Stellar orbits

4th part

Outlines

Surfaces of section

- Integral of motions
- Poincaré maps

Orbits in planar non-axisymmetric potential

- Surface of sections
    - energy dependency
    - flattening dependency
- Integrals of motions

Orbits in planar non-axisymmetric rotating potential

- The Jacobi integral
- Lagrange points
- Orbits around Lagrange points
- Orbits not confined to Lagrange points

Surfaces of section

Can we visualize the phase phase and check if an additional integral of motion exists ?

<u>Idea</u> :

We study the orbits in the meridional plane

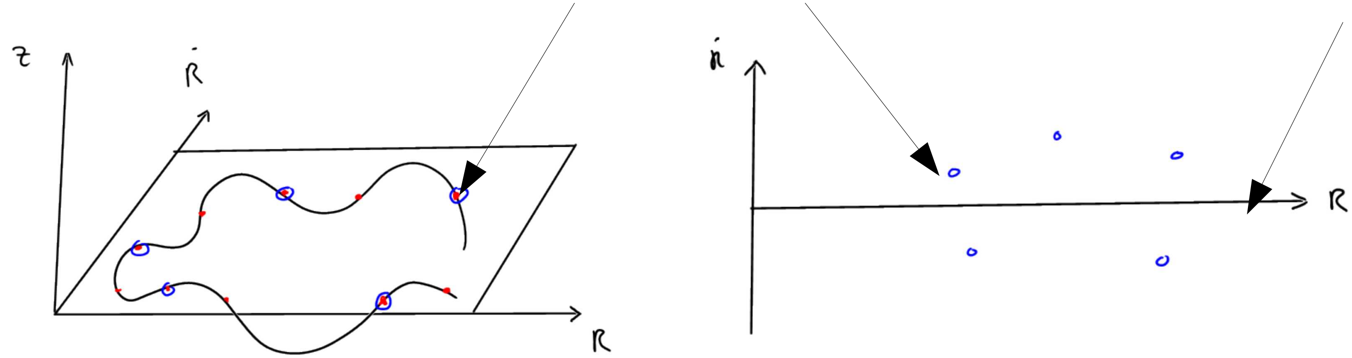
- 4-D — 4 indep. variables $(R, z, \dot{R}, \dot{z})$
- Energy E

$\rightarrow$ 3-D — 3 indep. variables $(R, z, \dot{R})$

- Drawing a 3-D phase space is still not easy. Instead, we draw slices of the phase space. We plot only phase space points that:
  - cross the $z = 0$ plane
  - have $\dot{z} > 0$

consequents

Surface of section or Poincaré maps



Surfaces of section

- A point in the surface of section (for a given E and L_z) defines an orbit as the three independent variables $(R, \dot{R}, z = 0)$ are defined.

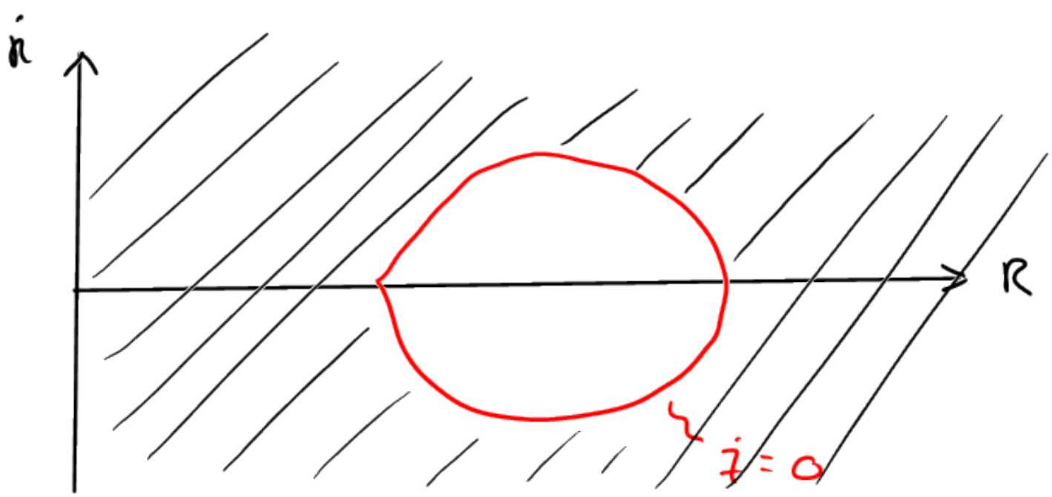
- Even if orbits have the same energy, they will never intersect in the plane (EoM are first order diff. equations).

- <u>Zero velocity curve</u> : curve defined by $\dot{z} = 0$

$$E = \frac{1}{2}\dot{R}^2 + \frac{1}{2}\dot{z}^2 + \Phi_{\rm eff}(R, z=0) \qquad \Rightarrow \qquad \dot{R} \leqslant \pm\sqrt{2\left[E - \Phi_{\rm eff}(R, z=0)\right]}$$

$$\dot{R}(R) = \pm\sqrt{2\left[E - \Phi_{\rm eff}(R, z=0)\right]}$$

defines the accessible region of the phase space

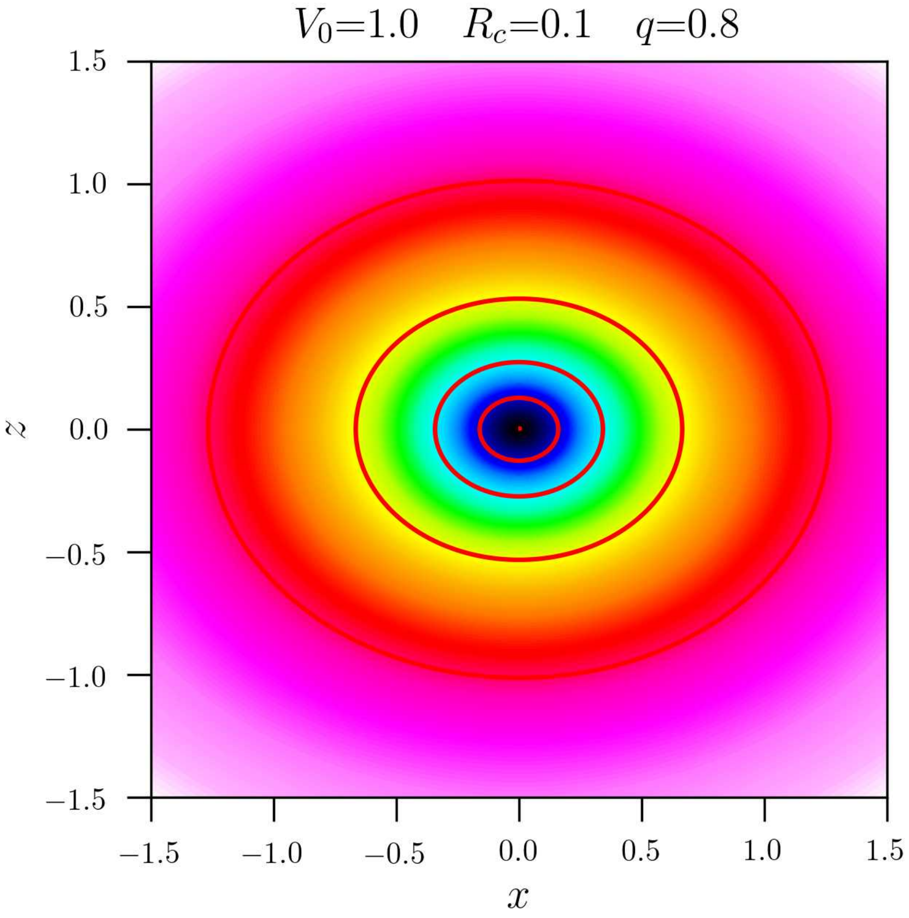


Surfaces of section

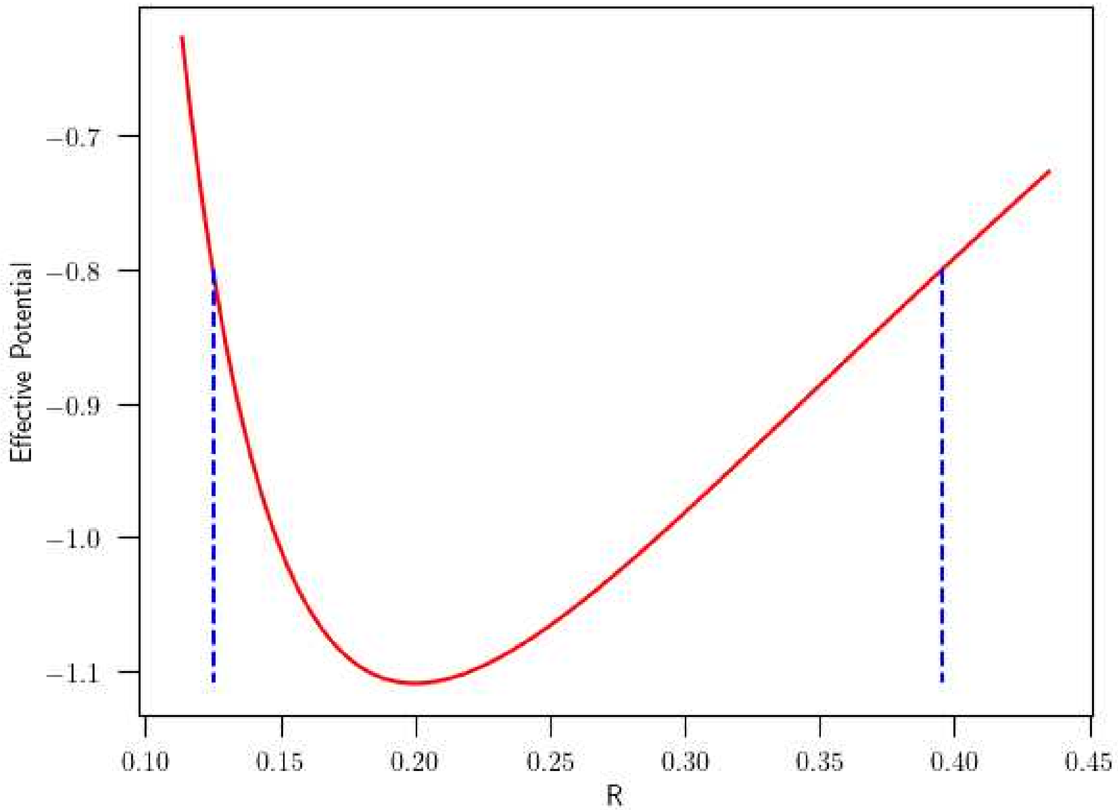
Examples

Logarithmic potential

$$\Phi_{\log}(R, z) = \frac{1}{2} V_0^2 \ln\left(R_c^2 + R^2 + \frac{z^2}{q^2}\right)$$

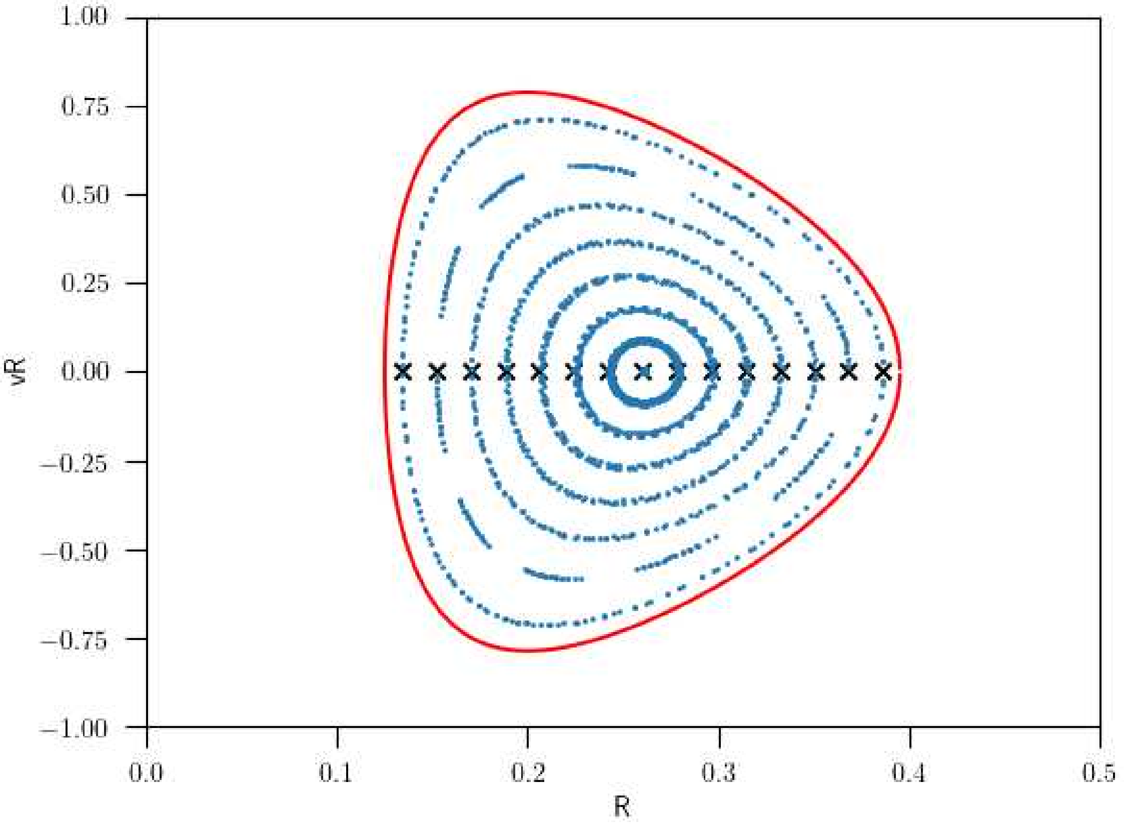


Effective Potential




```
./mapping-Rz.py --V0 1. --Rc 0.0 --p 0.9 --Lz 0.2 -E -0.8 --plotpotential
```

Invariant curves : Third Integral




```
./mapping-Rz.py --V0 1. --Rc 0.0 --p 0.9 --Lz 0.2 -E -0.8   --norbits 30 --nlaps 1000
```

The Third Integral I (I is in general non analytical)

Spherical systems : $|\vec{L}| \equiv L$ is conserved

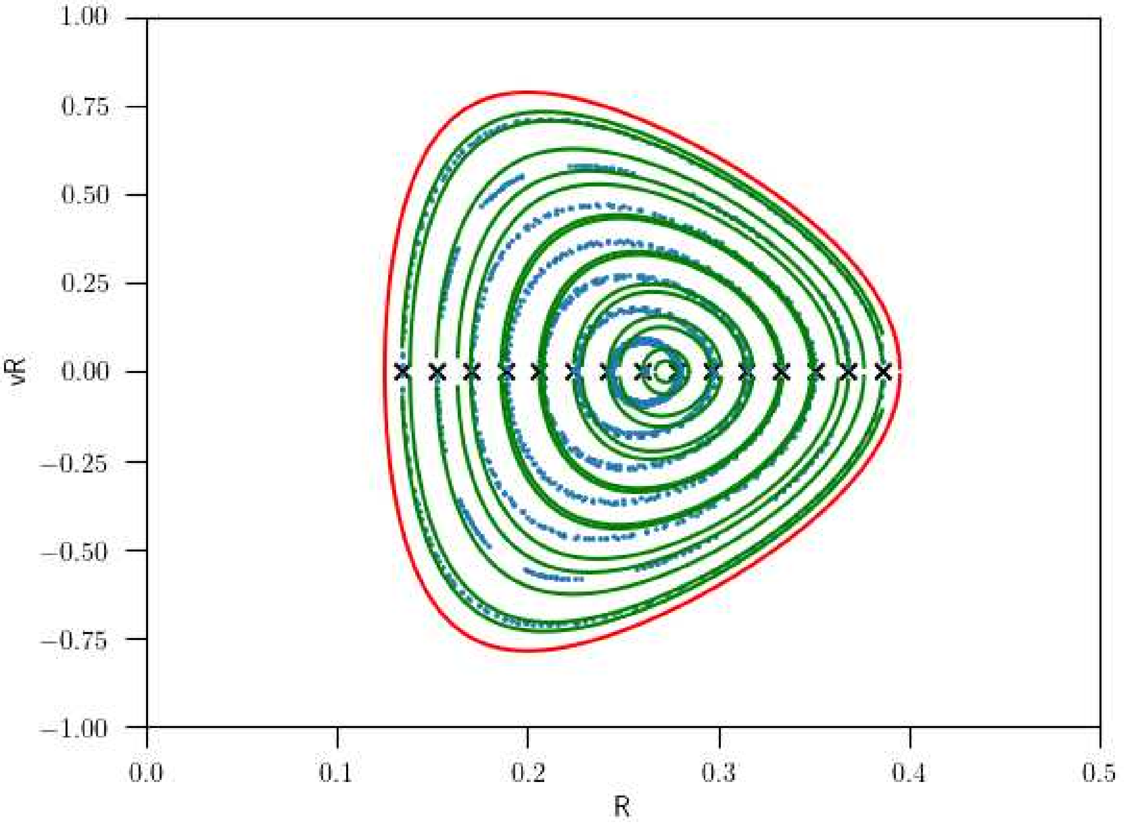
Nearly spherical potential : L is nearly an integral $\equiv I$?

What is the curve in the Poincaré map that satisfies $L = cte$?

EXERCICE

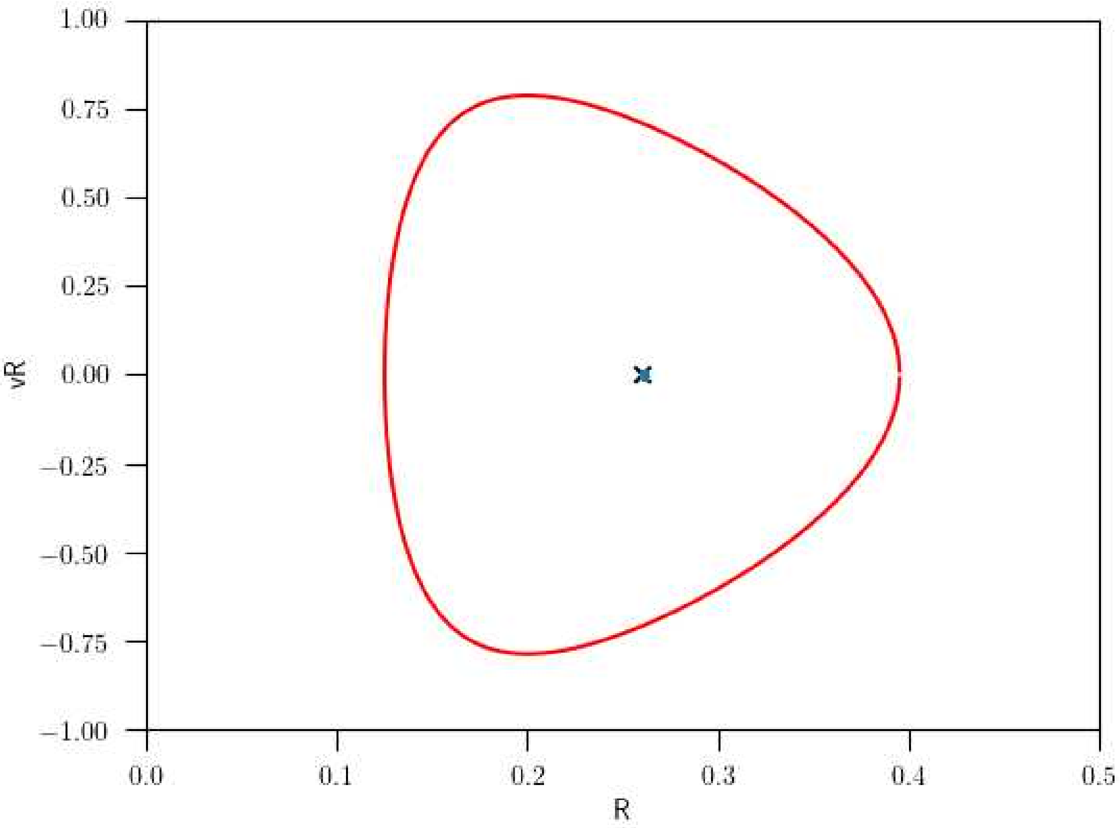
Invariant curves : Third Integral

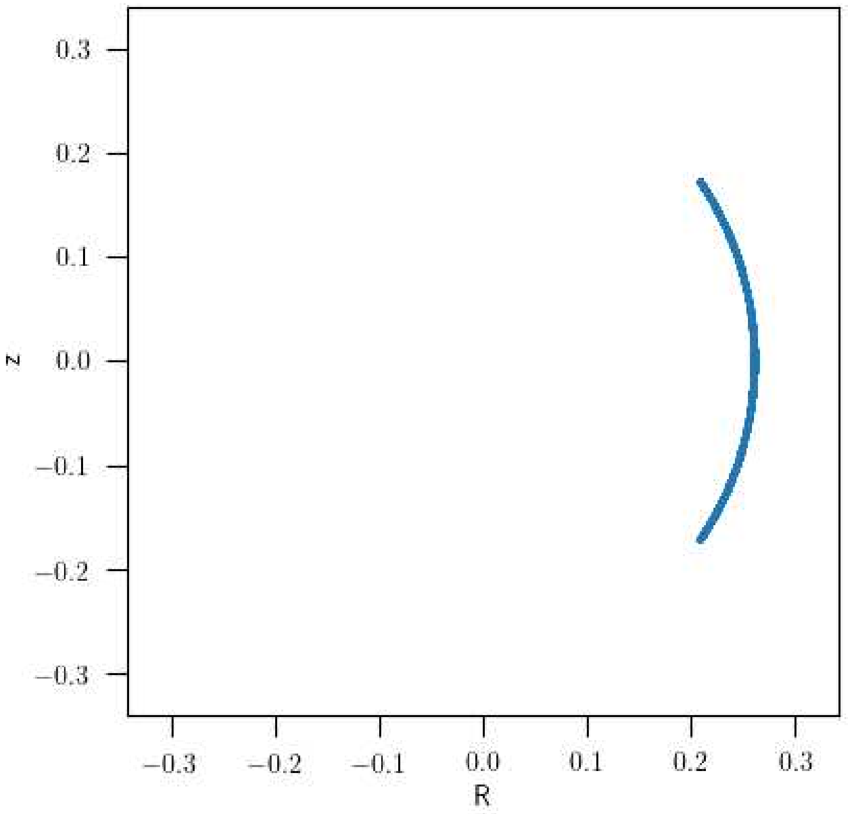
green : contours of constant total
angular momentum



