

The distribution function remains constant along the flow

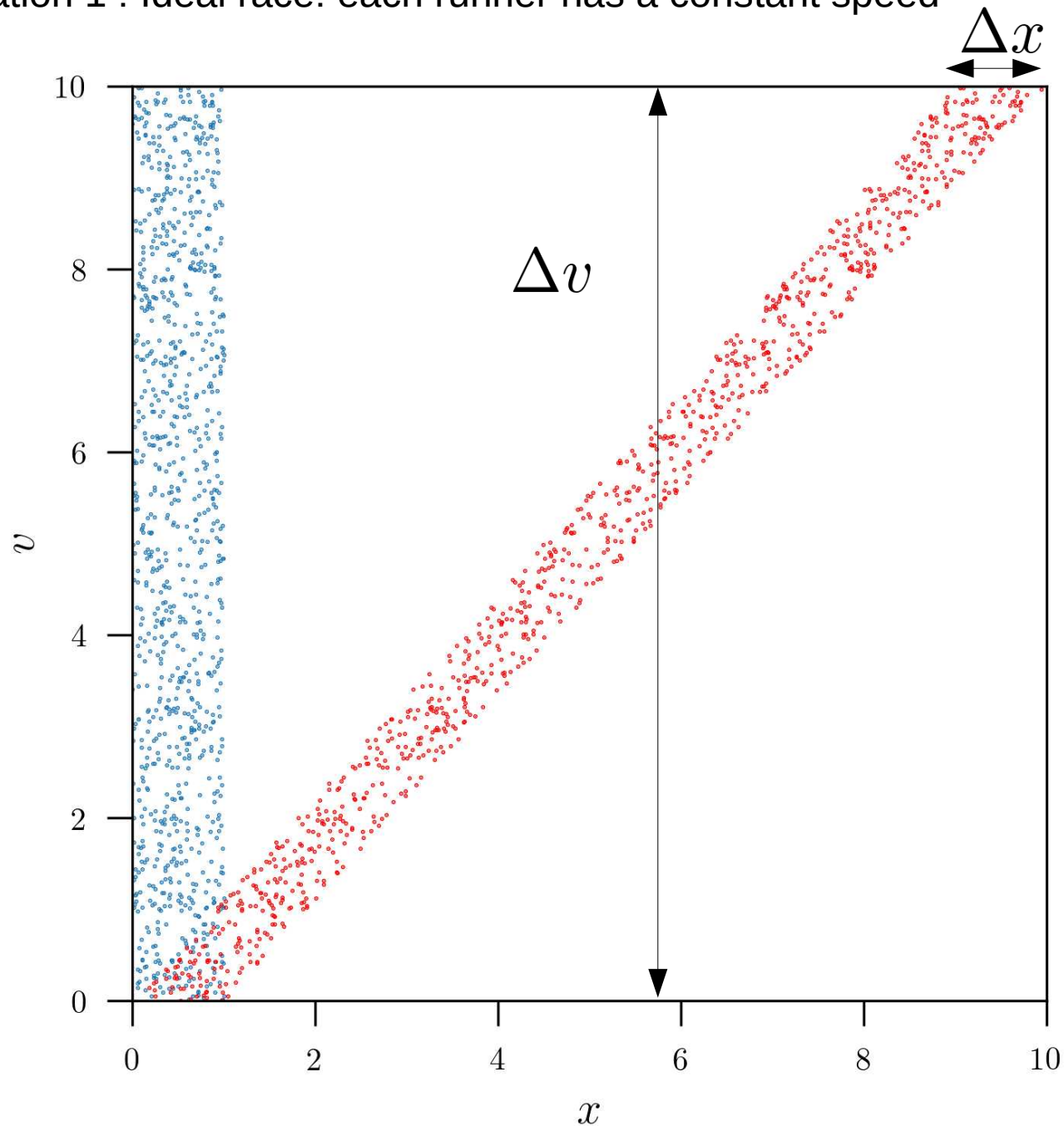
Illustration 1 : Ideal race: each runner has a constant speed





