Astrophysics IV: Stellar and galactic dynamics

Solutions

Problem 1:

Using the definition

$$v_c = R\Omega(R) \ ,$$

it follows that

$$\frac{\mathrm{d}\Omega}{\mathrm{d}R} = \frac{1}{R}\frac{\mathrm{d}v_c}{\mathrm{d}R} - v_c\frac{1}{R^2} \ .$$

Then

$$A(R) \equiv \frac{1}{2}\left(\frac{v_c}{R} - \frac{\mathrm{d}v_c}{\mathrm{d}R}\right) = \frac{1}{2}\left(-R\left(\frac{1}{R}\frac{\mathrm{d}v_c}{\mathrm{d}R} - \frac{v_c}{R^2}\right)\right) = -\frac{1}{2}R\frac{\mathrm{d}\Omega}{\mathrm{d}R}$$

$$B(R) \equiv -\frac{1}{2}\left(\frac{v_c}{R} + \frac{\mathrm{d}v_c}{\mathrm{d}R}\right) = -\frac{1}{2}\left(\Omega + \left(\frac{1}{R}\frac{\mathrm{d}v_c}{\mathrm{d}R} - \frac{v_c}{R^2}\right) + \frac{v_c}{R}\right) = -\left(\Omega + \frac{1}{2}R\frac{\mathrm{d}\Omega}{\mathrm{d}R}\right)$$

$$\Omega = A - B = \frac{1}{2}\left(\frac{v_c}{R} - \frac{\mathrm{d}v_c}{\mathrm{d}R}\right) + \frac{1}{2}\left(\frac{v_c}{R} + \frac{\mathrm{d}v_c}{\mathrm{d}R}\right) = \frac{v_c}{R} = \Omega$$

$$\begin{aligned}\kappa^2 &= \left(R\frac{\mathrm{d}(\Omega^2)}{\mathrm{d}R} + 4\Omega^2\right) = \left(2R\Omega\frac{\mathrm{d}\Omega}{\mathrm{d}R} + 4\Omega^2\right) = 2\Omega\left(R\frac{\mathrm{d}\Omega}{\mathrm{d}R} + 2\Omega\right) \\ &= 2\Omega\,(-2B) = -4B(A-B)\end{aligned}$$

Problem 2:

We have:

$$R^2 = x^2 + y^2$$

$$\vec{L} = \vec{r} \times \vec{v} = \vec{x} \times \dot{\vec{x}} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \times \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} y\dot{z} - z\dot{y} \\ z\dot{x} - x\dot{z} \\ x\dot{y} - y\dot{x} \end{pmatrix}$$

Since we're working in the $z = 0$ plane, and the z component of $\vec{L}$ is given by $L_z = x\dot{y} - y\dot{x}$, inserting this to compute L^2 gives:

$$L^2 = (y\dot{z} - z\dot{y})^2 + (z\dot{x} - x\dot{z})^2 + (x\dot{y} - y\dot{x})^2 = (x^2 + y^2)\dot{z}^2 + L_z^2 = R^2\dot{z}^2 + L_z^2 \; .$$

Now we use the energy conservation:

$$E = \frac{1}{2}\dot{R}^2 + \frac{1}{2}\dot{z}^2 + \Phi_{eff}(R, z) \; .$$

Now, we can eliminate $\dot{z}^2$ by using our expression for L^2:

$$E = \frac{1}{2}\dot{R}^2 + \frac{1}{2}\frac{1}{R^2}(L^2 - L_z^2) + \Phi_{eff}(R, z) \; .$$

Using $\Phi_{eff} = \frac{1}{2}\frac{L_z^2}{R^2} + \Phi$ and solving for $\dot{R}$ gives:

$$\dot{R} = \pm\sqrt{2\left(E - \frac{1}{2}\frac{L^2 - L_z^2}{R^2} - \Phi_{eff}\right)} = \pm\sqrt{2\left(E - \frac{L^2}{2R^2} - \Phi\right)} \; .$$

<u>Problem 3</u>:

We start from the Lagrangian:

$$L(\vec{x}, \dot{\vec{x}}) = \frac{1}{2}\left(\dot{\vec{x}} + \Omega \times \vec{x}\right)^2 - \Phi(\vec{x}) \; . \tag{1}$$

From the derivative of this Lagrangian, we can write the momentum $\vec{p}$ as :

$$\vec{p} = \dot{\vec{x}} + \Omega \times \vec{x} \; . \tag{2}$$

Using the Legendre transformation, we obtain the Hamiltonian:

$$H(q, p) = \frac{1}{2}\vec{p}^2 + \Phi(\vec{q}) - \vec{\Omega} \cdot (\vec{q} \times \vec{p}) \; , \tag{3}$$

where we renamed $\vec{x}$ by $\vec{q}$.

We set the rotation to be along the z axis, and for it to be uniformly rotating, it needs to be constant, i.e.

$$\vec{\Omega} = \begin{pmatrix} 0 \\ 0 \\ \Omega \end{pmatrix} \quad \Rightarrow \vec{\Omega} \cdot (\vec{q} \times \vec{p}) = \Omega(q_x p_y - p_x q_y) \; .$$

The equations of motion in canonical coordinates are given by Hamilton's equations:

$$\dot{p} = -\frac{\partial}{\partial q}H(p, q), \quad \dot{q} = \frac{\partial}{\partial p}H(p, q) \; . \tag{4}$$

In our case, we get:

$$
\begin{aligned}
\dot{q}_x &= p_x + \Omega q_y \\
\dot{q}_y &= p_y - \Omega q_x \\
\dot{p}_x &= -\frac{\partial}{\partial q_x}\Phi(q,p) + \Omega p_y \\
\dot{p}_y &= -\frac{\partial}{\partial q_y}\Phi(q,p) - \Omega p_x \; .
\end{aligned}
$$

The relations between Cartesian and canonical coordinates are:

$$
\begin{aligned}
q_x &= x \\
q_y &= y \\
p_x &= \dot{x} - \Omega y \\
p_y &= \dot{y} + \Omega x \; .
\end{aligned}
$$