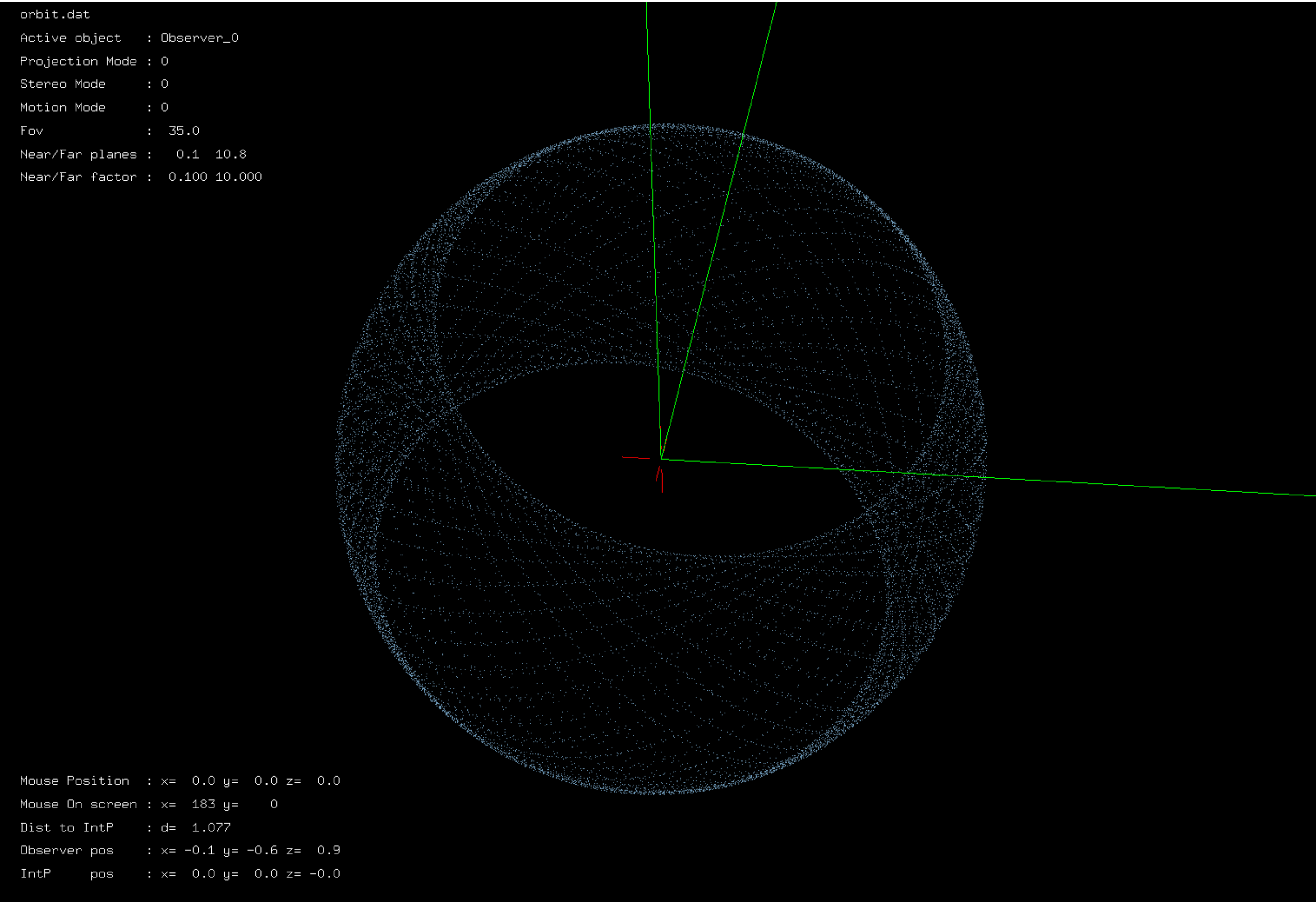

```
./mapping-Rz.py --V0 1. --Rc 0.0 --p 0.9 --Lz 0.2 -E -0.8   --norbits 15 --nlaps 100 --add_IL
```

shell orbit

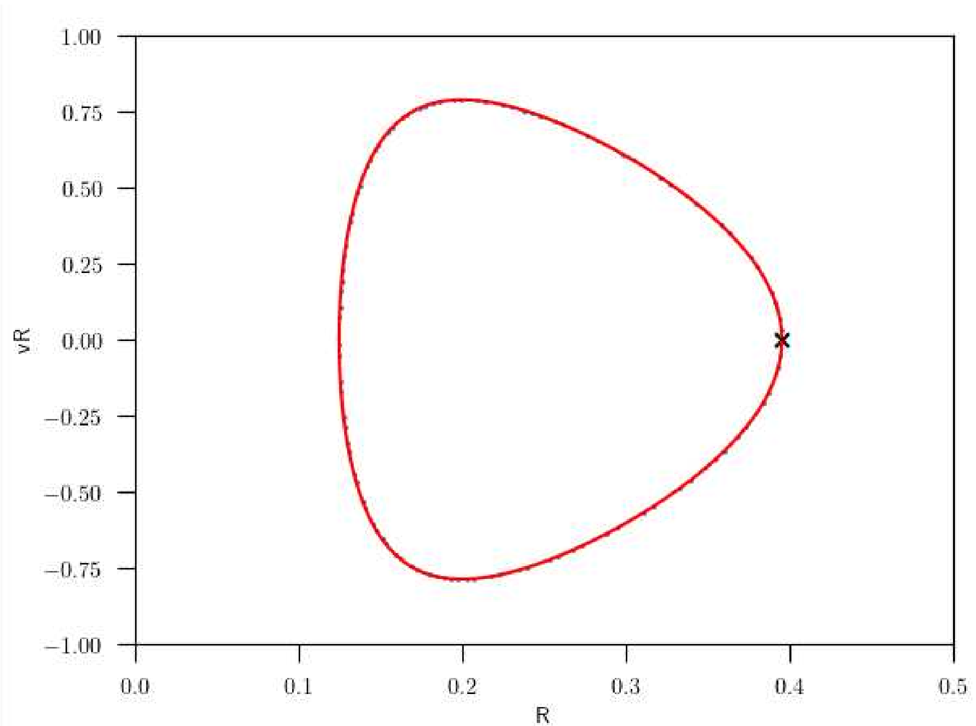


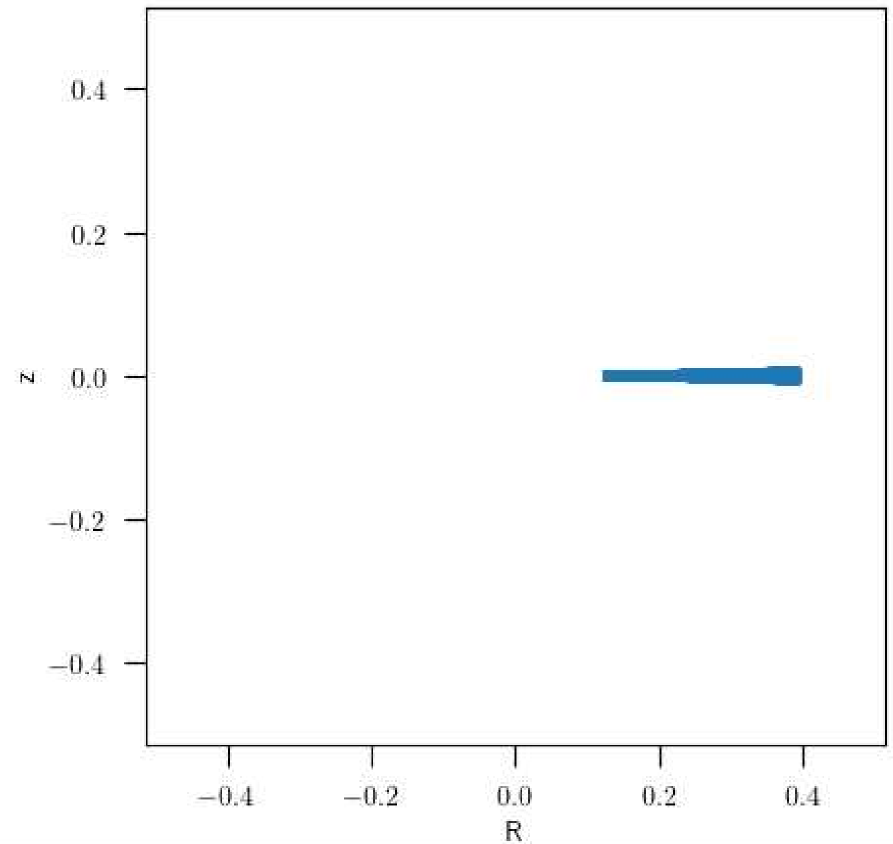



```
./mapping-Rz.py --V0 1. --Rc 0.0 --p 0.9 --Lz 0.2 -E -0.8  --nlaps 100 --R 0.2612
```

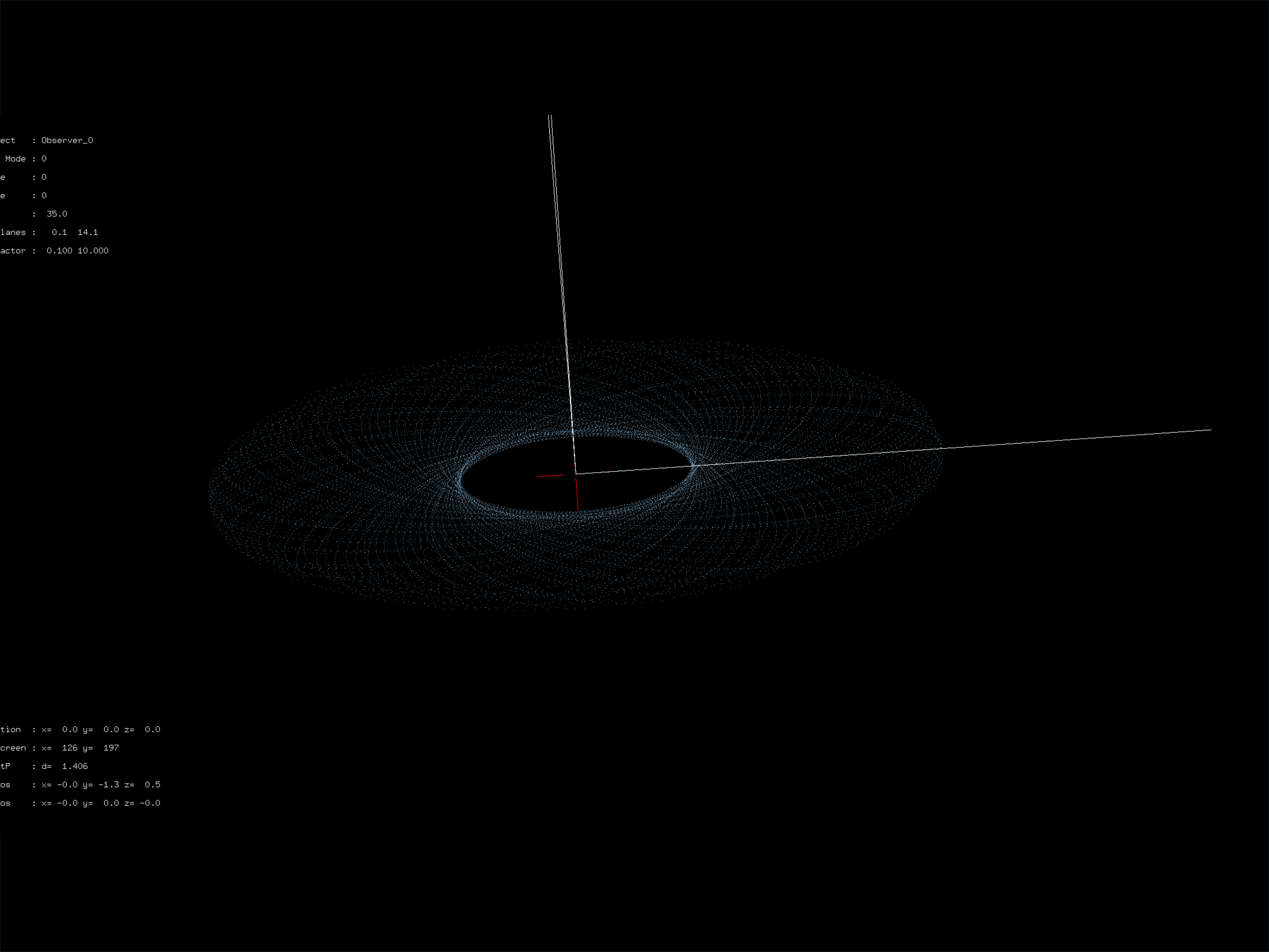

orbit.dat
Active object : Observer_0
Projection Mode : 0
Stereo Mode : 0
Motion Mode : 0
Fov : 35.0
Near/Far planes : 0.1 10.8
Near/Far factor : 0.100 10.000
Mouse Position : x= 0.0 y= 0.0 z= 0.0
Mouse On screen : x= 183 y= 0
Dist to IntP : d= 1.077
Observer pos : x= -0.1 y= -0.6 z= 0.9
IntP pos : x= 0.0 y= 0.0 z= -0.0

Large radius

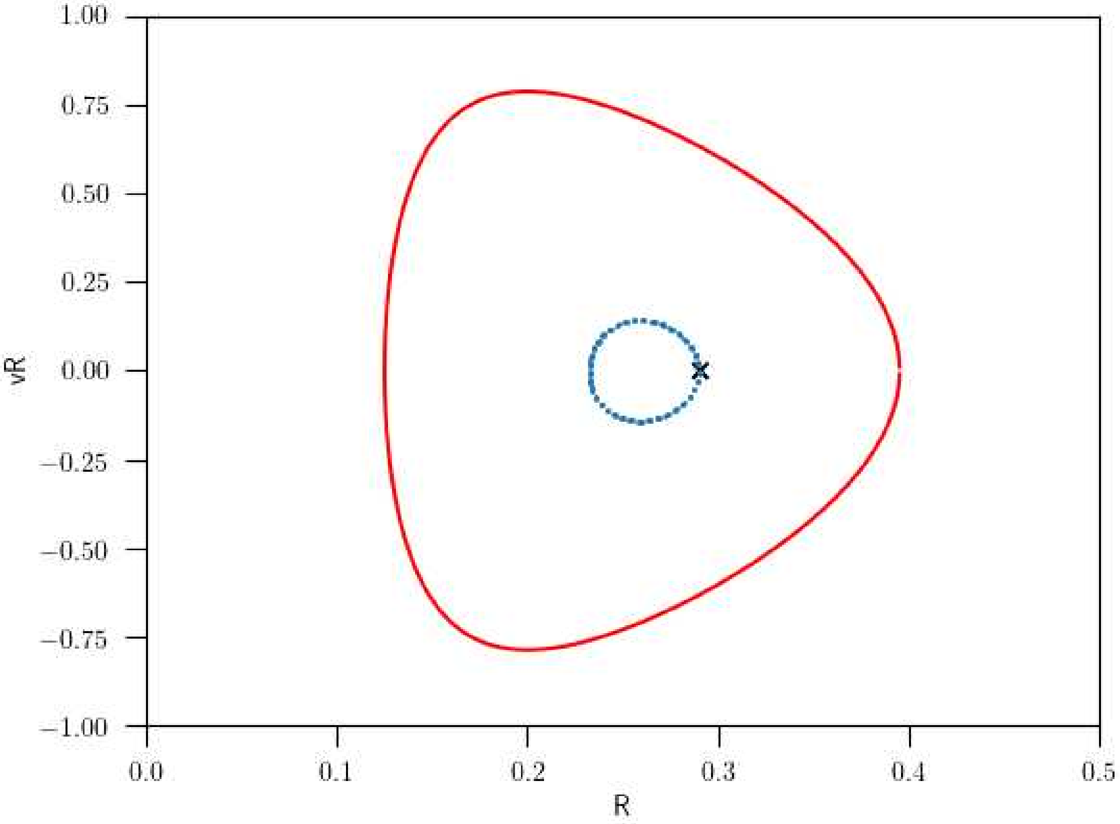


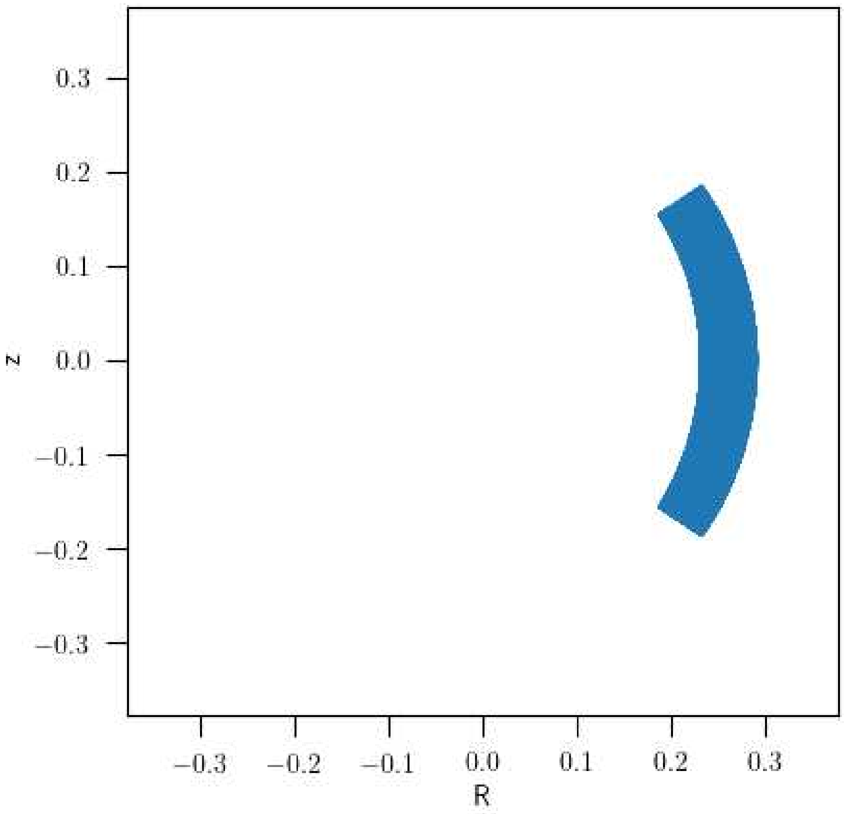



```
./mapping-Rz.py --V0 1. --Rc 0.0 --p 0.9 --Lz 0.2 -E -0.8   --norbits 15 --nlaps 100 --R 0.3953
```

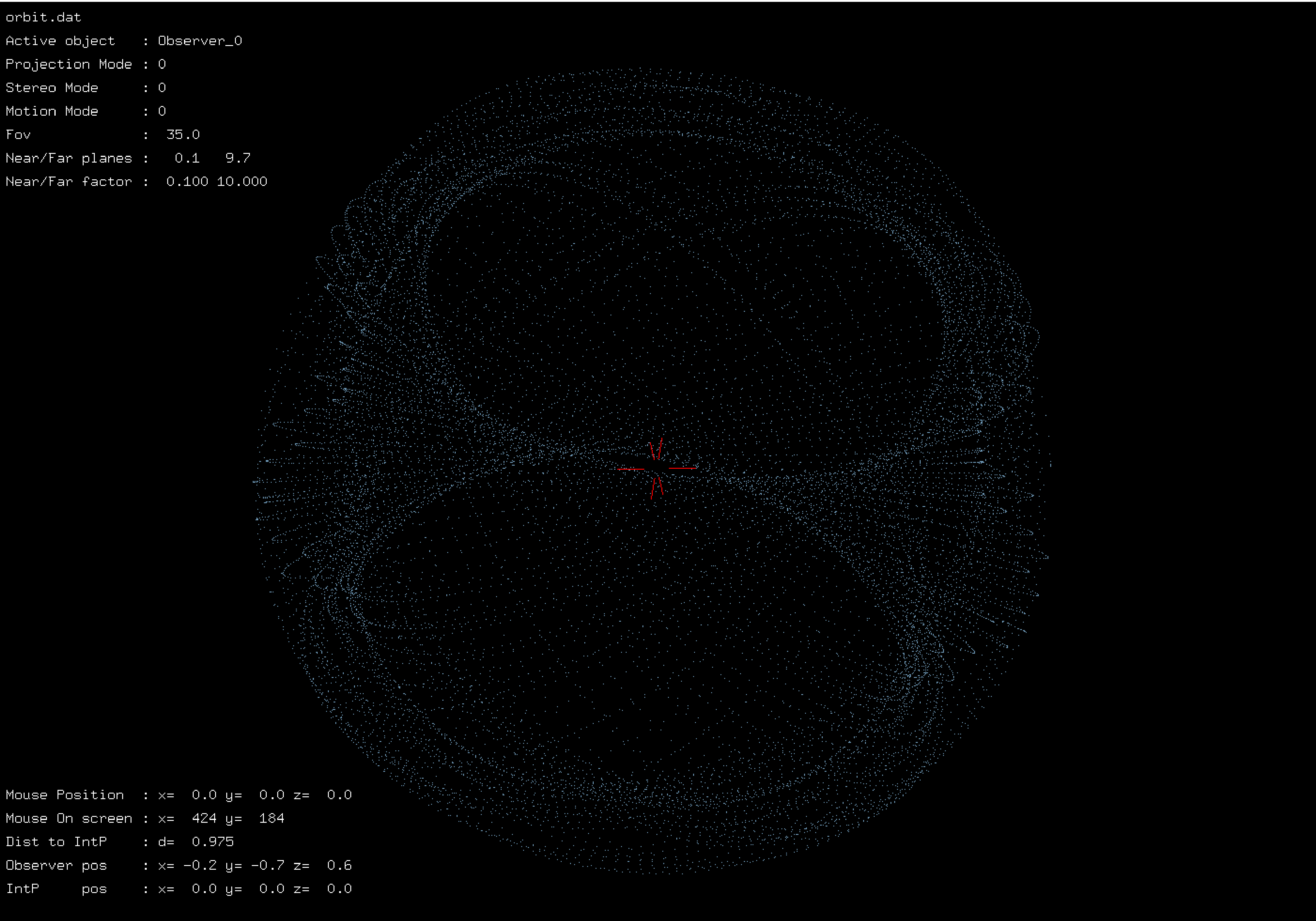

ect : Observer_0
Mode : 0
e : 0
e : 0
: 35.0
lanes : 0.1 14.1
actor : 0.100 10.000
tion : x= 0.0 y= 0.0 z= 0.0
creen : x= 126 y= 197
tP : d= 1.406
os : x= -0.0 y= -1.3 z= 0.5
os : x= -0.0 y= 0.0 z= -0.0