# The distribution function remains constant along the flow

Illustration 1 : Ideal race: each runner has a constant speed



$\nu$ : the phase space volume

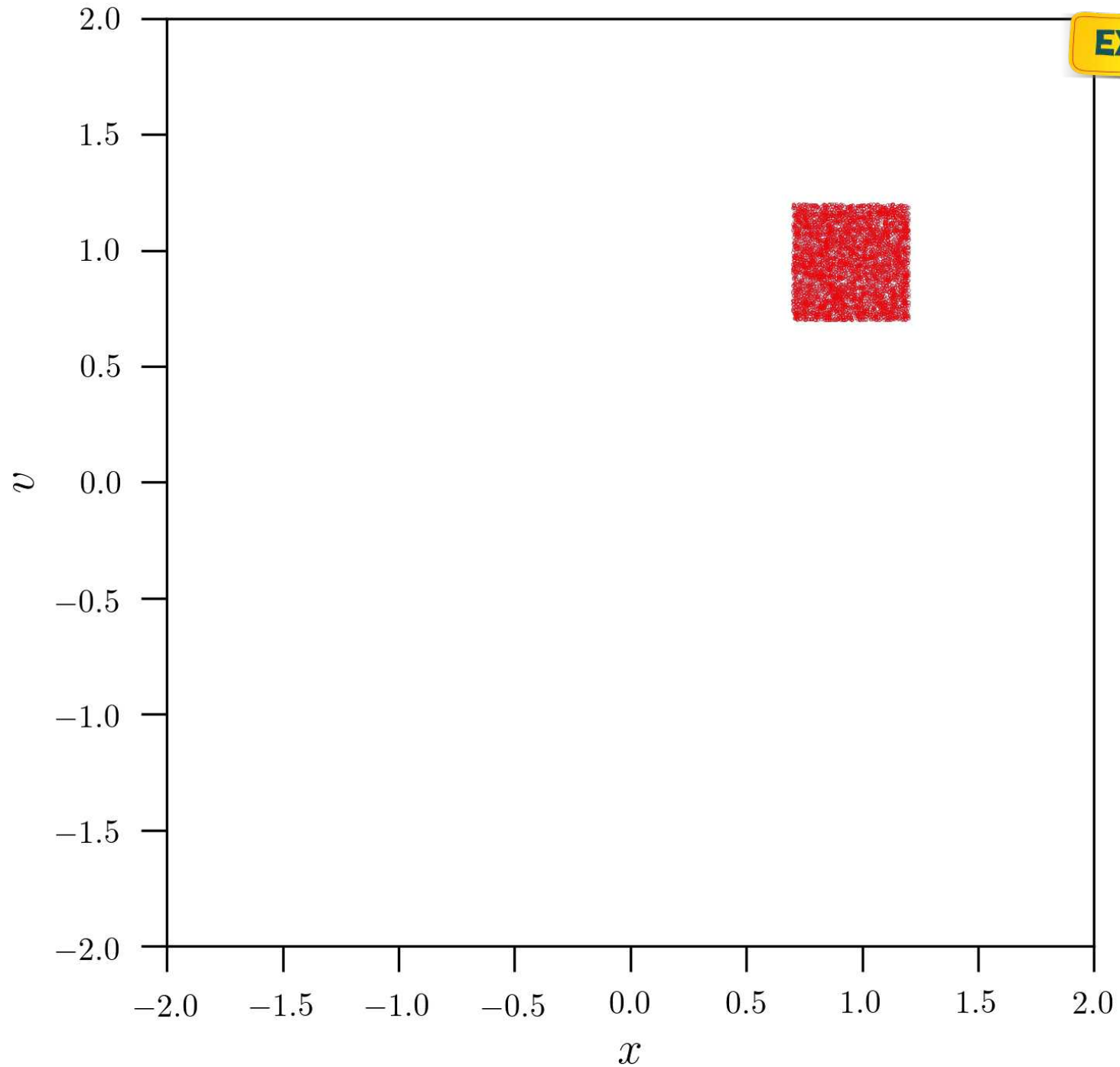
$$\tilde{f}(t=0) = \frac{N}{\nu_0} = \frac{N}{\Delta x \Delta v}$$

$$\tilde{f}(t=t) = \frac{N}{\nu_t} = \frac{N}{\Delta x \Delta v}$$

Illustration 2 : Harmonic oscillator

$$H(x, \dot{x}) = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\omega^2 x^2$$

$$\omega = 1$$



**EXERCICE**

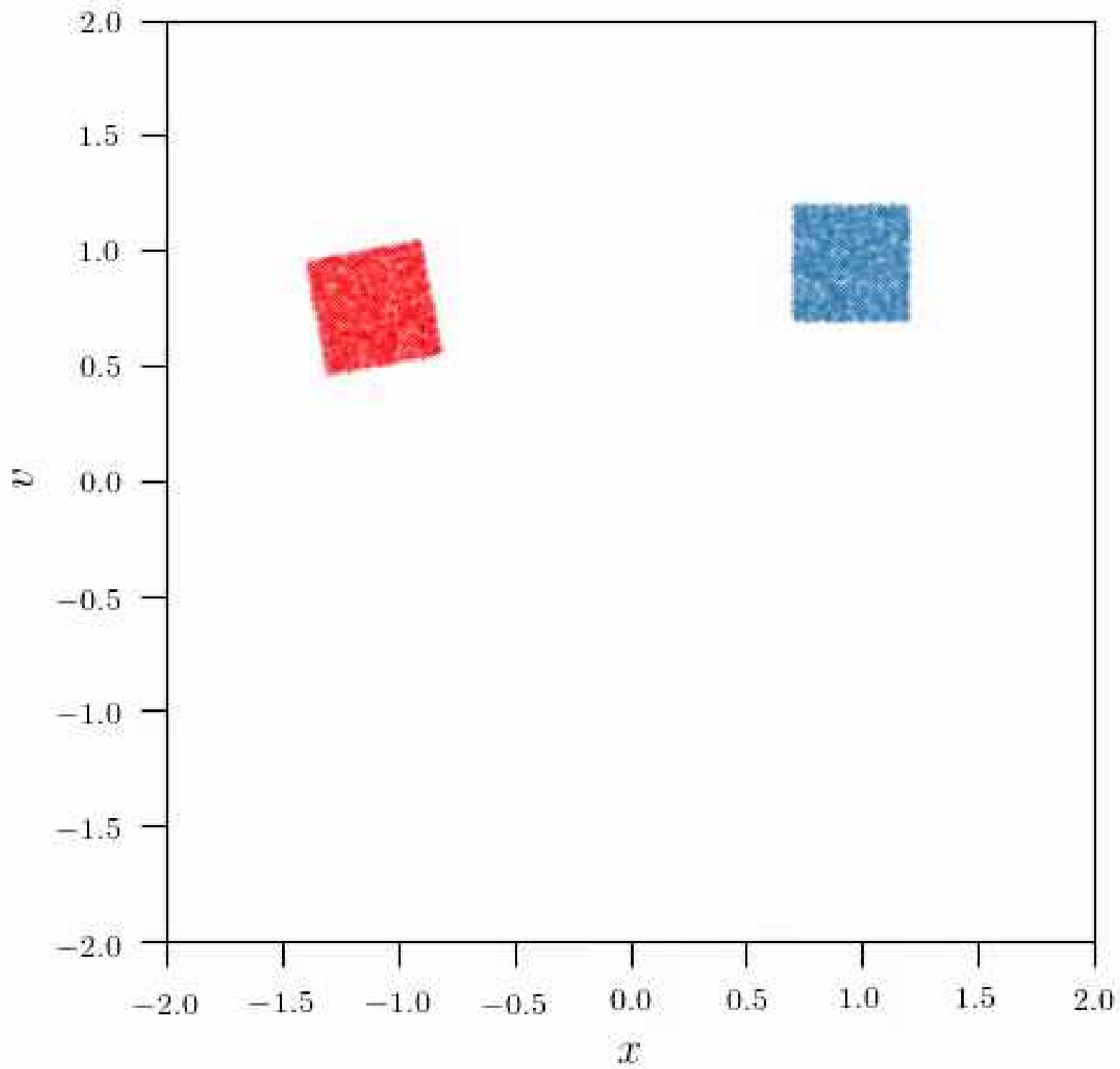
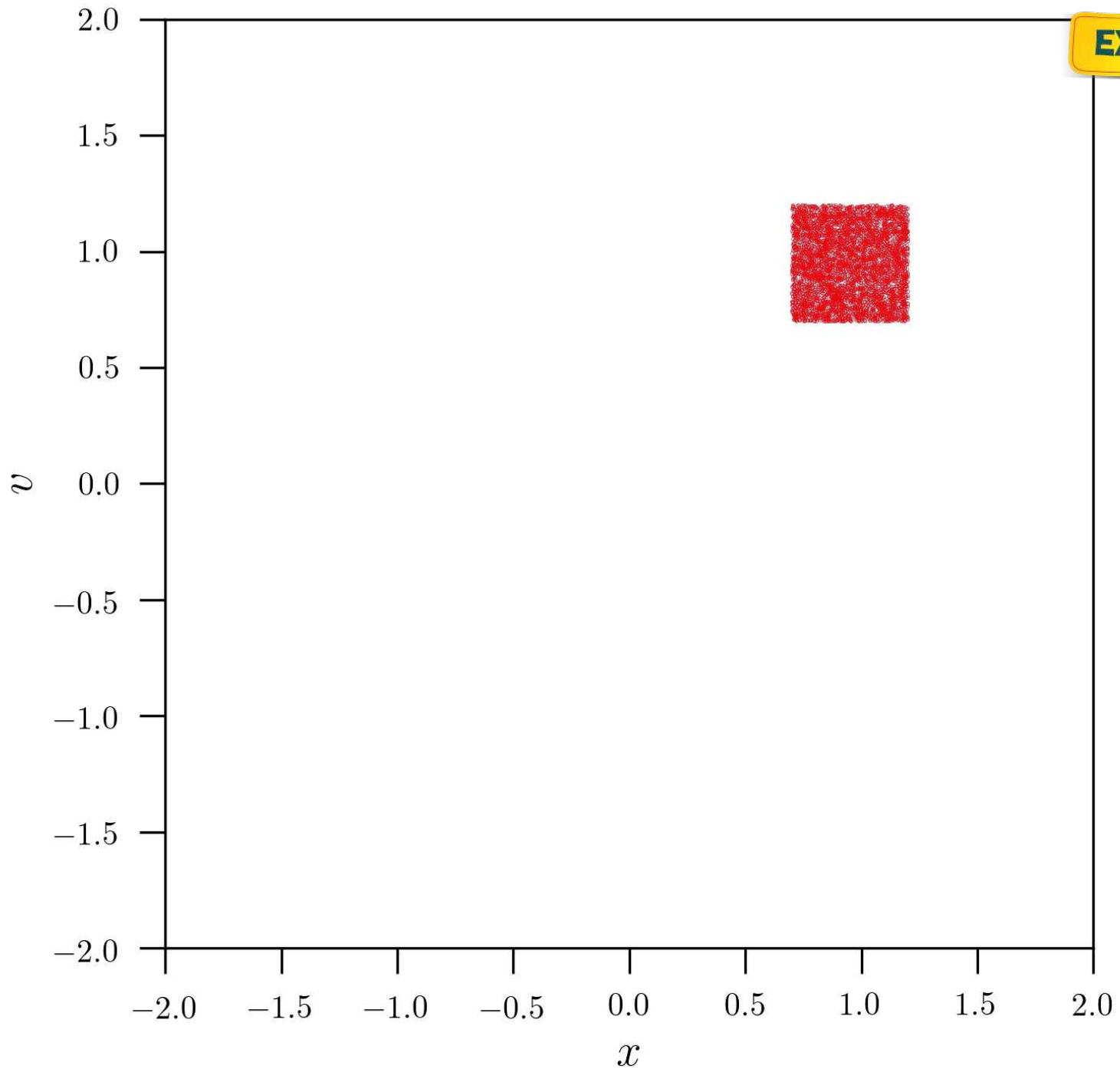