Smaller radius

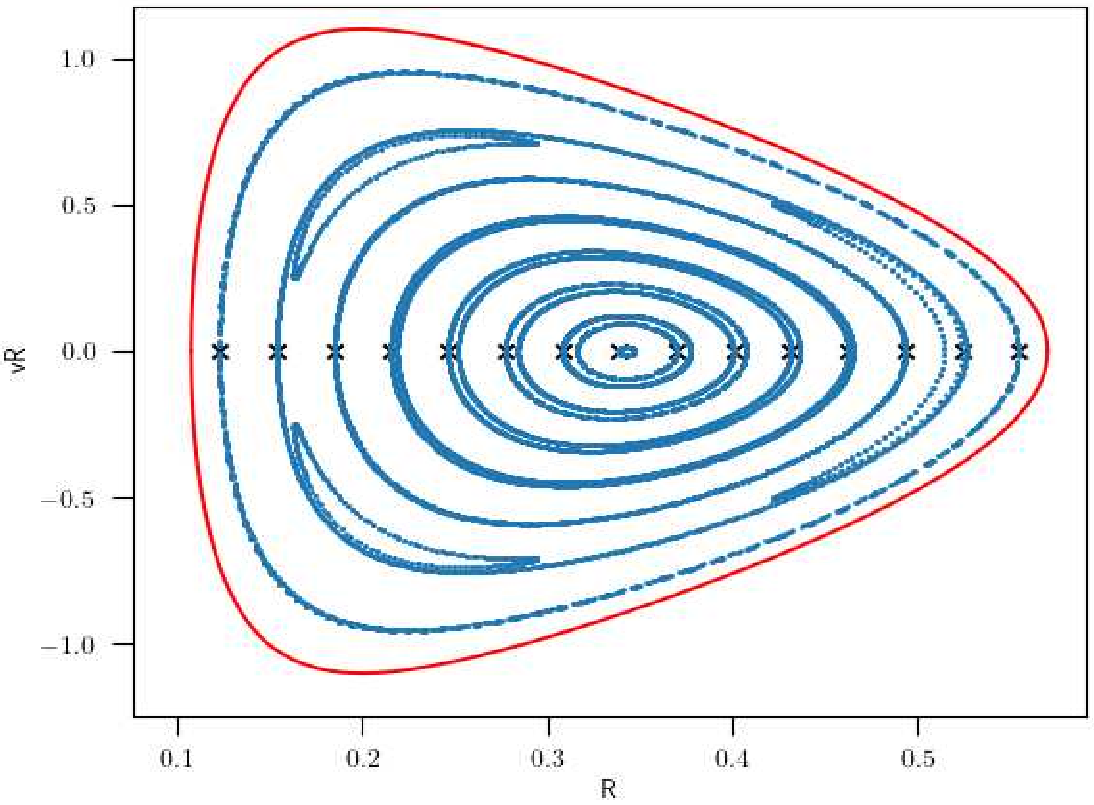





```
./mapping-Rz.py --V0 1. --Rc 0.0 --p 0.9 --Lz 0.2 -E -0.8   --norbits 15 --nlaps 100   --R 0.29
```

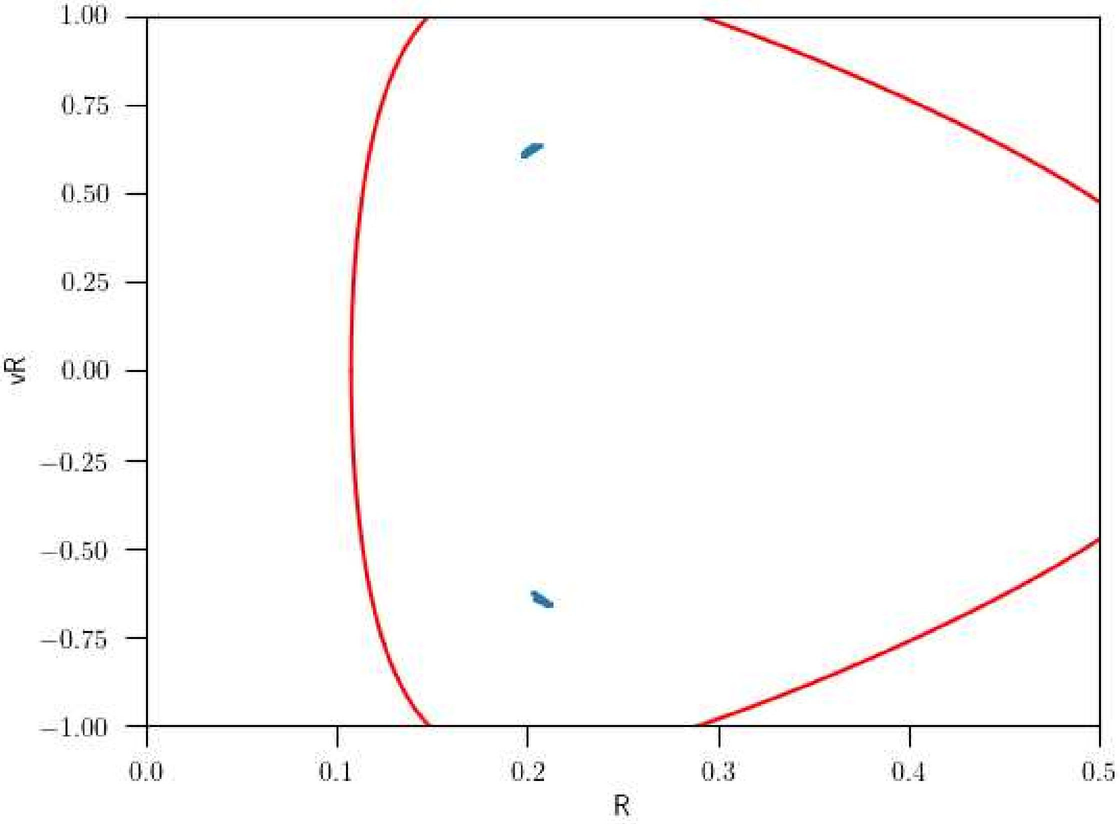

orbit.dat
Active object : Observer_0
Projection Mode : 0
Stereo Mode : 0
Motion Mode : 0
Fov : 35.0
Near/Far planes : 0.1 9.7
Near/Far factor : 0.100 10.000
Mouse Position : x= 0.0 y= 0.0 z= 0.0
Mouse On screen : x= 424 y= 184
Dist to IntP : d= 0.975
Observer pos : x= -0.2 y= -0.7 z= 0.6
IntP pos : x= 0.0 y= 0.0 z= 0.0

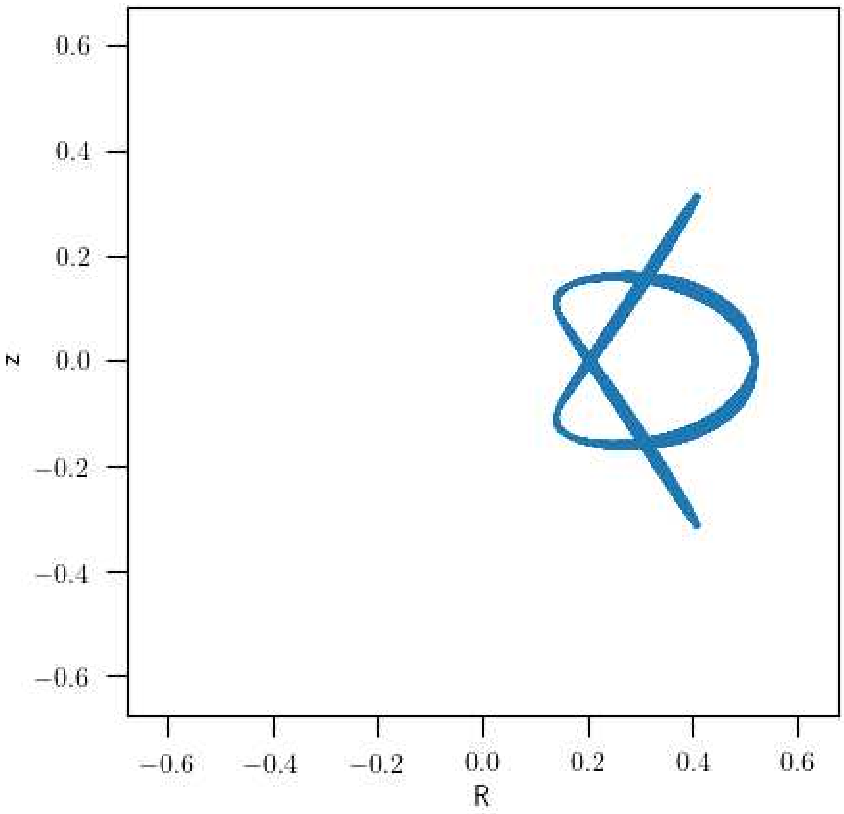
At higher energy



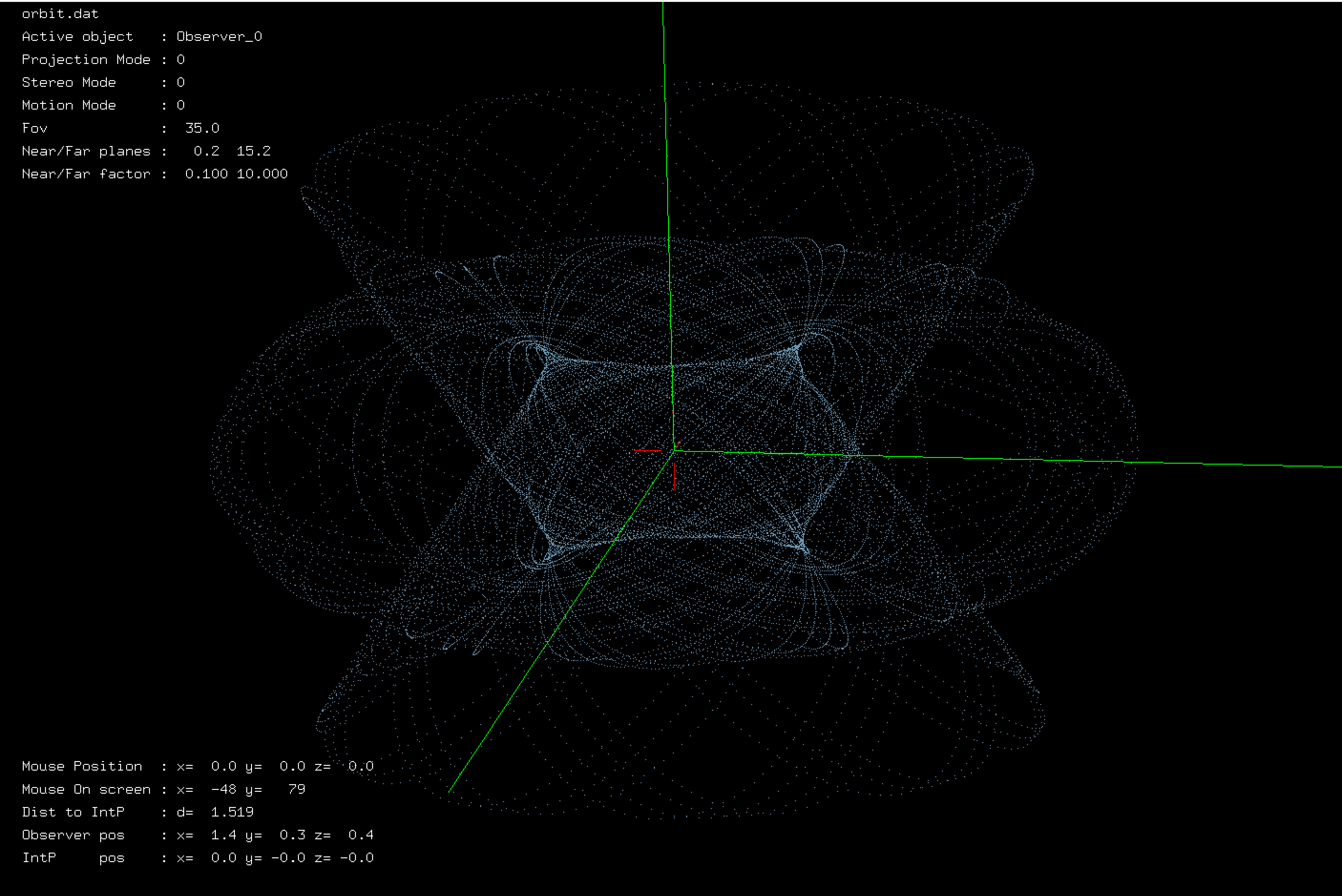

```
./mapping-Rz.py --V0 1. --Rc 0.0 --p 0.9 --Lz 0.2 -E -0.5 --norbits 15 --nlaps 1000
```

Bifurcation (resonance) : new orbit family

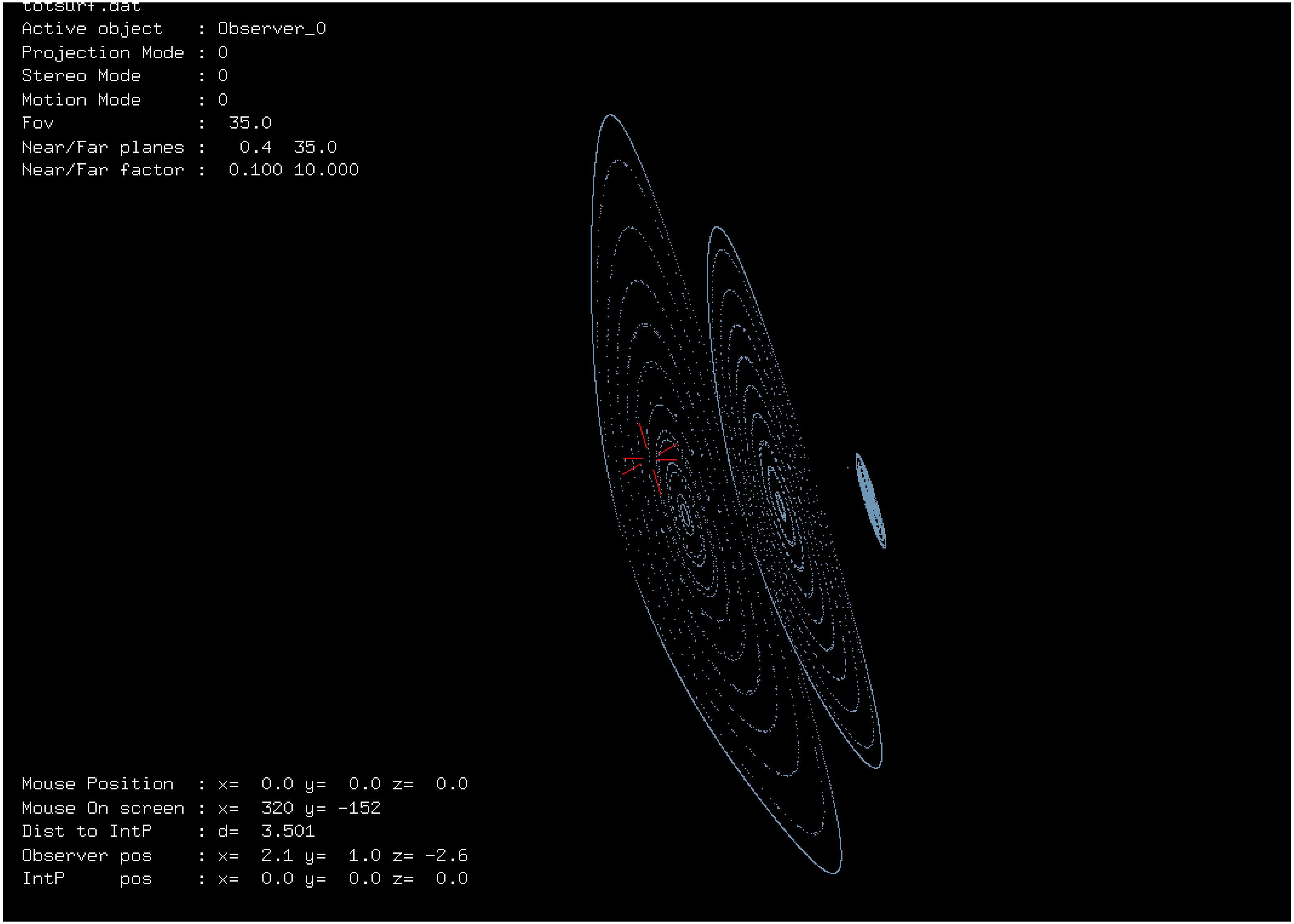





```
./mapping-Rz.py --V0 1. --Rc 0.0 --p 0.9 --Lz 0.2 -E -0.5 --vR 0 --R 0.52   --nlaps 100
```


orbit.dat
Active object : Observer_0
Projection Mode : 0
Stereo Mode : 0
Motion Mode : 0
Fov : 35.0
Near/Far planes : 0.2 15.2
Near/Far factor : 0.100 10.000
Mouse Position : x= 0.0 y= 0.0 z= 0.0
Mouse On screen : x= -48 y= 79
Dist to IntP : d= 1.519
Observer pos : x= 1.4 y= 0.3 z= 0.4
IntP pos : x= 0.0 y= -0.0 z= -0.0

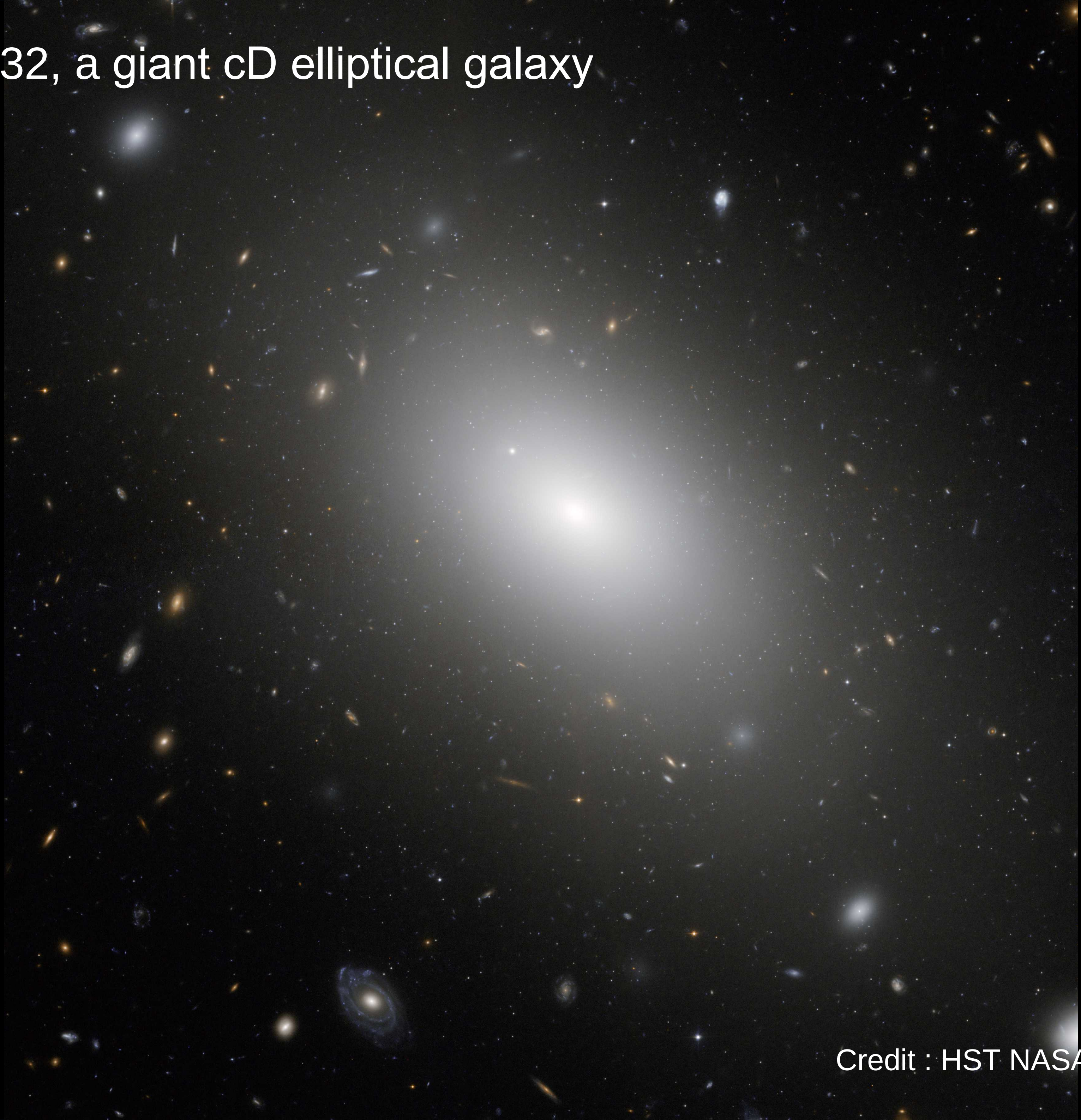
Slices of different energies




```
rm surf-*.dat
./mapping-Rz.py --V0 1. --Rc 0.0 --p 0.9 --Lz 0.2 -E -1.1 --vR 0 --norbits 50
./mapping-Rz.py --V0 1. --Rc 0.0 --p 0.9 --Lz 0.2 -E -0.8 --vR 0 --norbits 50
./mapping-Rz.py --V0 1. --Rc 0.0 --p 0.9 --Lz 0.2 -E -0.5 --vR 0 --norbits 50
./concatenate.py surf-0*
glups --fullscreen -p glparameters  totsurf.dat
```

Orbits in planar non-axisymmetric potentials

NGC 1132, a giant cD elliptical galaxy

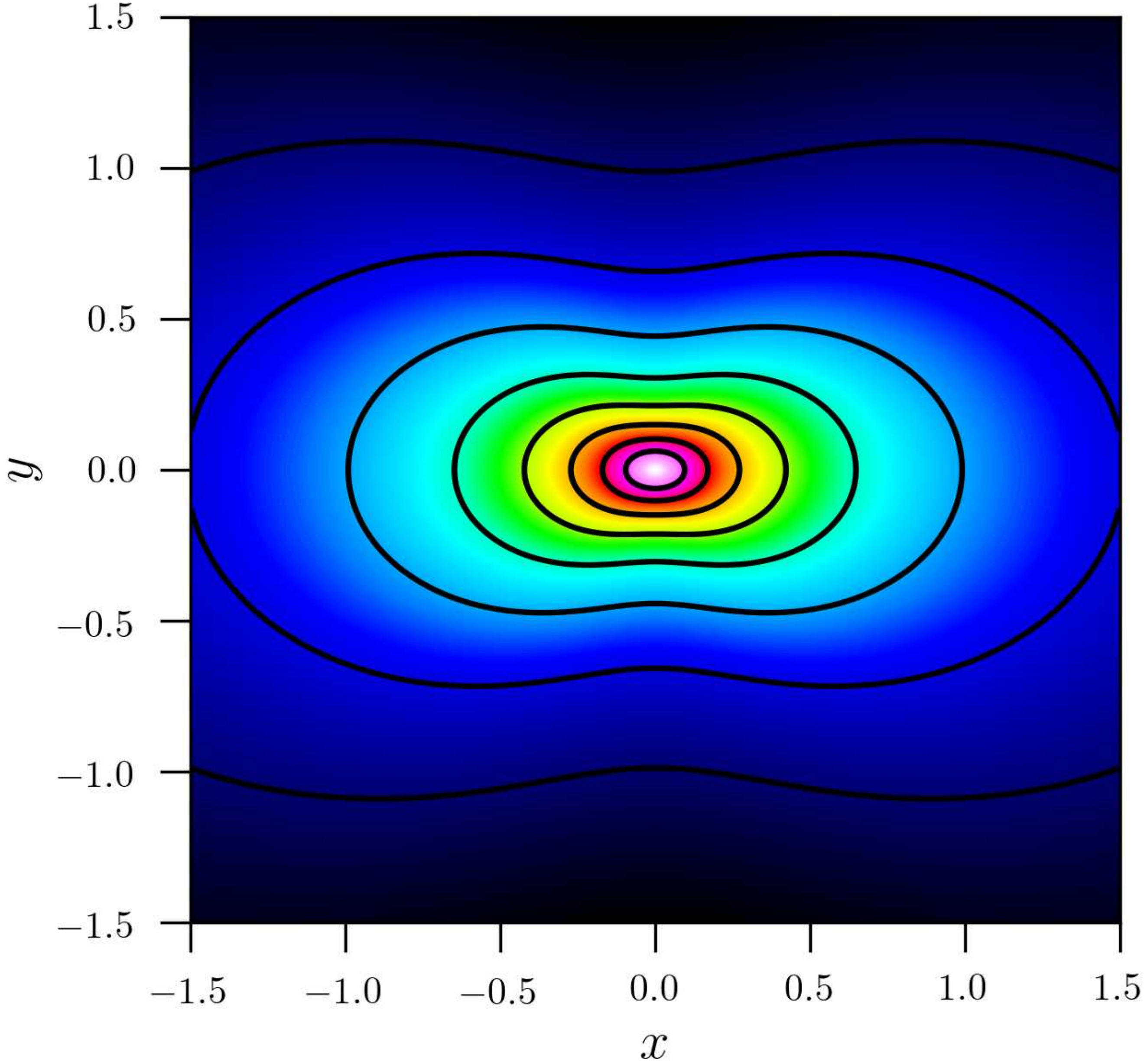


Credit : HST NASA/ESA

NGC 1300 SBb

Bar model : Logarithmic potential:
Vo=1 Rc=0.13 q=0.8)

$$\Phi_{\log}(x,y) = \frac{1}{2}V_0^2 \ln\left(R_c^2 + x^2 + \left(\frac{y}{q}\right)^2\right)$$



$$R \ll R_c$$

Orbits in planar non-axisymmetric static potential

Model : logarithmic potential

$$\boxed{\phi(x,y) = \frac{1}{2} V_0^2 \ln\left(R_c + x^2 + \frac{y^2}{q^2}\right)}$$

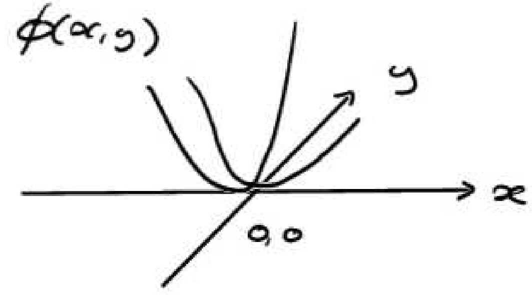
q : flattening parameter
(equipotential axis ratio)

Motions for $R \ll R_c$

$$\phi(x,y) \simeq \phi(0,0) + \underbrace{\cancel{\frac{\partial\phi}{\partial x}}}_{=0}\Big|_{0,0} x + \underbrace{\cancel{\frac{\partial\phi}{\partial y}}}_{=0}\Big|_{0,0} y + \frac{1}{2}\frac{\partial^2\phi}{\partial x^2}\Big|_{0,0} x^2 + \frac{1}{2}\frac{\partial^2\phi}{\partial y^2}\Big|_{0,0} y^2$$

$$\frac{\partial^2\phi}{\partial x^2}\Big|_{0,0} = \frac{V_0^2}{R_c^2}$$

$$\frac{\partial^2\phi}{\partial y^2}\Big|_{0,0} = \frac{V_0^2}{R_c^2}\,\frac{1}{q^2}$$



Equations of motion

$$\ddot{x} = -\frac{\partial \phi}{\partial x} \qquad \ddot{y} = -\frac{\partial \phi}{\partial y}$$

$$\rightarrow$$

$$\ddot{x} = -\frac{V_0^2}{R_c^2}\, x \qquad \ddot{y} = -\frac{V_0^2}{q^2 R_c^2}\, y$$

$$\omega_x = \frac{V_0}{R_c} \qquad \omega_y = \frac{V_0}{q R_c}$$

2 decoupled harmonic oscillators with different frequencies

$$\omega_y = \frac{1}{q}\,\omega_x \quad (q < 1)$$

if $q = \frac{n}{m}$ $\quad n, m \in \mathbb{N}$

$\Rightarrow$ closed orbit

Integrals of motions (Hamiltonians)

$$H_x = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\omega_x^2 x^2$$

$$H_y = \frac{1}{2}\dot{y}^2 + \frac{1}{2}\omega_y^2 y^2$$

$$R < R_c$$

Surfaces of section (in planar potentials)

Can we visualize the phase phase and check if an
additional integral of motion exists ?

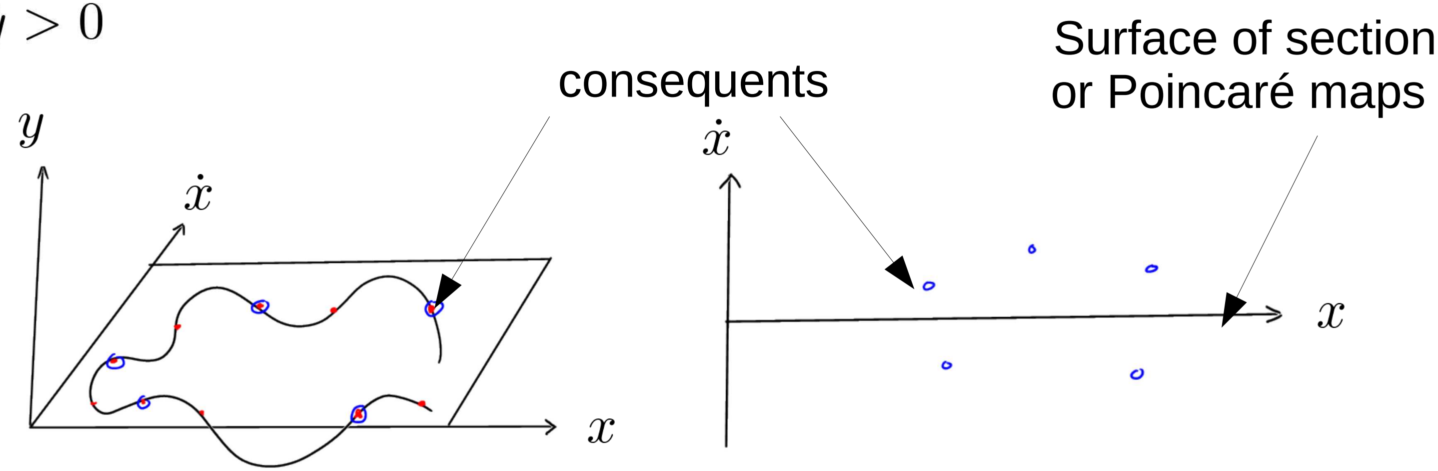
Idea :

We study the orbits in the plane z=0

- 4-D — 4 indep. variables $(x, y, \dot{x}, \dot{y})$
- Energy E

→ 3-D — 3 indep. variables $(x, y, \dot{x})$

- Drawing a 3-D phase space is still not easy. Instead, we draw slices of the phase space. We plot only phase space points that:
  - cross the $y = 0$ plane
  - have $\dot{y} > 0$



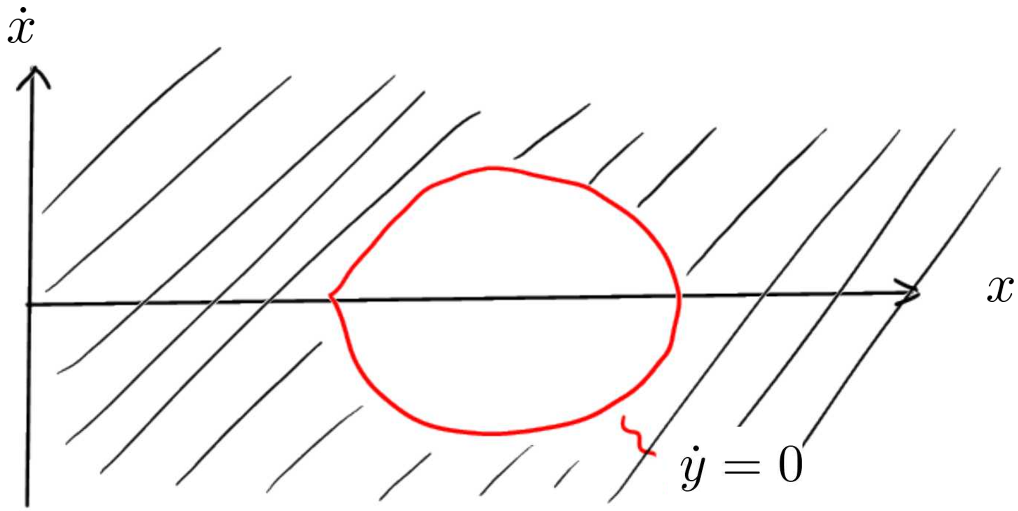
Surfaces of section (in planar potentials)

- A point in the surface of section (for a given E) defines an orbit as the three independent variables $(x, \dot{x}, y = 0)$ are defined.
- Even if orbits have the same energy, they will never intersect in the plane.
- Zero velocity curve : curve defined by $\dot{y} = 0$

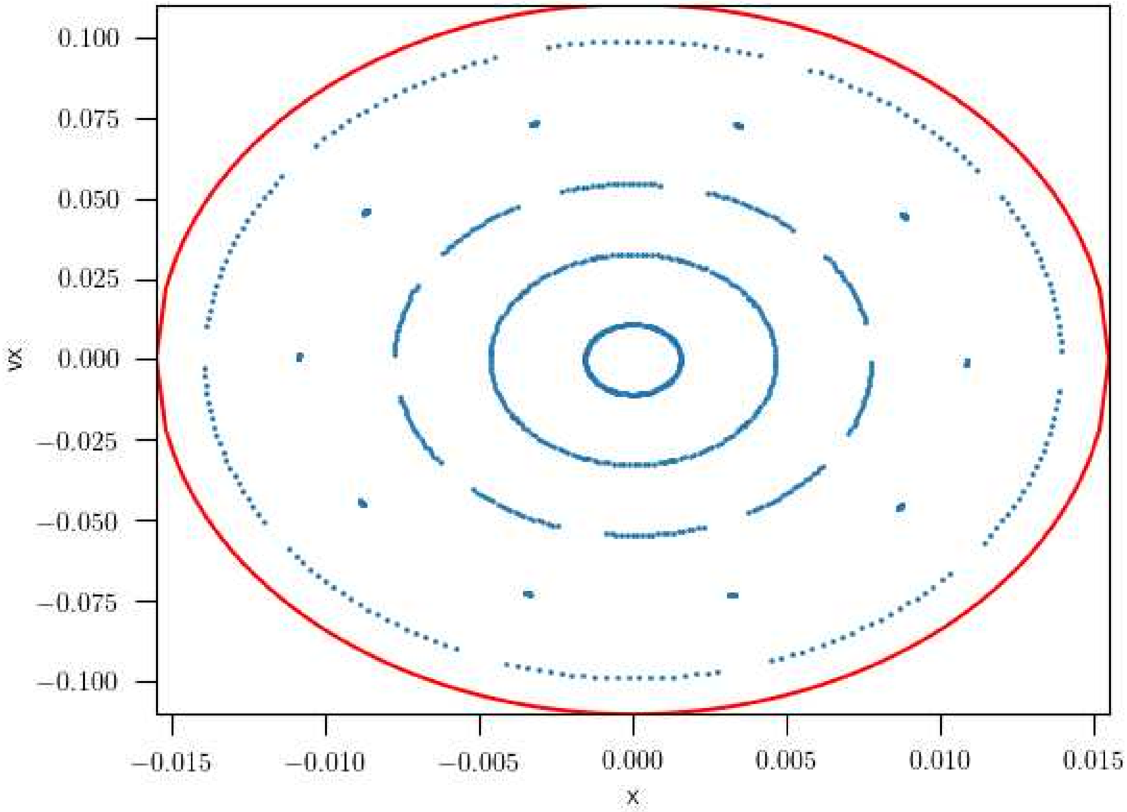
$$E = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\dot{y}^2 + \Phi(x, y = 0) \qquad \Rightarrow \qquad \dot{x} \leqslant \pm\sqrt{2\left[E - \Phi(x, y = 0)\right]}$$

$$\dot{x}(x) = \pm\sqrt{2\left[E - \Phi(x, y = 0)\right]}$$

defines the accessible region of the phase space

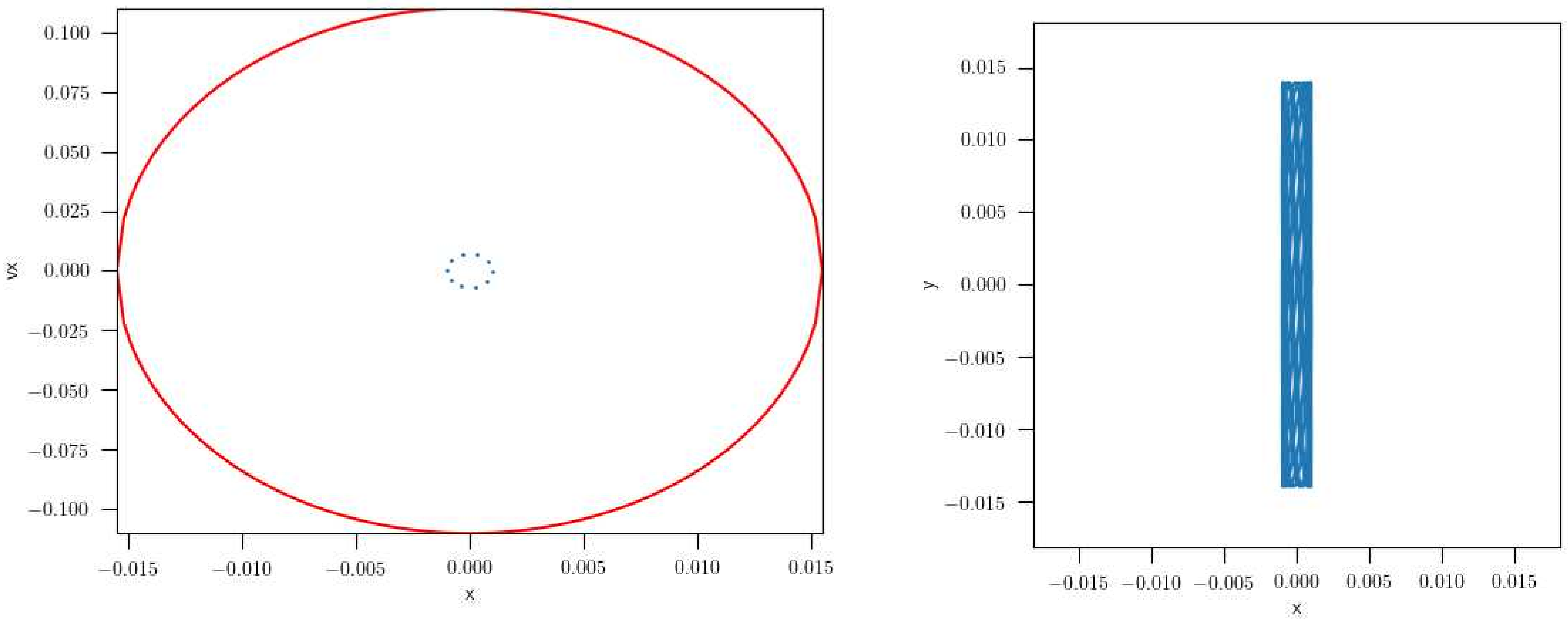


Increasing energy : perturbed harmonic oscillator (coupling terms)



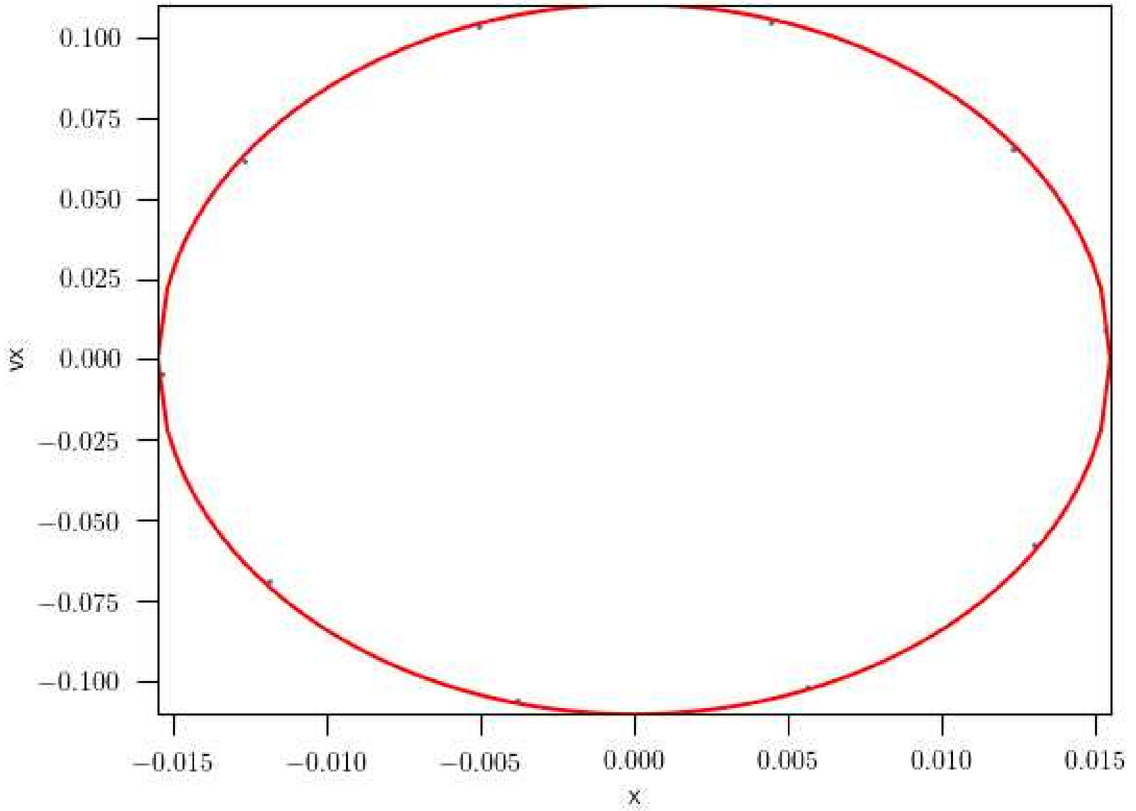

```
./mapping.py --V0 1. --Rc 0.14 --q 0.9  -E -1.96
```