Illustration 2 : Harmonic oscillator

$$H(x, \dot{x}) = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\omega^2 x^2$$

$$\omega = 0.75$$



**EXERCICE**

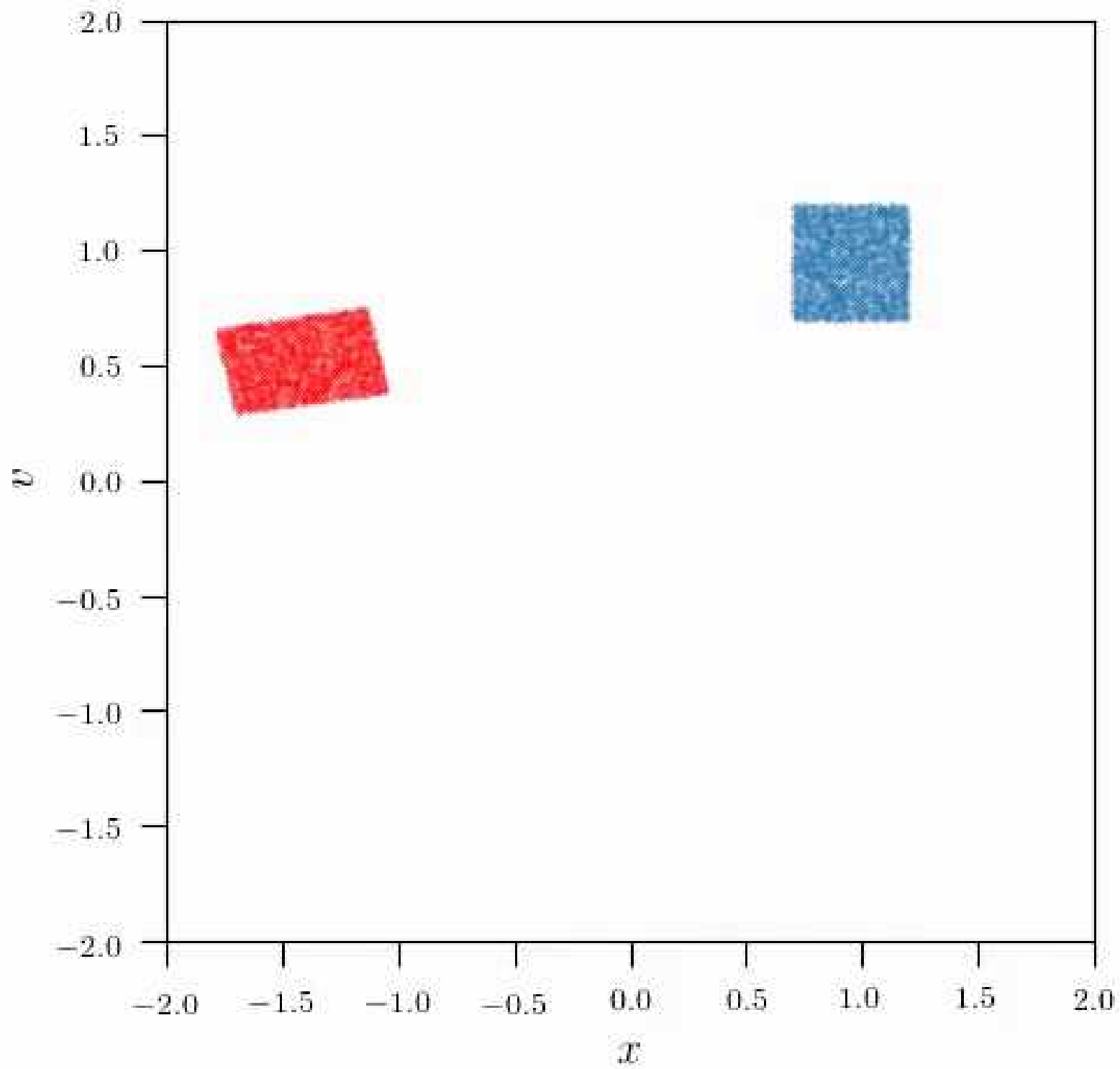
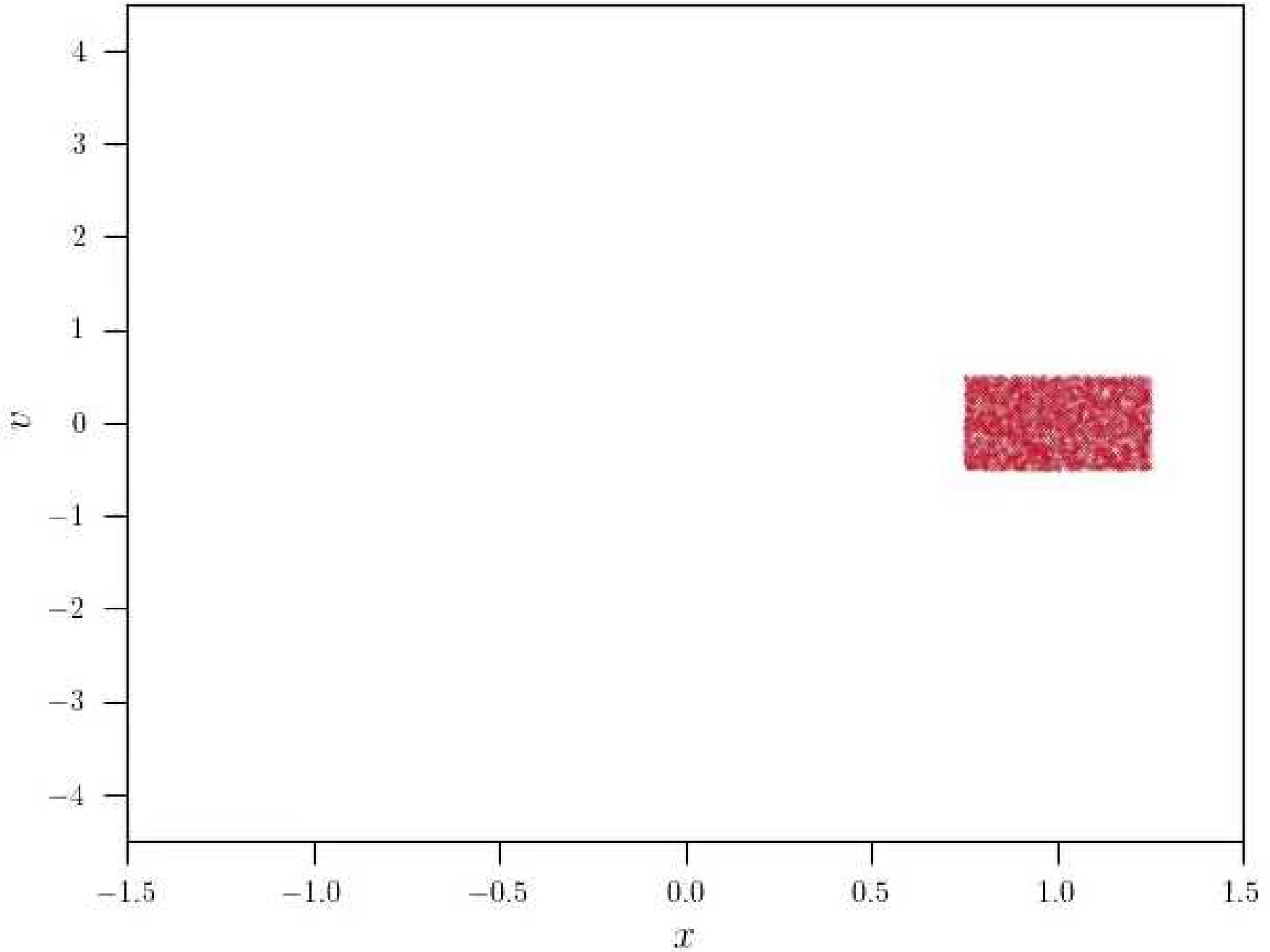
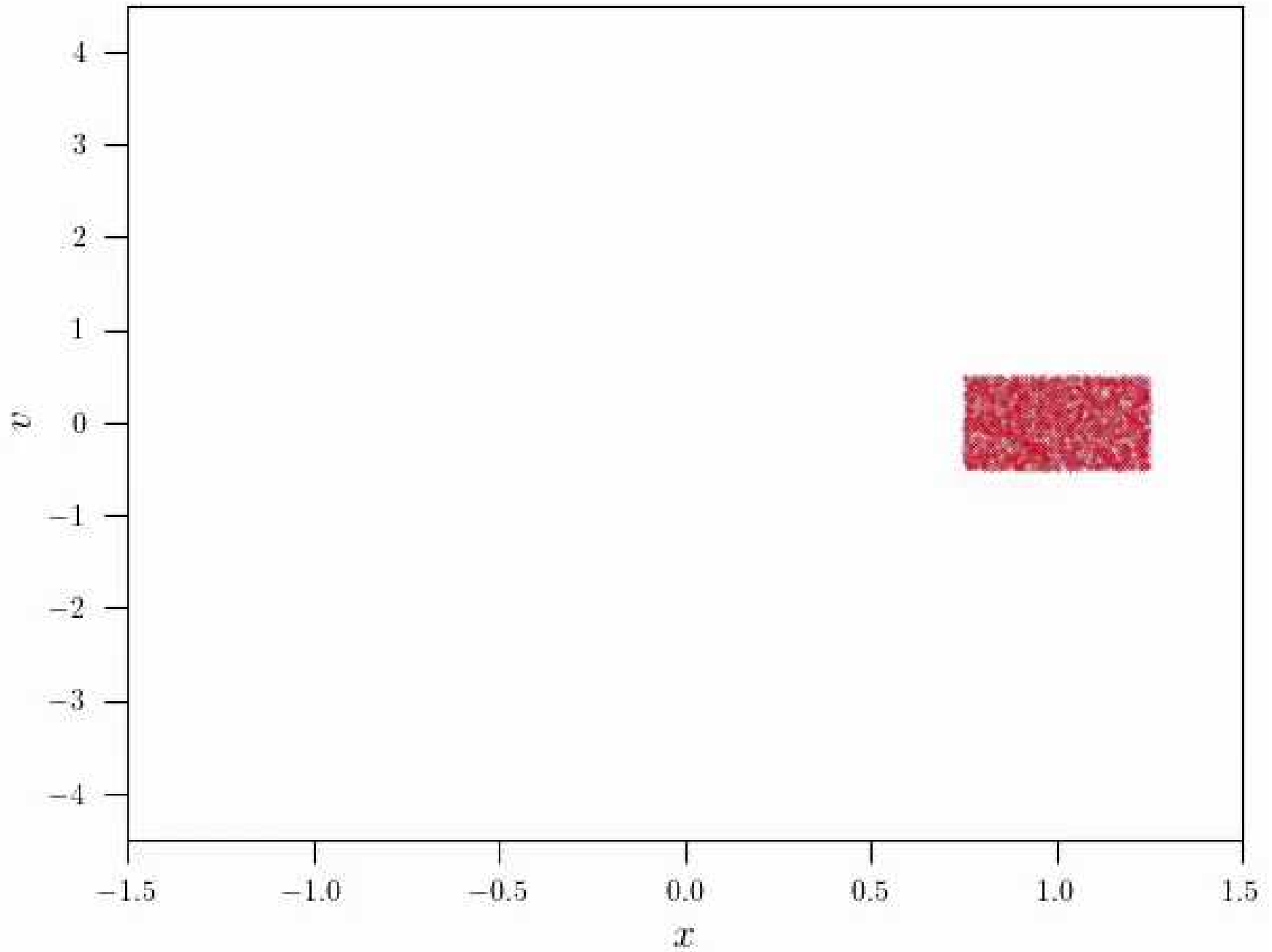


Illustration 3 : Plummer

$$H(x, \dot{x}) = \frac{1}{2}\dot{x}^2 - \frac{GM}{x}$$



### Illustration 3 : Plummer



Expressing the continuity equation using  $\tilde{\omega} = (\tilde{q}, \tilde{p})$

---

$$\begin{aligned}\frac{d}{dt} f(\tilde{\omega}, t) &= \frac{\partial f(\tilde{\omega}, t)}{\partial t} + \vec{\nabla}_{\tilde{\omega}} \cdot (f(\tilde{\omega}, t) \dot{\tilde{\omega}}) = 0 \\ &= \frac{\partial f(\tilde{\omega}, t)}{\partial t} + \dot{\tilde{\omega}} \cdot \vec{\nabla}_{\tilde{\omega}} f(\tilde{\omega}, t) = 0 \\ &= \frac{\partial f(\tilde{q}, \tilde{p})}{\partial t} + \sum_i \dot{q}_i \frac{\partial}{\partial q_i} f(\tilde{q}, \tilde{p}) + \sum_i \dot{p}_i \frac{\partial}{\partial p_i} f(\tilde{q}, \tilde{p})\end{aligned}$$

$$\frac{d}{dt} f(\tilde{\omega}, t) = \frac{\partial f(\tilde{q}, \tilde{p})}{\partial t} + \dot{\tilde{q}} \cdot \frac{\partial}{\partial \tilde{q}} f(\tilde{q}, \tilde{p}) + \dot{\tilde{p}} \cdot \frac{\partial}{\partial \tilde{p}} f(\tilde{q}, \tilde{p}) = 0$$

The Collisionless Boltzmann Equation

## Using the Hamilton Equations

$$\dot{\vec{q}} = \frac{\partial H}{\partial \vec{p}} \quad \dot{\vec{p}} = - \frac{\partial H}{\partial \vec{q}}$$

Then

$$\frac{\partial}{\partial t} f + \dot{\vec{q}} \frac{\partial}{\partial \vec{q}} f + \dot{\vec{p}} \frac{\partial}{\partial \vec{p}} f = 0$$

becomes

$$\frac{\partial}{\partial t} f + \frac{\partial H}{\partial \vec{p}} \frac{\partial f}{\partial \vec{q}} - \frac{\partial H}{\partial \vec{q}} \frac{\partial f}{\partial \vec{p}} = 0$$

$$\frac{\partial}{\partial t} f + [f, H] = 0$$

Poisson brackets

$$[A, B] := \frac{\partial A}{\partial \vec{q}} \frac{\partial B}{\partial \vec{p}} - \frac{\partial A}{\partial \vec{p}} \frac{\partial B}{\partial \vec{q}}$$
$$= \sum_i^n \frac{\partial A}{\partial q_i} \frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i} \frac{\partial B}{\partial q_i}$$

# The Collisionless Boltzmann equation in various coordinates

**EXERCICE**

## Generalized coordinates

$$\vec{p} = \frac{\partial L(\vec{q}, \dot{\vec{q}})}{\partial \dot{\vec{q}}}$$

$$\frac{\partial f}{\partial t} + \dot{\vec{q}} \cdot \frac{\partial f}{\partial \vec{q}} + \dot{\vec{p}} \cdot \frac{\partial f}{\partial \vec{p}} \equiv \frac{\partial f}{\partial t} + \frac{\partial f}{\partial \vec{q}} \cdot \frac{\partial H}{\partial \vec{p}} - \frac{\partial f}{\partial \vec{p}} \cdot \frac{\partial H}{\partial \vec{q}} = 0$$

## Cartesian coordinates

$$\begin{cases} p_x = \dot{x} = v_x \\ p_y = \dot{y} = v_y \\ p_z = \dot{z} = v_z \end{cases} \quad H = \frac{1}{2} (v_x^2 + v_y^2 + v_z^2) + \Phi(x, y, z)$$

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \frac{\partial f}{\partial \vec{x}} - \frac{\partial \Phi}{\partial \vec{x}} \cdot \frac{\partial f}{\partial \vec{v}} = 0$$