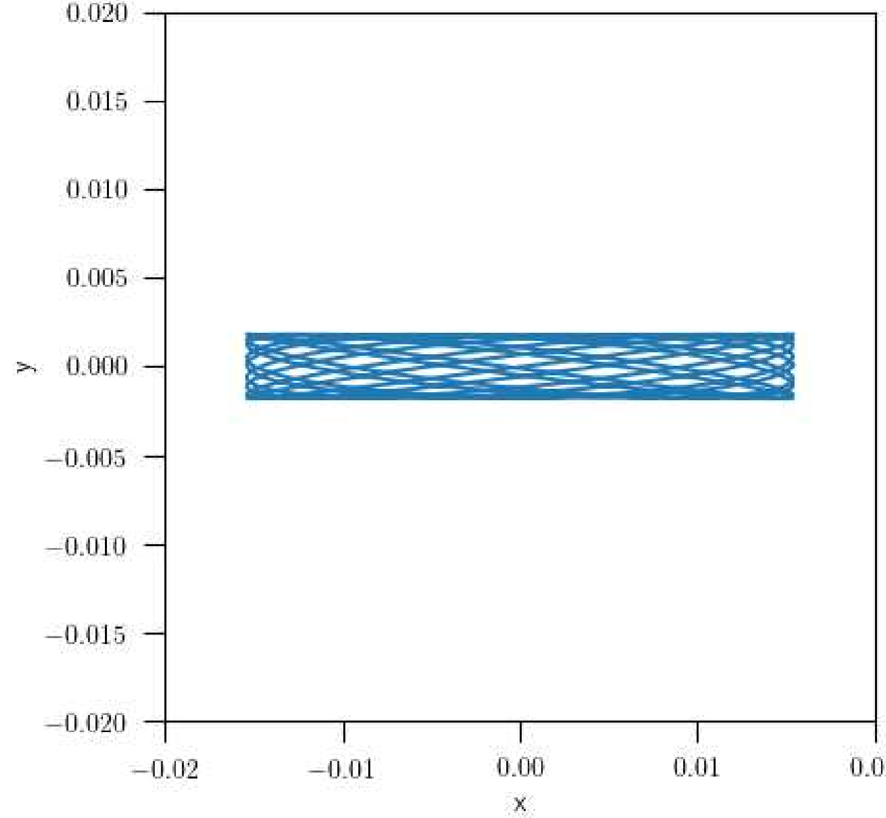
small x, Y-elongated orbits (box orbit)




```
./mapping.py --V0 1. --Rc 0.14 --q 0.9  -E -1.96   --x 0.001  --nlaps 10
```

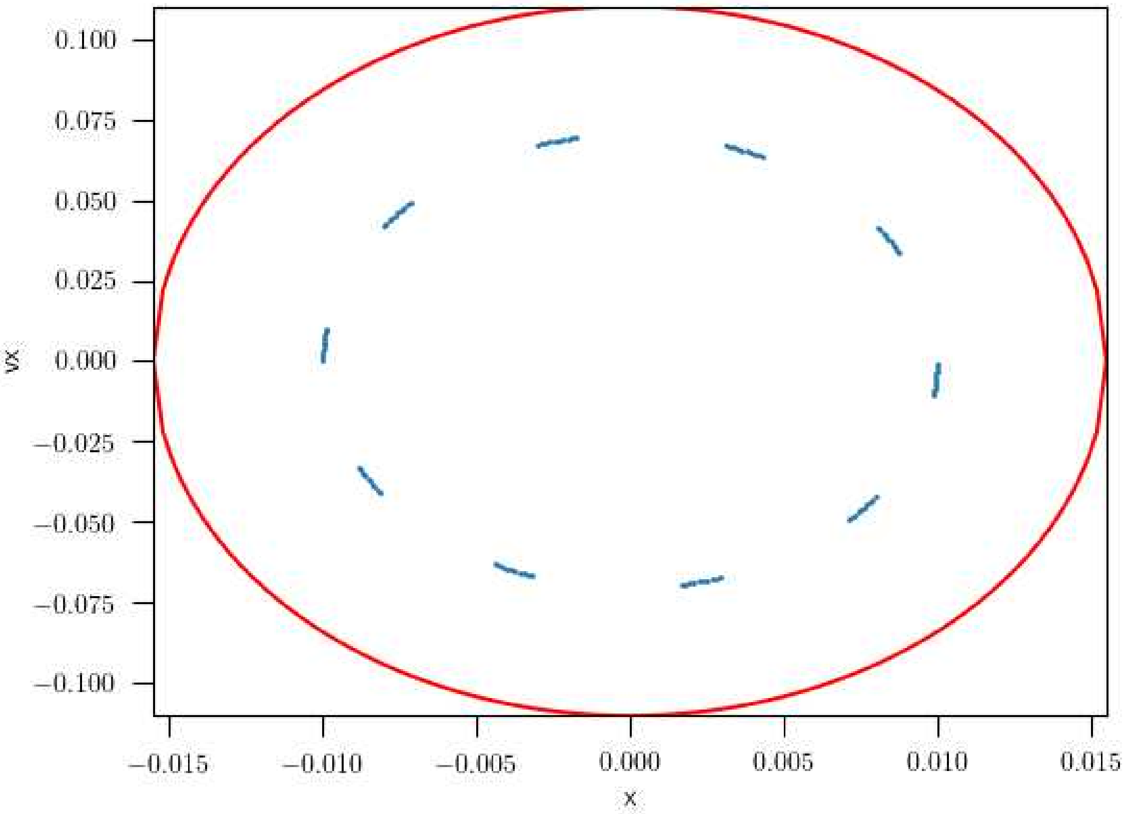
large x, X-elongated orbits (box orbit)

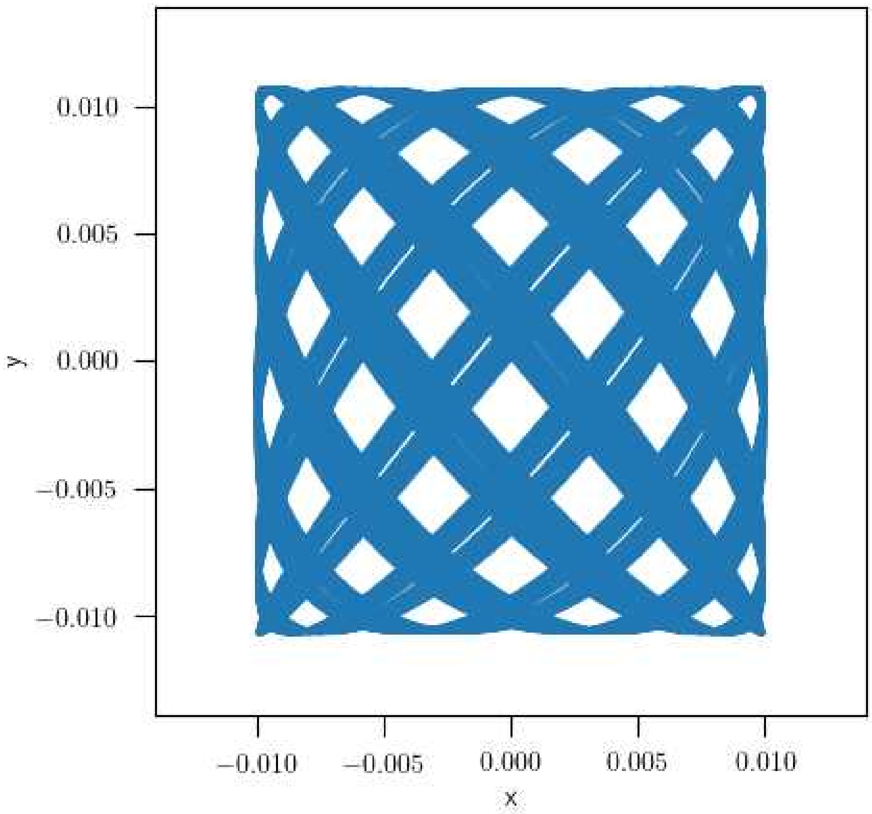





```
./mapping.py --V0 1. --Rc 0.14 --q 0.9  -E -1.96   --x 0.0154 --nlaps 10
```

Increasing energy : perturbed harmonic oscillator

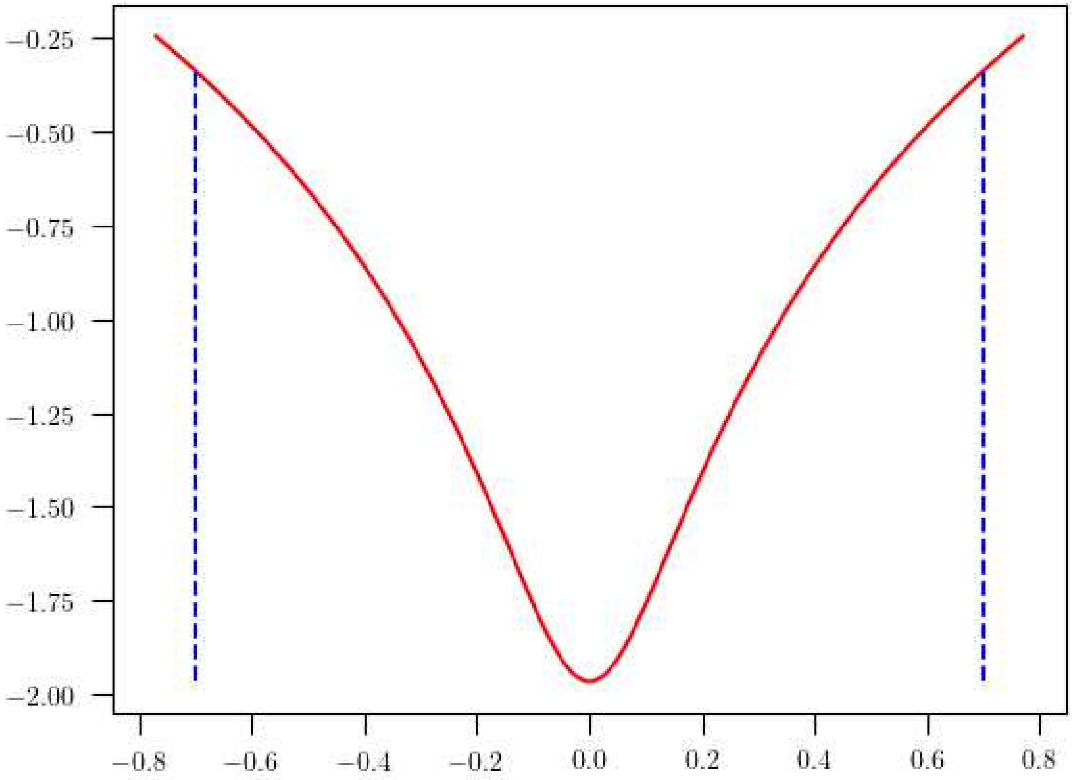





```
./mapping.py --V0 1. --Rc 0.14 --q 0.9  -E -1.96   --x 0.01
```

$$R > R_c$$

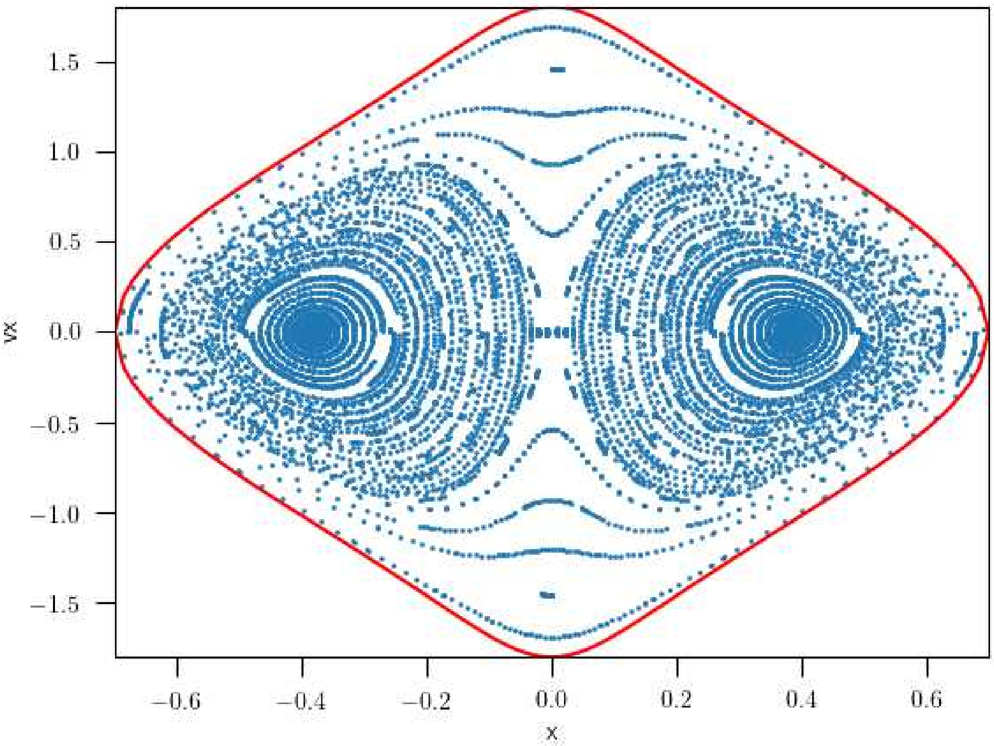
Potential and energy



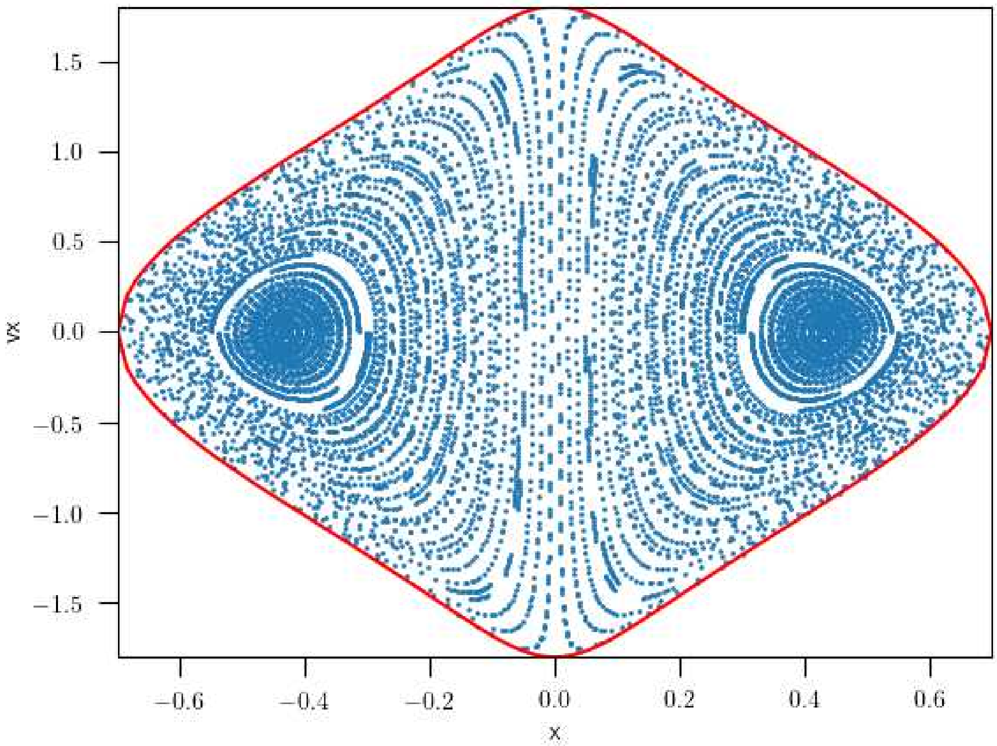

```
./mapping.py --V0 1. --Rc 0.14 --q 0.9  -E -0.337  --plotpotential
```

Phase space

$q = 0.9$

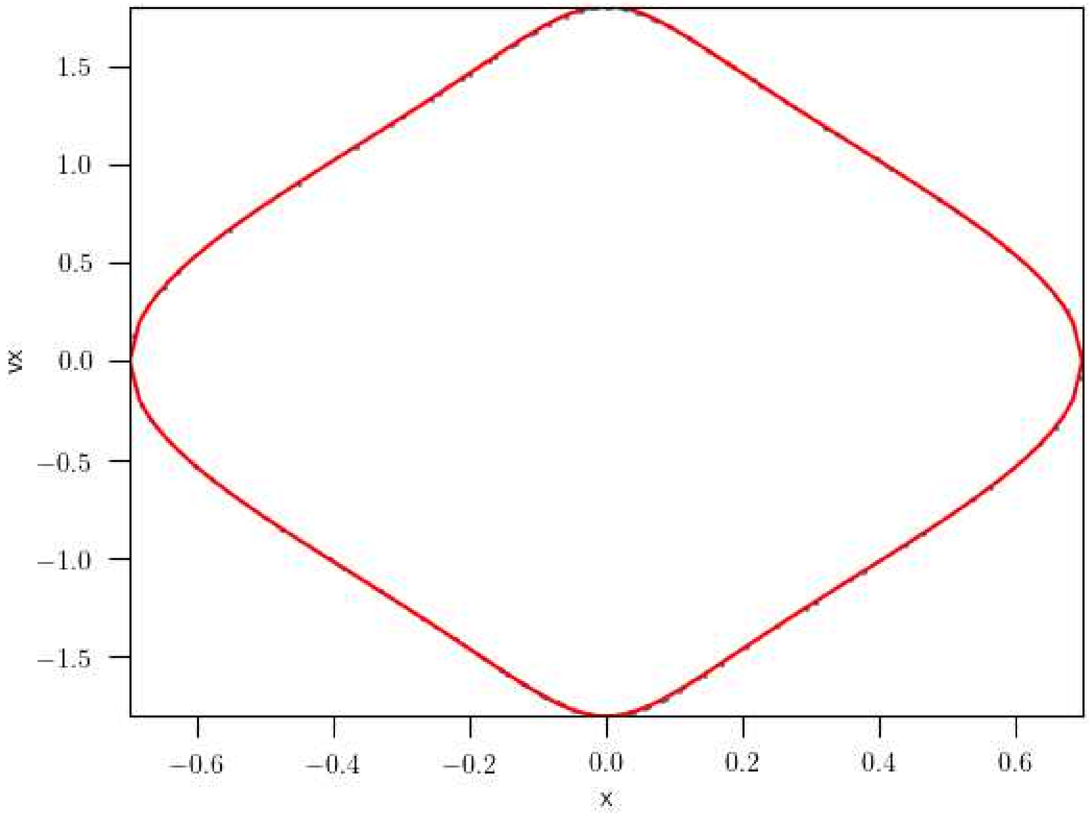


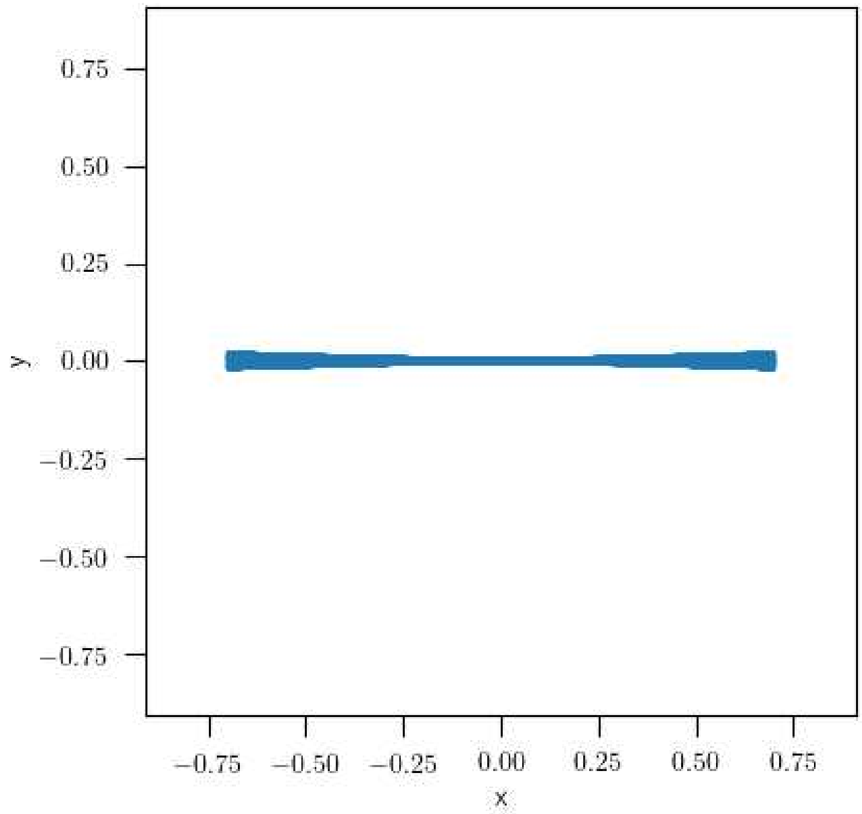
$q = 1.0$




```
./mapping.py --V0 1. --Rc 0.14 --q 0.9  -E -0.337 --norbits 100
./mapping.py --V0 1. --Rc 0.14 --q 1.0  -E -0.337 --norbits 100
```

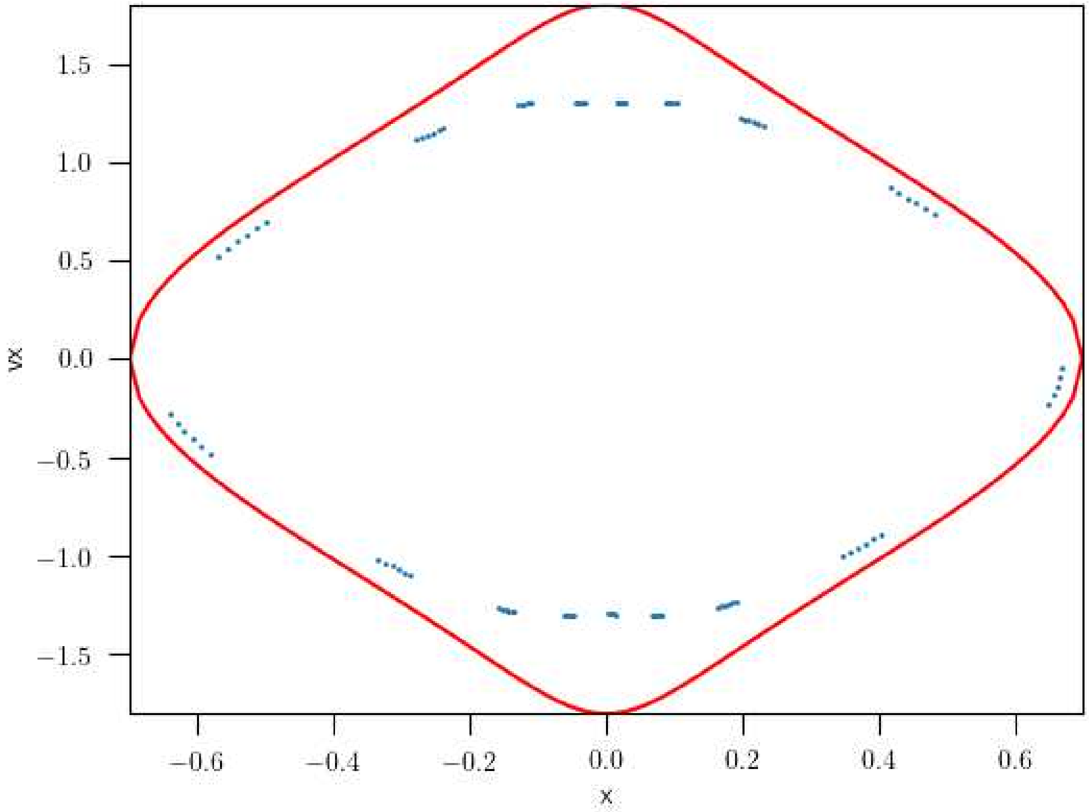
Box orbits

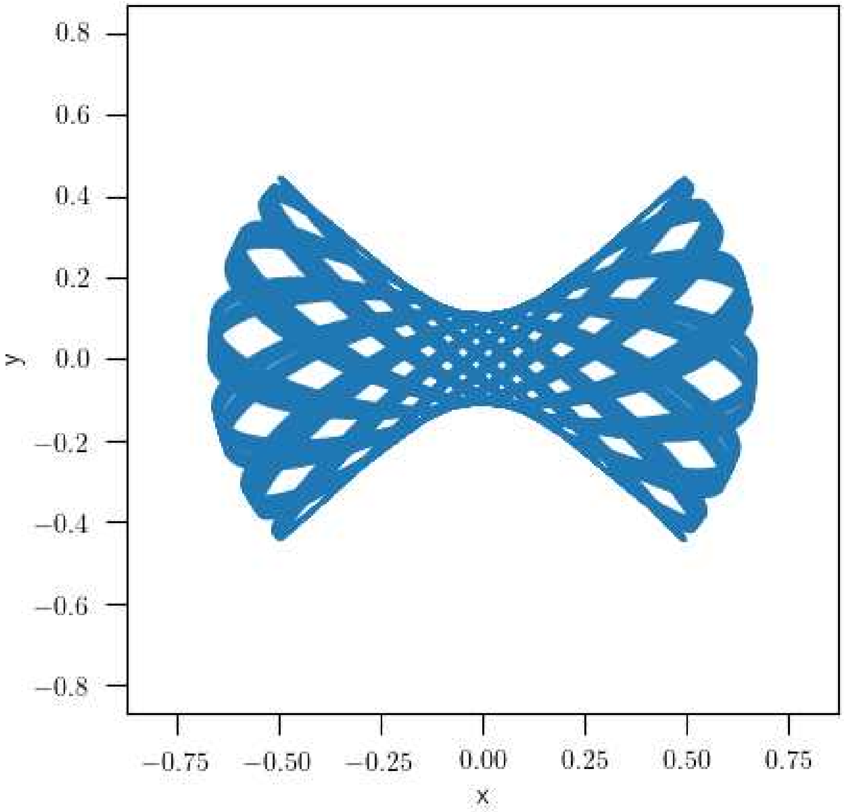





```
./mapping.py --V0 1. --Rc 0.14 --q 0.9    -E -0.337 --x 0.7
```

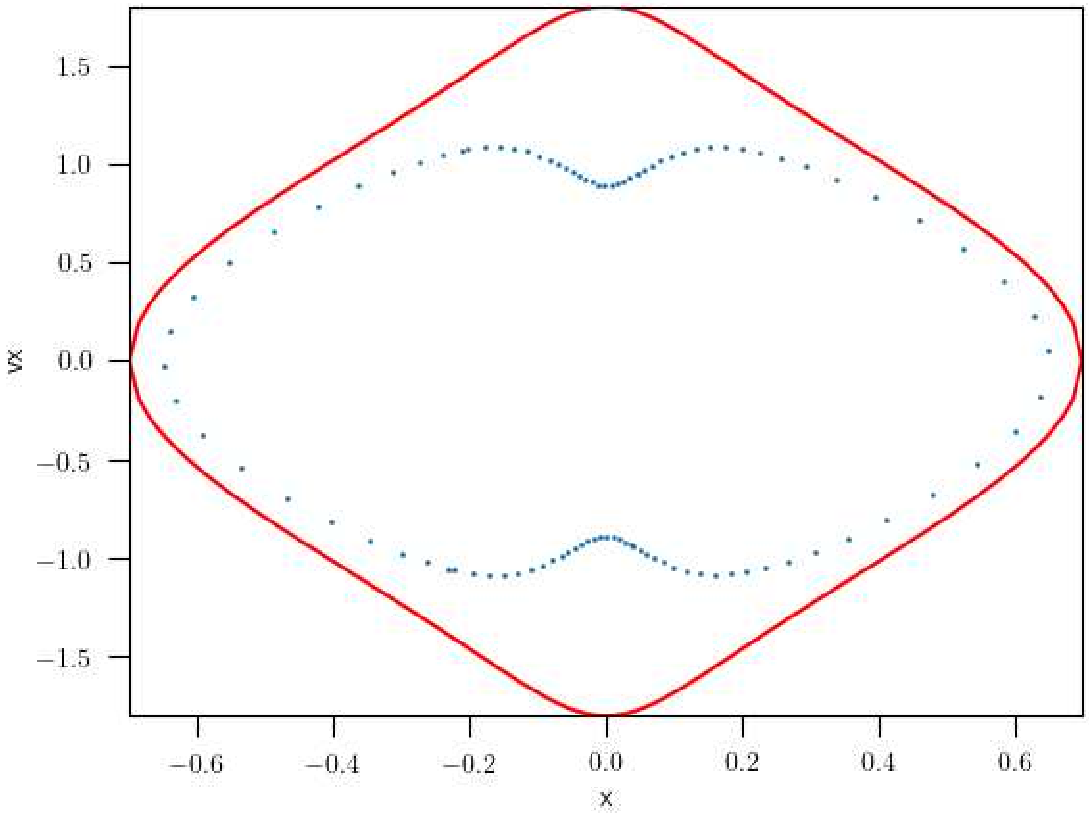
Box orbits

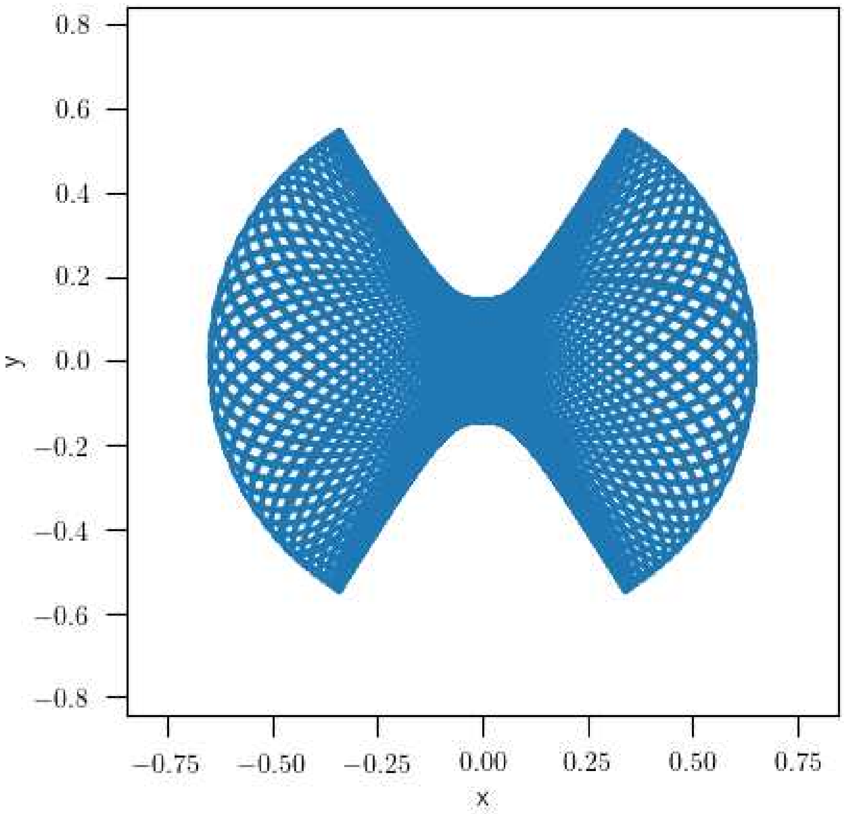





```
./mapping.py --V0 1. --Rc 0.14 --q 0.9    -E -0.337 --x 0.67
```

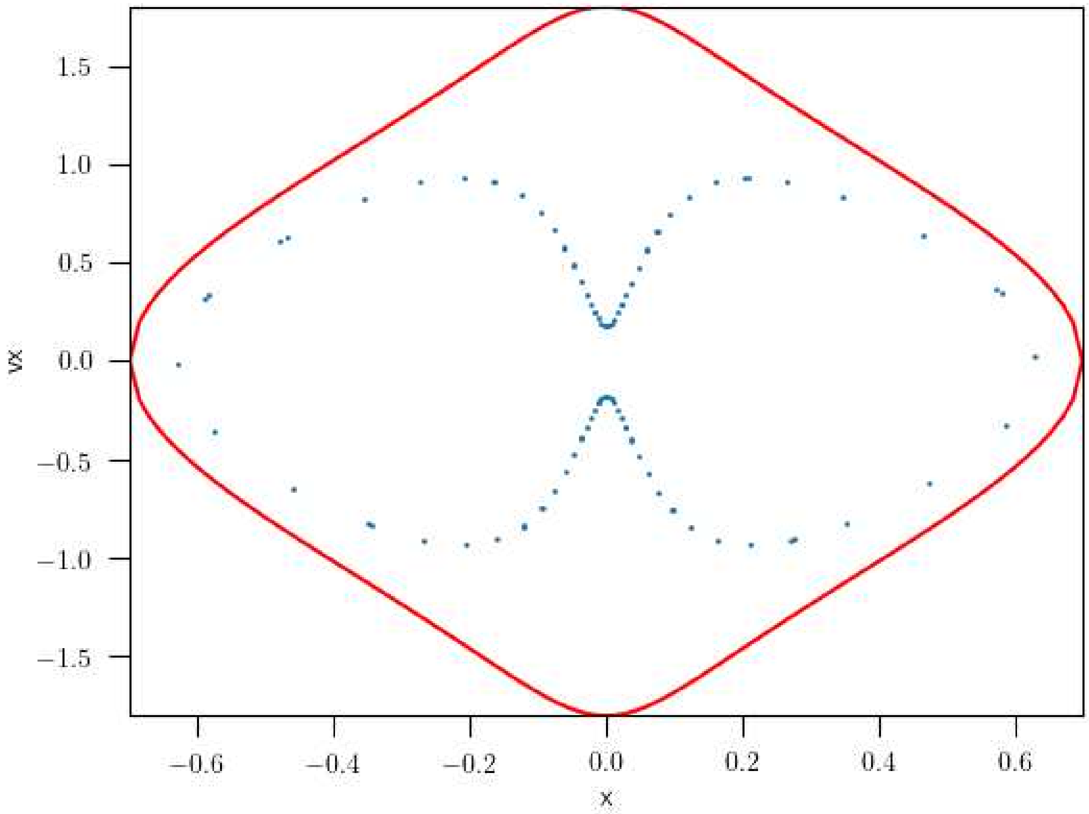
Box orbits

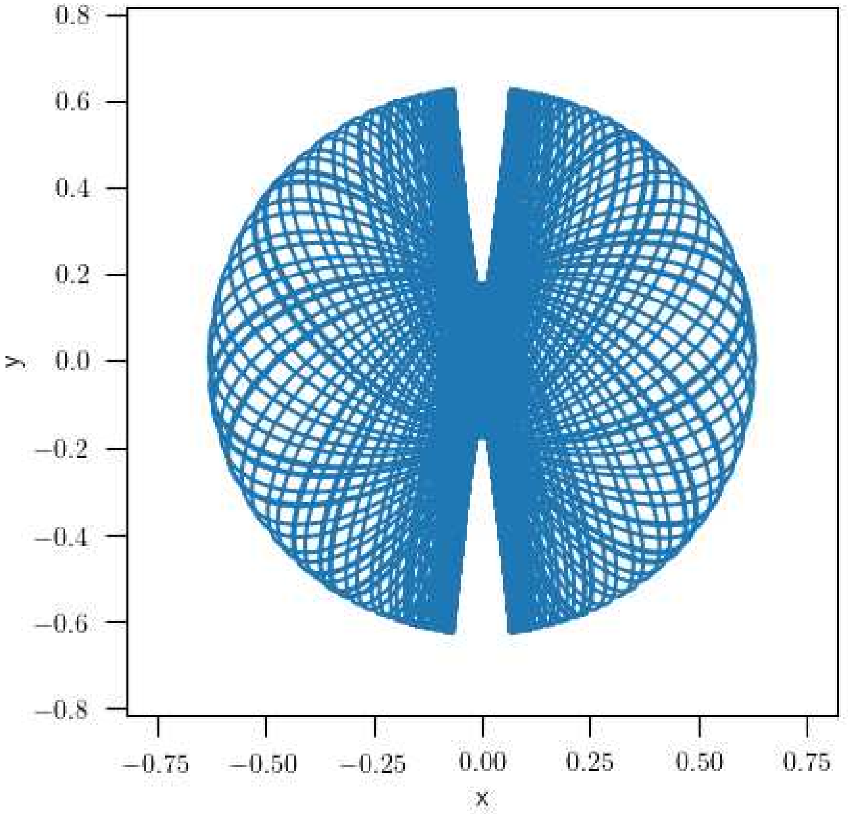





```
./mapping.py --V0 1. --Rc 0.14 --q 0.9    -E -0.337 --x 0.65
```

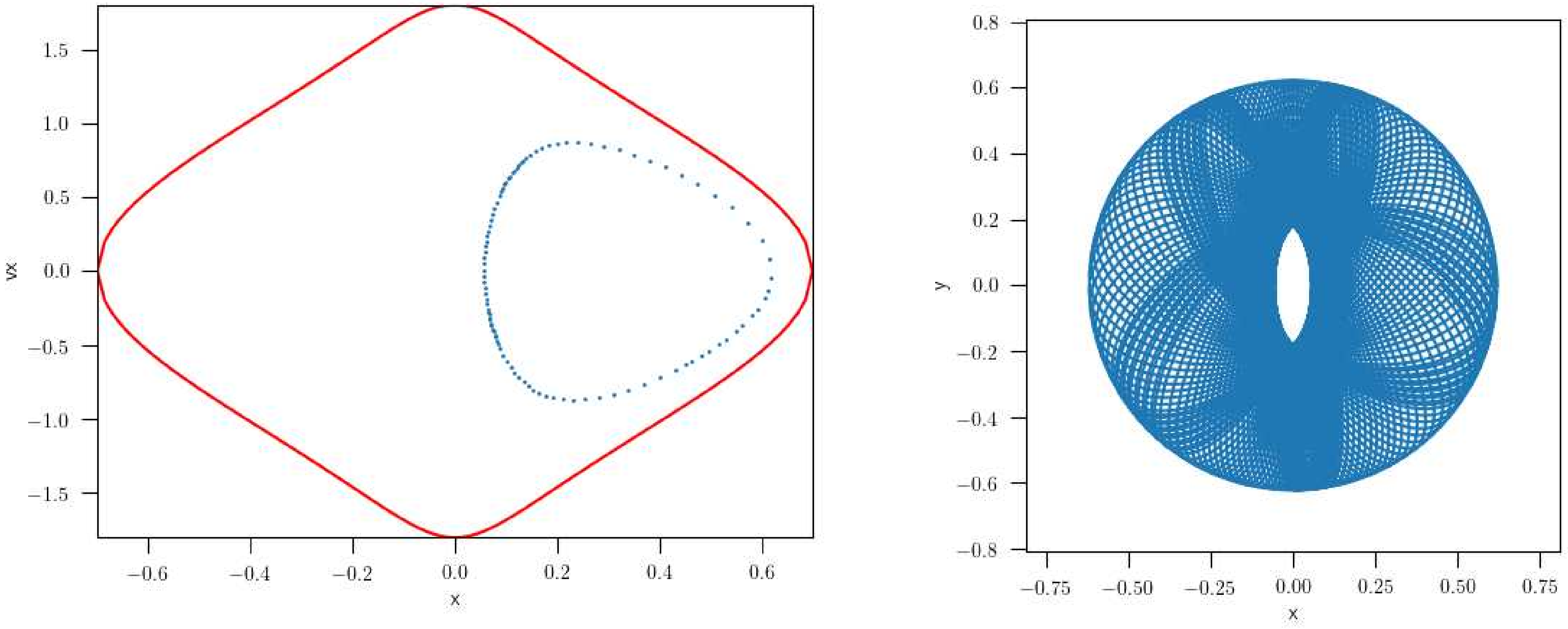
Box orbits





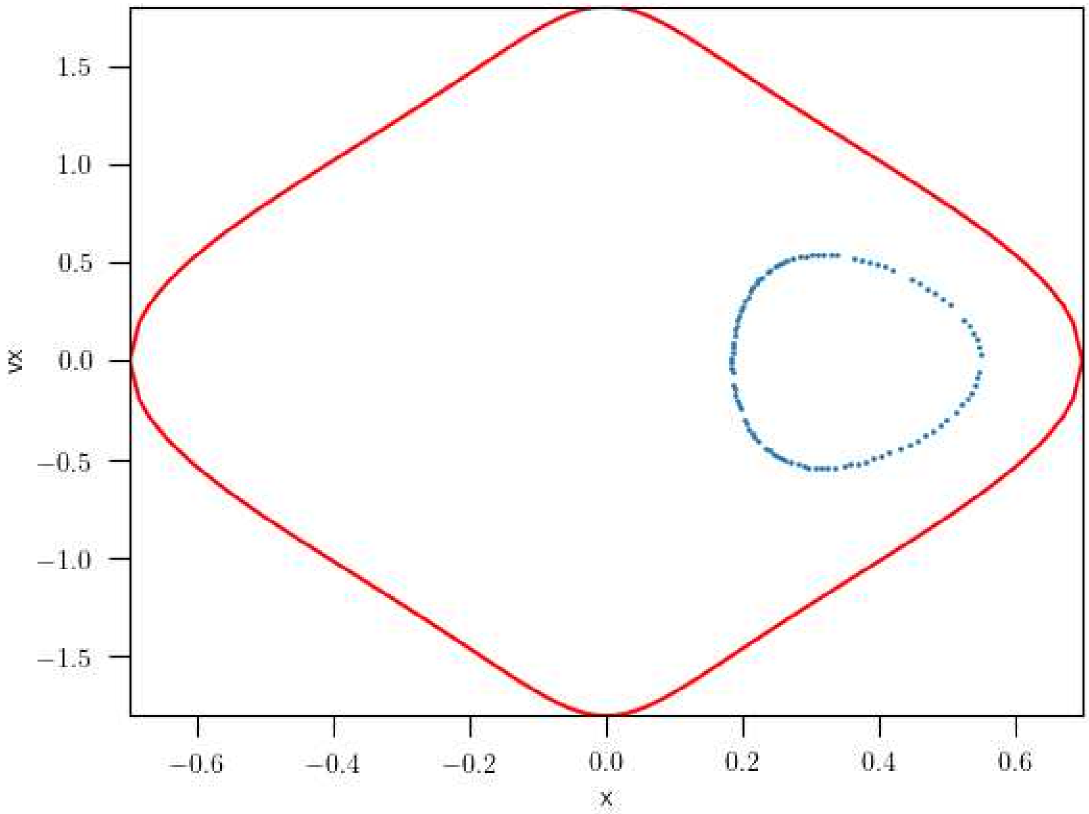

```
./mapping.py --V0 1. --Rc 0.14 --q 0.9    -E -0.337 --x 0.63
```