## Spherical coordinates

$$\begin{cases} p_r = \dot{r} = v_r \\ p_\theta = r^2 \dot{\theta} = r v_\theta \\ p_\phi = r^2 \sin^2(\theta) \dot{\phi} = r \sin(\theta) v_\phi \end{cases}$$

$$H = \frac{1}{2} \left( p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2 \sin^2(\theta)} \right) + \Phi(R, \theta, \phi)$$

$$\frac{\partial f}{\partial t} + p_r \frac{\partial f}{\partial r} + \frac{p_\theta}{r^2} \frac{\partial f}{\partial \theta} + \frac{p_\phi}{r^2 \sin^2(\theta)} \frac{\partial f}{\partial \phi} - \left( \frac{\partial \Phi}{\partial r} - \frac{p_\theta^2}{r^3} - \frac{p_\phi^2}{r^3 \sin^2(\theta)} \right) \frac{\partial f}{\partial p_r} - \left( \frac{\partial \Phi}{\partial \theta} - \frac{p_\phi^2 \cos(\theta)}{r^2 \sin^3(\theta)} \right) \frac{\partial f}{\partial p_\theta} - \frac{\partial \Phi}{\partial \phi} \frac{\partial f}{\partial p_\phi} = 0$$

## Cylindrical coordinates

$$\begin{cases} p_R = \dot{R} = v_R \\ p_\phi = R^2 \dot{\phi} = R v_\phi \\ p_z = \dot{z} = v_z \end{cases}$$

$$H = \frac{1}{2} \left( p_R^2 + \frac{p_\phi^2}{R^2} + p_z^2 \right) + \Phi(R, \phi, z)$$

$$\frac{\partial f}{\partial t} + p_R \frac{\partial f}{\partial R} + \frac{p_\phi}{R^2} \frac{\partial f}{\partial \phi} + p_z \frac{\partial f}{\partial z} - \left( \frac{\partial \Phi}{\partial R} - \frac{p_\phi^2}{R^3} \right) \frac{\partial f}{\partial p_R} - \frac{\partial \Phi}{\partial \phi} \frac{\partial f}{\partial p_\phi} - \frac{\partial \Phi}{\partial z} \frac{\partial f}{\partial p_z} = 0$$

# Limits of the Collisionless Boltzmann equation

## I. Finite stellar lifetime

- Stars are created and die. The hypothesis of conservation of the probability/number is violated.

We should better have (in Cartesian coordinates):

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \frac{\partial f}{\partial \vec{x}} - \frac{\partial \Phi}{\partial \vec{x}} \cdot \frac{\partial f}{\partial \vec{v}} = B(\vec{x}, \vec{v}, t) - D(\vec{x}, \vec{v}, t)$$

$$\begin{aligned} &\sim \frac{v}{R} f && \sim \frac{a}{v} f \\ &\sim \frac{1}{t_{\text{cross}}} f && \sim \frac{1}{t_{\text{cross}}} f \end{aligned}$$

Rate per unit phase-space volume at which stars are born and die

- Define

$$\gamma = \frac{|B - D|}{f} t_{\text{cross}}$$

If  $\gamma \ll 1$  the approximation is ok

i.e. : the fractional change in the number of stars per crossing time must be small.

# Limits of the Collisionless Boltzmann equation

$$T_{\text{cross}} \sim 300 \text{ Myr}$$

Examples:

- M-stars in an elliptical galaxies
  - Life time  $> 10 \text{ Gyr}$  ( $> t_{\text{cross}}$ )  $\gamma \cong 0$
  - $B=0$  (no star formation)
- O-stars in the Milky Way
  - Life time  $< 100 \text{ Myr}$  ( $< t_{\text{cross}}$ )  $\gamma \gg 1$
  - Do not move much, the phase space distribution will be dominated by star formation processes
- Main sequence stars ( $M < 1.5 M_{\odot}$ )
  - Life time  $> 1 \text{ Gyr}$  ( $> t_{\text{cross}}$ )  $\gamma \cong 0$

# Limits of the Collisionless Boltzmann equation

## II. Correlation between stars

- We assumed that the probability of finding one peculiar stars somewhere in the phase space is independent of the others. Mathematically: the probability of finding particle "i" in  $d^6\vec{\omega}$  and "j" in  $d^6\vec{\omega}'$  is :

$$f(\vec{\omega})d^6\vec{\omega} \cdot f(\vec{\omega}')d^6\vec{\omega}'$$

This is not completely true, as stars interact gravitationally and may generate correlations.

However, this is not a real problem as long as the forces between nearby stars do not dominate over the forces due to the rest of the system (the definition of a collisionless system).

**Equilibria of collisionless systems**

**Relations between the DFs  
and observables**

## Relations between the DF and observables

$$f(\vec{w})$$

- $f(\vec{w})$  : probability density  
in the phase space
- $f(\vec{w}) d^6\vec{w}$  : probability of finding 1 star  
in the phase space volume  $[\vec{w}, \vec{w} + d\vec{w}]$

## Distribution function in the configuration space

$$\nu(\vec{x}) = \int d^3\vec{v} f(\vec{x}, \vec{v})$$

- $\nu(\vec{x})$  : probability density  
in the configuration space
- $\nu(\vec{x}) d^3\vec{x}$  : probability of finding 1 star  
in the configuration space volume  $[\vec{x}, \vec{x} + d\vec{x}]$

## Distribution function in the configuration space

$$n(\vec{x}) = N \nu(\vec{x}) = \int d^3\vec{v} \hat{f}(\vec{x}, \vec{v})$$

- $n(\vec{x})$  : number density of star in the configuration space
- $n(\vec{x}) d^3\vec{x}$  : probability of finding  $N$  stars in the configuration space volume  $[\vec{x}, \vec{x} + d\vec{x}]$