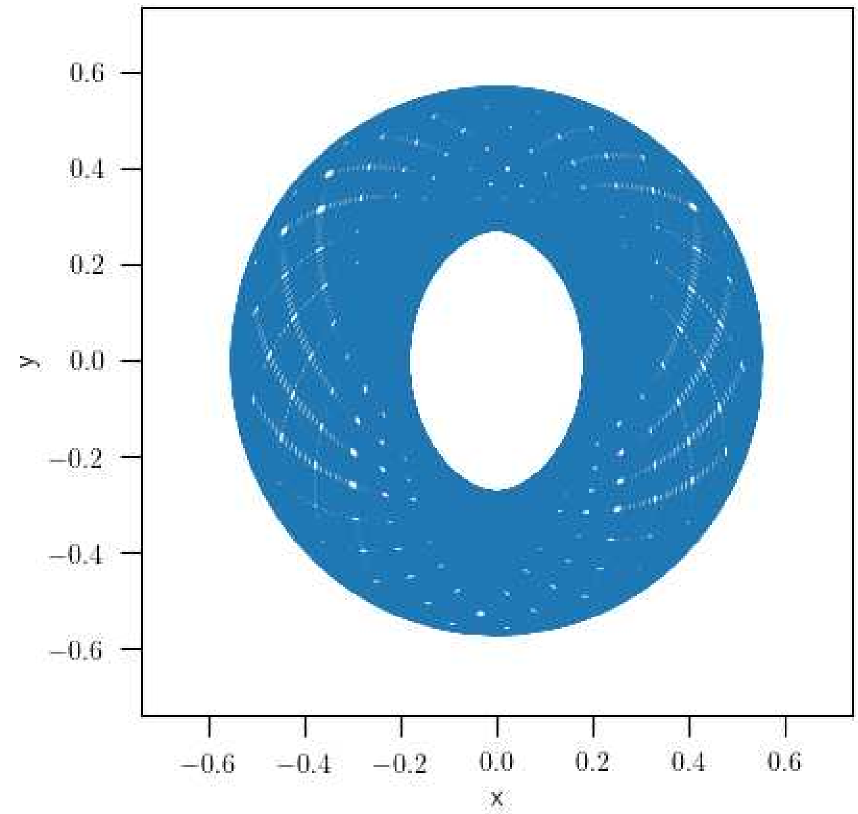
Loop orbits



./mapping.py --V0 1. --Rc 0.14 --q 0.9   -E -0.337 --x 0.62

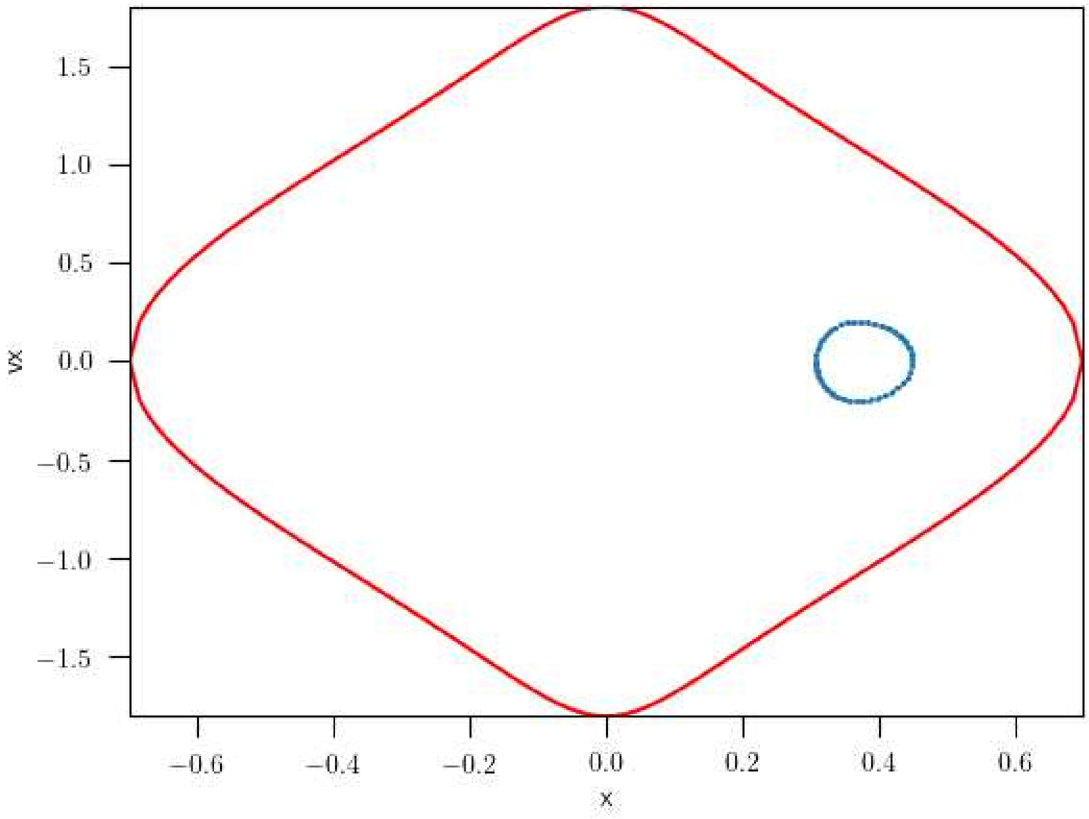
Loop orbits

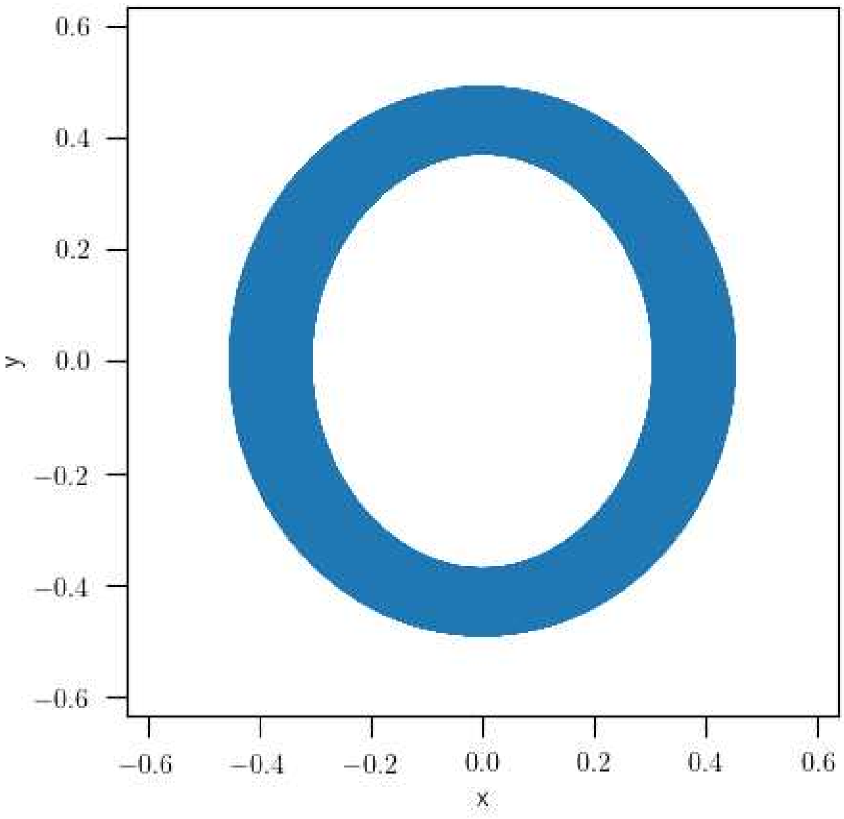





```
./mapping.py --V0 1. --Rc 0.14 --q 0.9    -E -0.337 --x 0.55
```

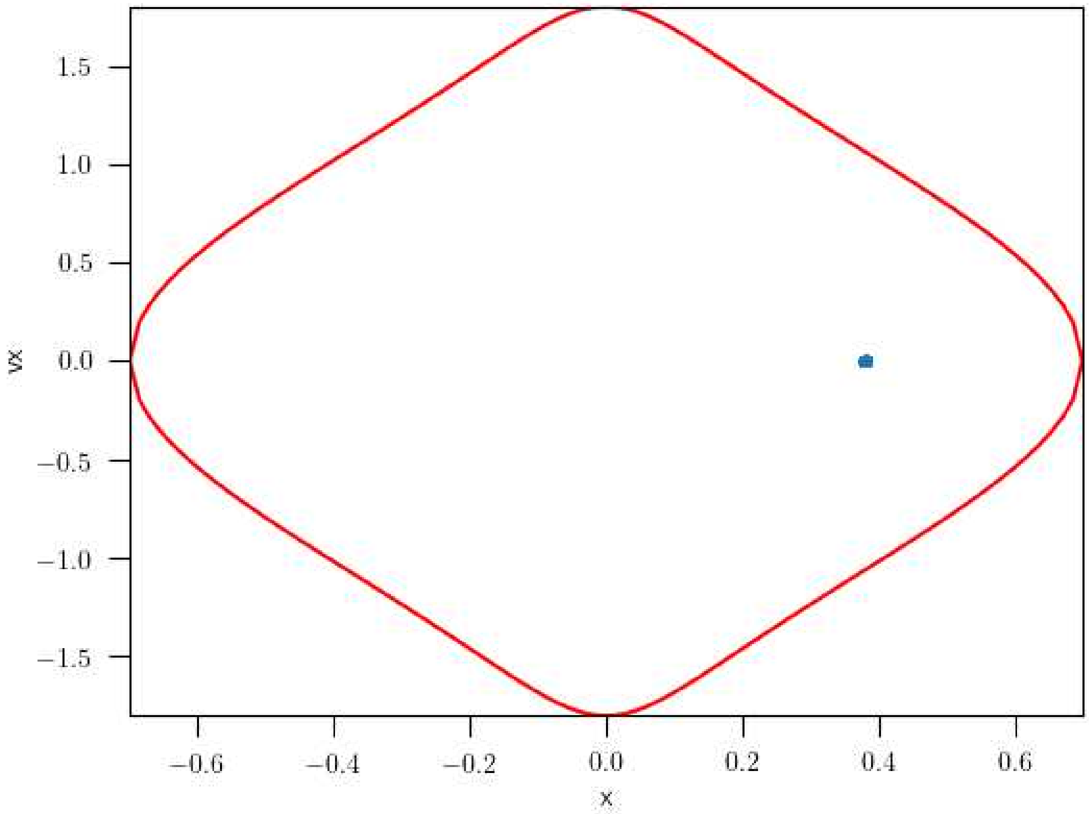
Loop orbits

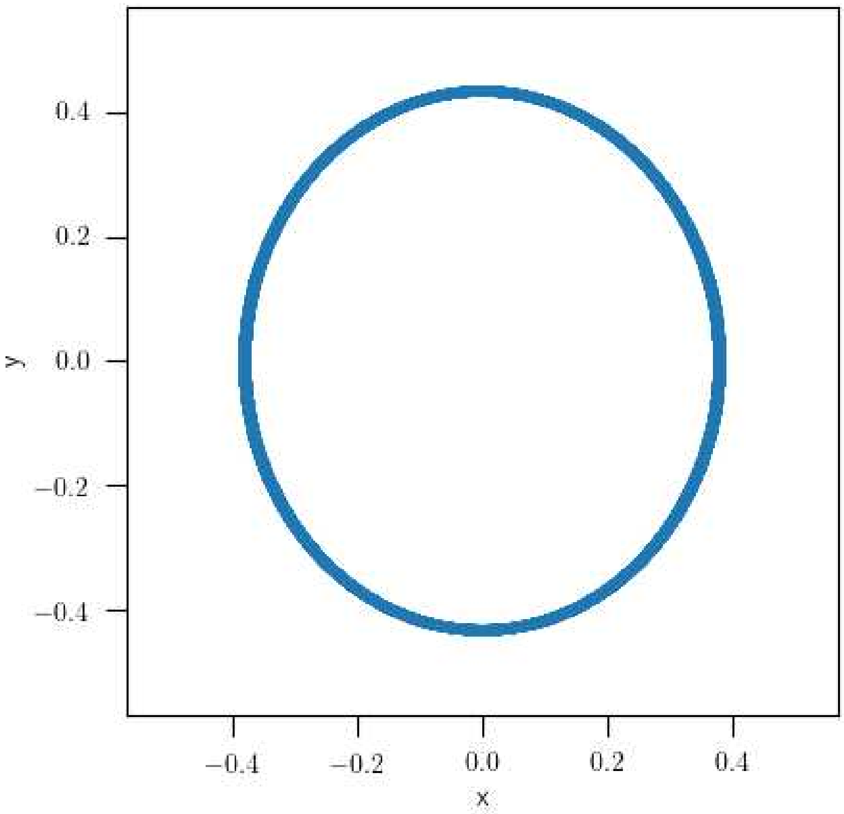




`./mapping.py --V0 1. --Rc 0.14 --q 0.9    -E -0.337 --x 0.45`

Loop orbits



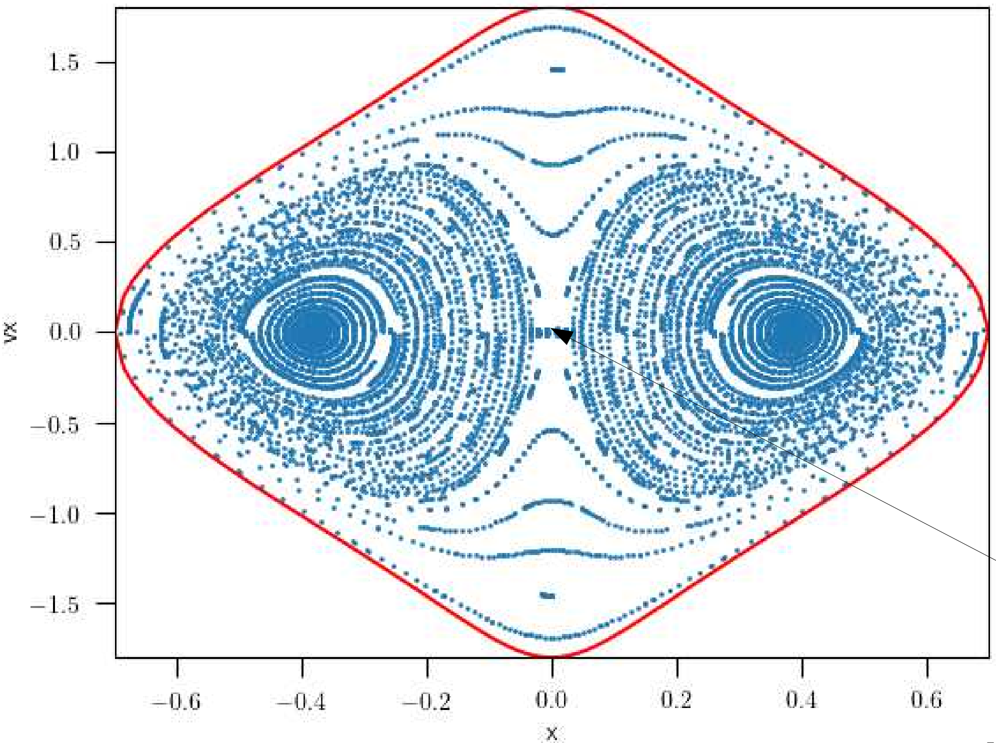


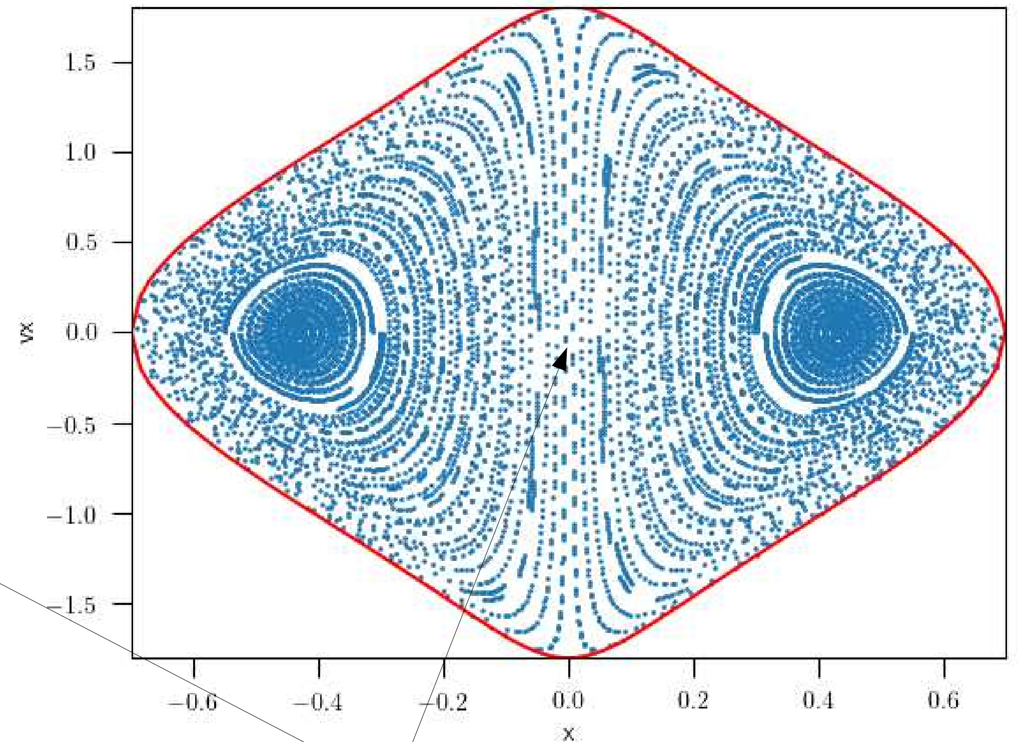

```
./mapping.py --V0 1. --Rc 0.14 --q 0.9    -E -0.337 --x 0.374
```

Box orbits elongated towards the y axis

$q = 0.9$

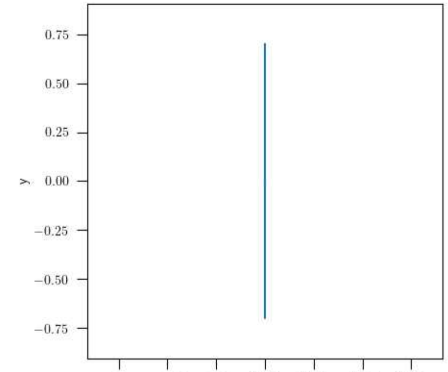
$q = 1.0$





unstable

stable

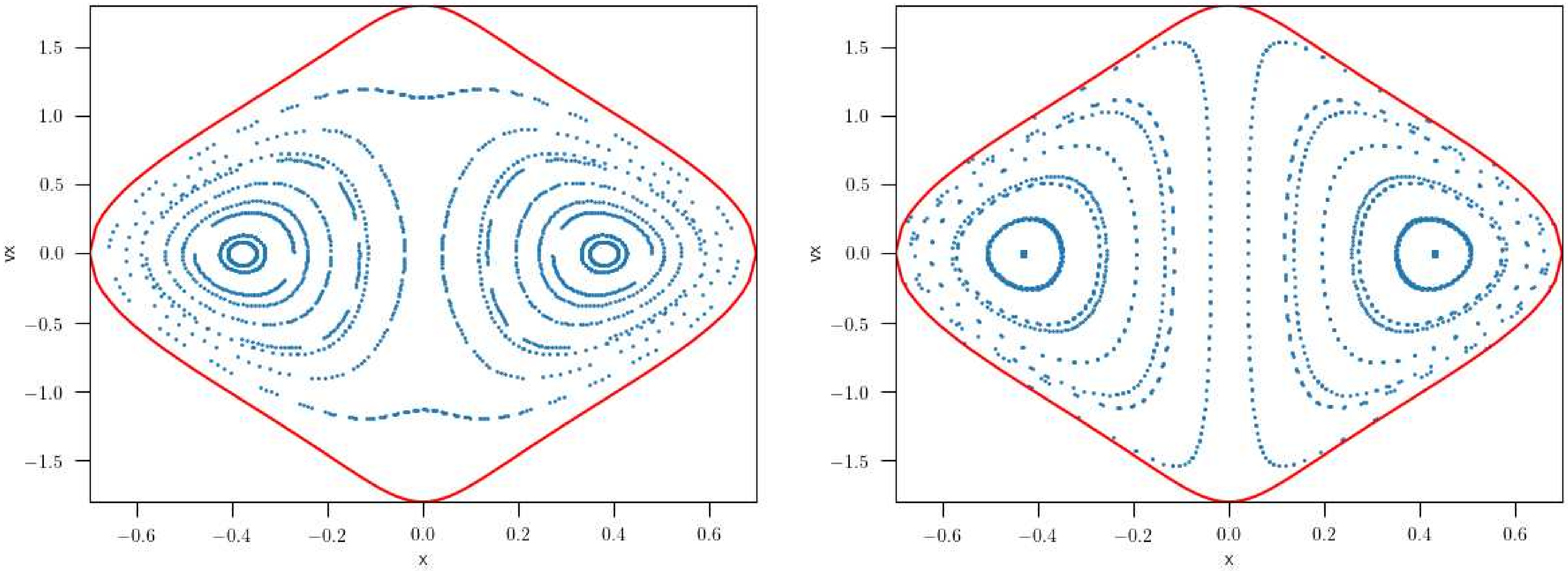



```
./mapping.py --V0 1. --Rc 0.14 --q 0.9  -E -0.337 --norbits 100
./mapping.py --V0 1. --Rc 0.14 --q 1.0  -E -0.337 --norbits 100
```

Integral of motions ?

Integral of motions ?

$q = 0.9$ $\quad\quad$ $q = 1.0$




```
./mapping.py --V0 1. --Rc 0.14 --q 0.9  -E -0.337  --norbits 18
./mapping.py --V0 1. --Rc 0.14 --q 1.0  -E -0.337  --norbits 18
```

Integrals of motions

① "nearly circular orbits"

Angular momentum conservation $\quad L_z = x\dot{y} - y\dot{x}$

can we compute $x = x(\dot{x})$ in the plane $y = 0$?