## Distribution function in the configuration space

$$\rho(\vec{x}) = N \cdot m \cdot \nu(\vec{x}) = m \int d^3\vec{v} \tilde{f}(\vec{x}, \vec{v})$$

$m$ : mass of particles

- $\rho(\vec{x})$  : mass density of star in the configuration space
- $\rho(\vec{x}) d^3\vec{x}$  : probability of finding a mass  $M = N \cdot m$  in the configuration space volume  $[\vec{x}, \vec{x} + d\vec{x}]$

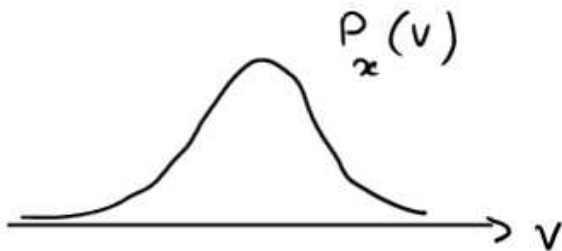
## Distribution function in the velocity space

$$P_{\vec{x}}(\vec{v}) = \frac{f(\vec{x}, \vec{v})}{v(\vec{x})}$$

$$\int P_{\vec{x}}(\vec{v}) d^3\vec{v} = \frac{1}{v(\vec{x})} \int \underbrace{f(\vec{x}, \vec{v})}_{:= v(\vec{x})} d^3\vec{v} = 1$$

$\equiv$  velocity distribution function (VDF)

- $P_{\vec{x}}(\vec{v})$  : probability density at the position  $\vec{x}$  in the velocity space
- $P_{\vec{x}}(\vec{v}) d^3\vec{v}$  : probability of finding 1 star in  $\vec{x}$  in the velocity space volume  $[\vec{v}, \vec{v} + d\vec{v}]$

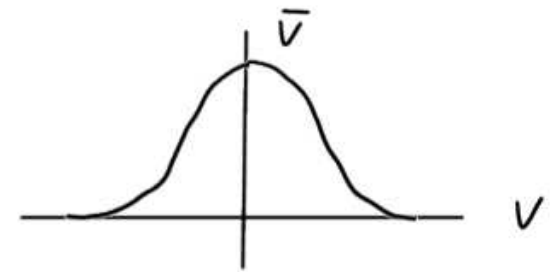


can be measured near the sun

Mean velocity (first moment of the VDF)

$$\vec{V}(\vec{x}) = \int \vec{v} P_{\vec{x}}(\vec{v}) d^3\vec{v} = \frac{1}{V(\vec{x})} \int \vec{v} f(\vec{x}, \vec{v}) d^3\vec{v}$$

- along one peculiar axis  $\vec{n}$



$$\vec{V}_{\vec{n}}(\vec{x}) = \int \vec{v} \cdot \vec{n} P_{\vec{x}}(\vec{v}) d^3\vec{v} = \frac{1}{V(\vec{x})} \int \vec{v} \cdot \vec{n} f(\vec{x}, \vec{v}) d^3\vec{v}$$

- if  $\vec{n} = \vec{e}_i$

$$\vec{V}_i(\vec{x}) = \int v_i P_{\vec{x}}(\vec{v}) d^3\vec{v} = \frac{1}{V(\vec{x})} \int v_i f(\vec{x}, \vec{v}) d^3\vec{v}$$

## Velocity dispersion tensor (second moment of the VDF)

$$\sigma_{ij}^2 = \int (v_i - \bar{v}_i)(v_j - \bar{v}_j) P_{\vec{x}}(\vec{v}) d^3\vec{v}$$

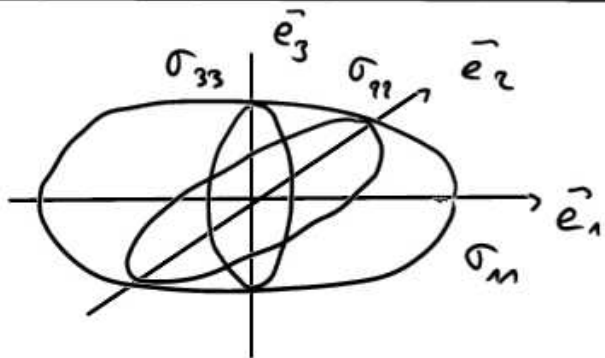
$$= \frac{1}{N(\vec{x})} \int (v_i - \bar{v}_i)(v_j - \bar{v}_j) f(\vec{x}, \vec{v}) d^3\vec{v}$$

$$= \frac{1}{N(\vec{x})} \left[ \int v_i v_j f(\vec{x}, \vec{v}) d^3\vec{v} - \left( \int v_i f(\vec{x}, \vec{v}) d^3\vec{v} \right) \left( \int v_j f(\vec{x}, \vec{v}) d^3\vec{v} \right) \right]$$

$$= \overline{v_i v_j} - \bar{v}_i \bar{v}_j$$

3x3 symmetric tensor  
⇒ may be diagonalised

Describe an ellipsoid (velocity ellipsoid)



$$\sigma_{ij}^2 = \sigma_{ii}^2 \delta_{ij} = \begin{pmatrix} \sigma_{11} & 0 & 0 \\ 0 & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{pmatrix}$$

**The End**