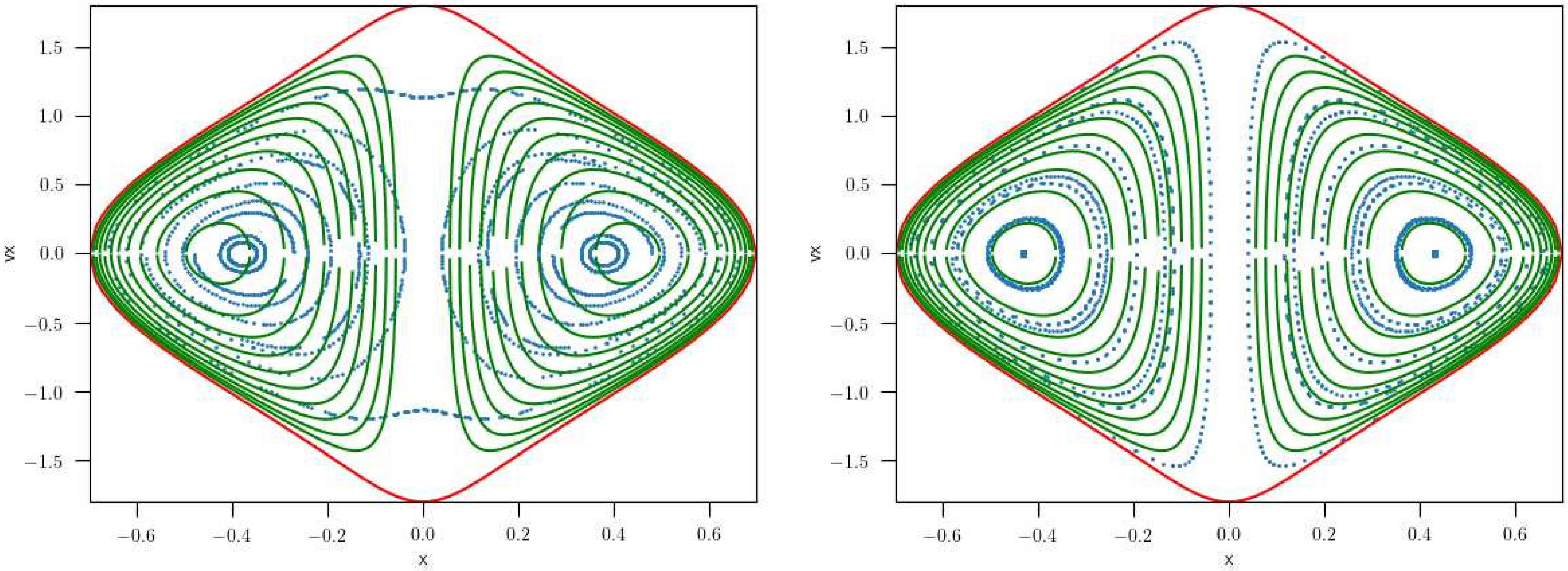
$$L_z = x\dot{y}$$

$$E = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\dot{y}^2 + \phi(x, y=0)$$

$$\boxed{\dot{x} = \sqrt{2(E-\phi) - \dot{y}^2} = \sqrt{2\left(E - \phi - \frac{L_z^2}{x^2}\right)}}$$

$$L_z$$

$q = 0.9$ $\quad$ $q = 1.0$




```
./mapping.py --V0 1. --Rc 0.14 --q 0.9  -E -0.337  --norbits 18  --add_ILz
./mapping.py --V0 1. --Rc 0.14 --q 1.0  -E -0.337  --norbits 18  --add_ILz
```

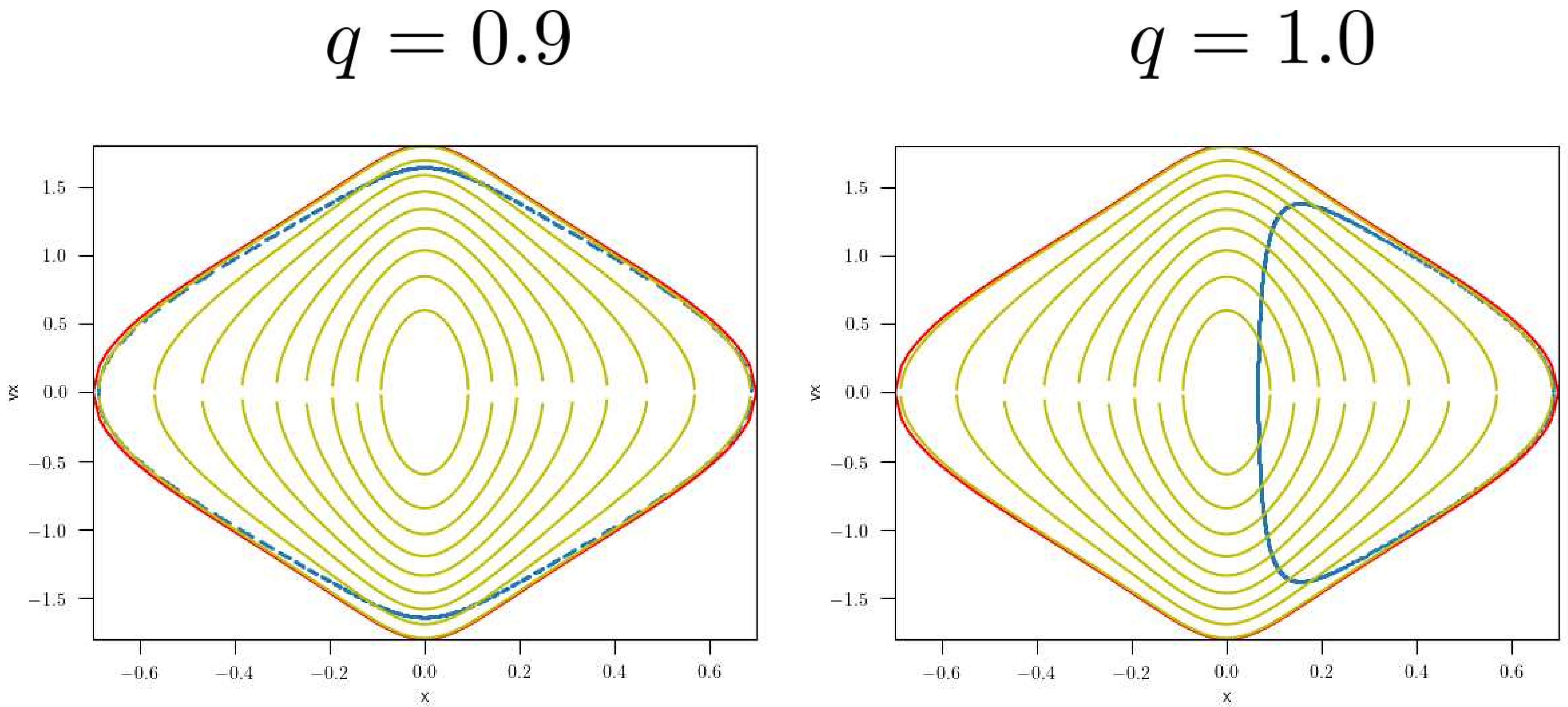
Integrals of motions

② Motion parallel to the long axis $\quad (y = \dot{y} = 0)$

$$H_x = \frac{1}{2}\dot{x}^2 + \phi(x, y=0) = E_x \quad \text{(harmonic oscillator)}$$

$$\boxed{\dot{x} = \sqrt{2\left(E_x - \phi(x, y=0)\right)}}$$

H_x



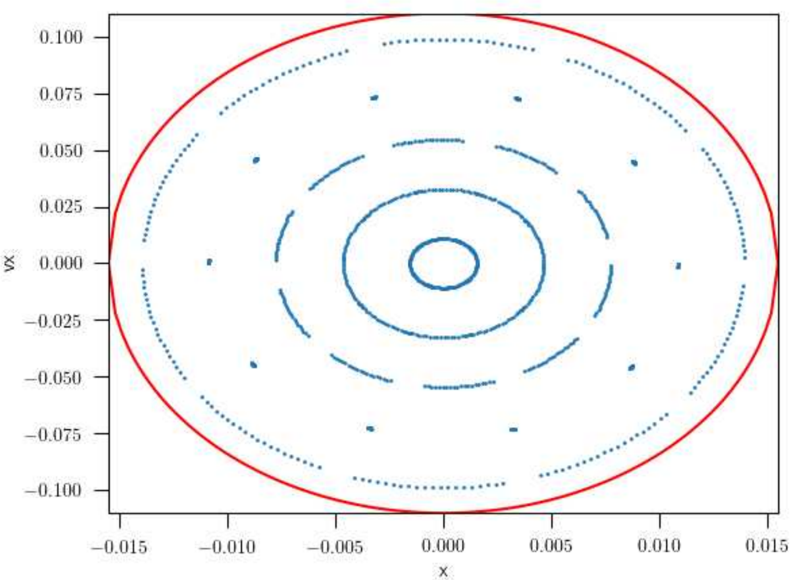

```
./mapping.py --V0 1. --Rc 0.14 --q 0.9  -E -0.337  --x 0.69  --nlaps 1000    --add_Ix
./mapping.py --V0 1. --Rc 0.14 --q 1.0  -E -0.337  --x 0.69  --nlaps 1000    --add_Ix
```

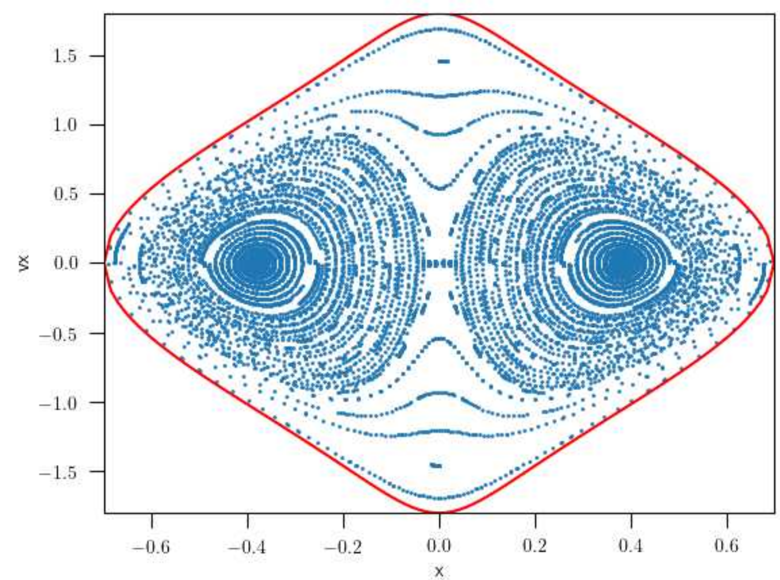
$$R \sim R_c$$

Family decoupling

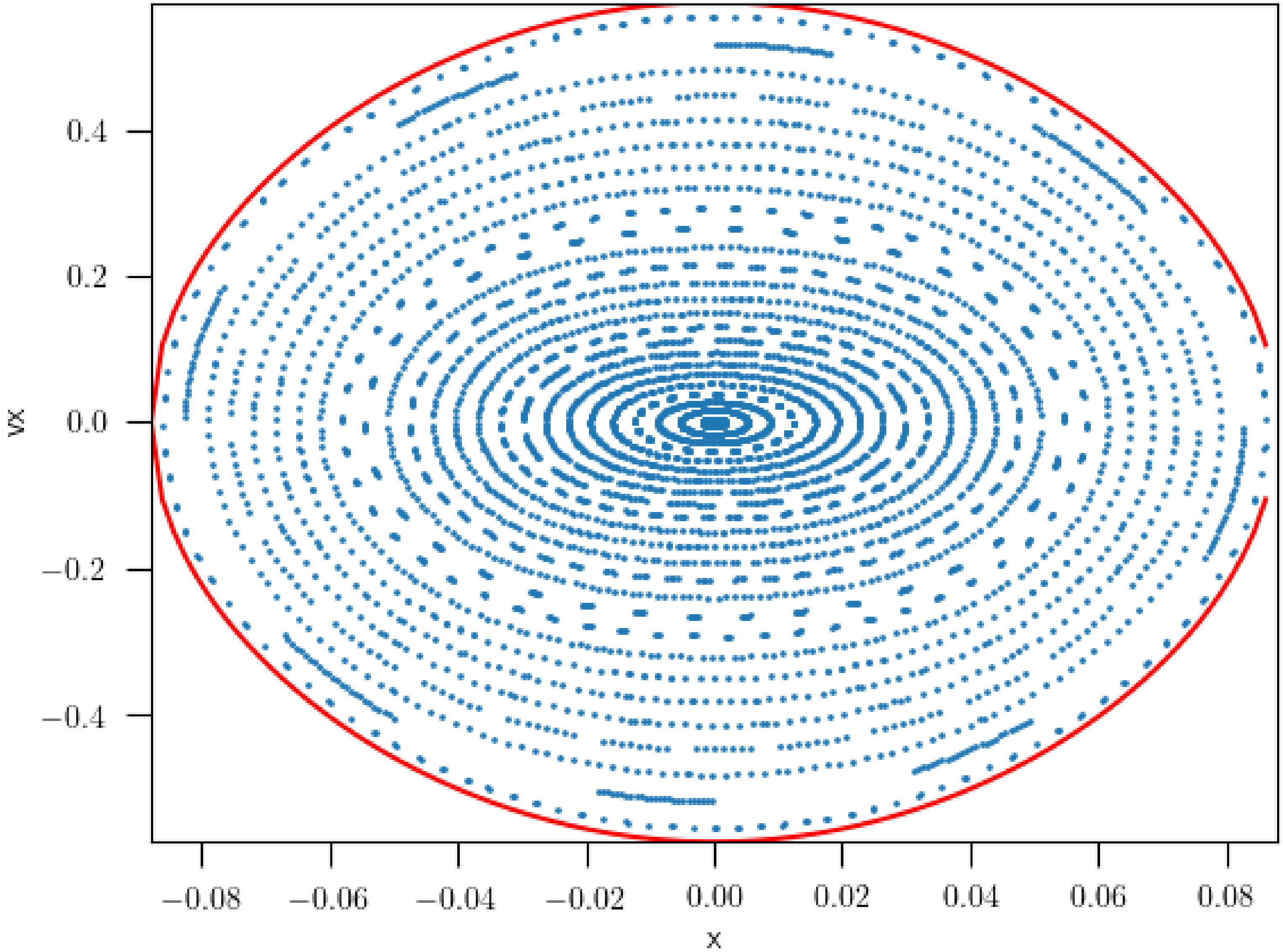
from low energy — to — high energy

1 family — 2 famillies



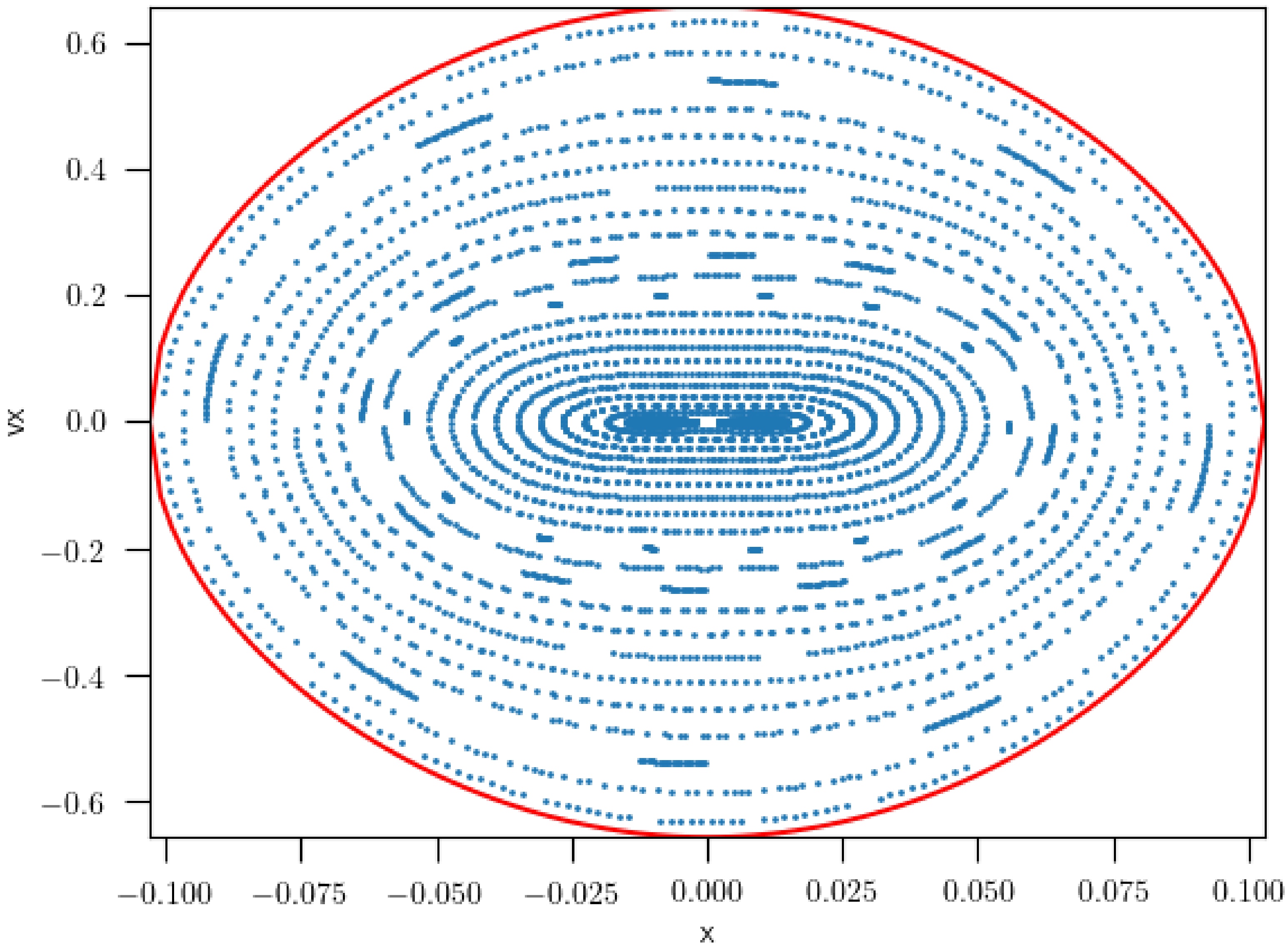


$$E = -1.8$$



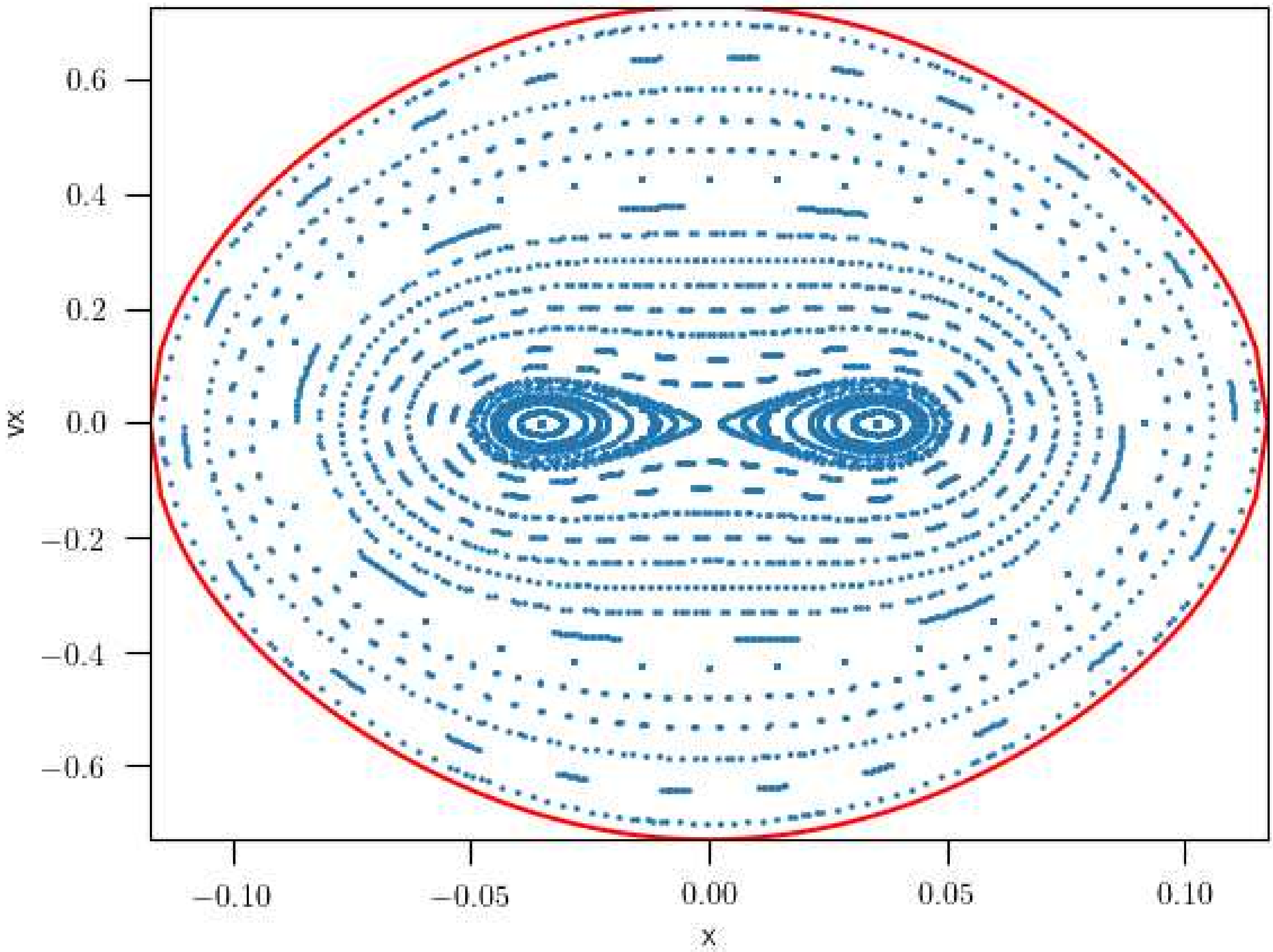

```
./mapping.py --V0 1. --Rc 0.14 --q 0.9  -E -1.80  --norbits 50
```

$$E = -1.75$$



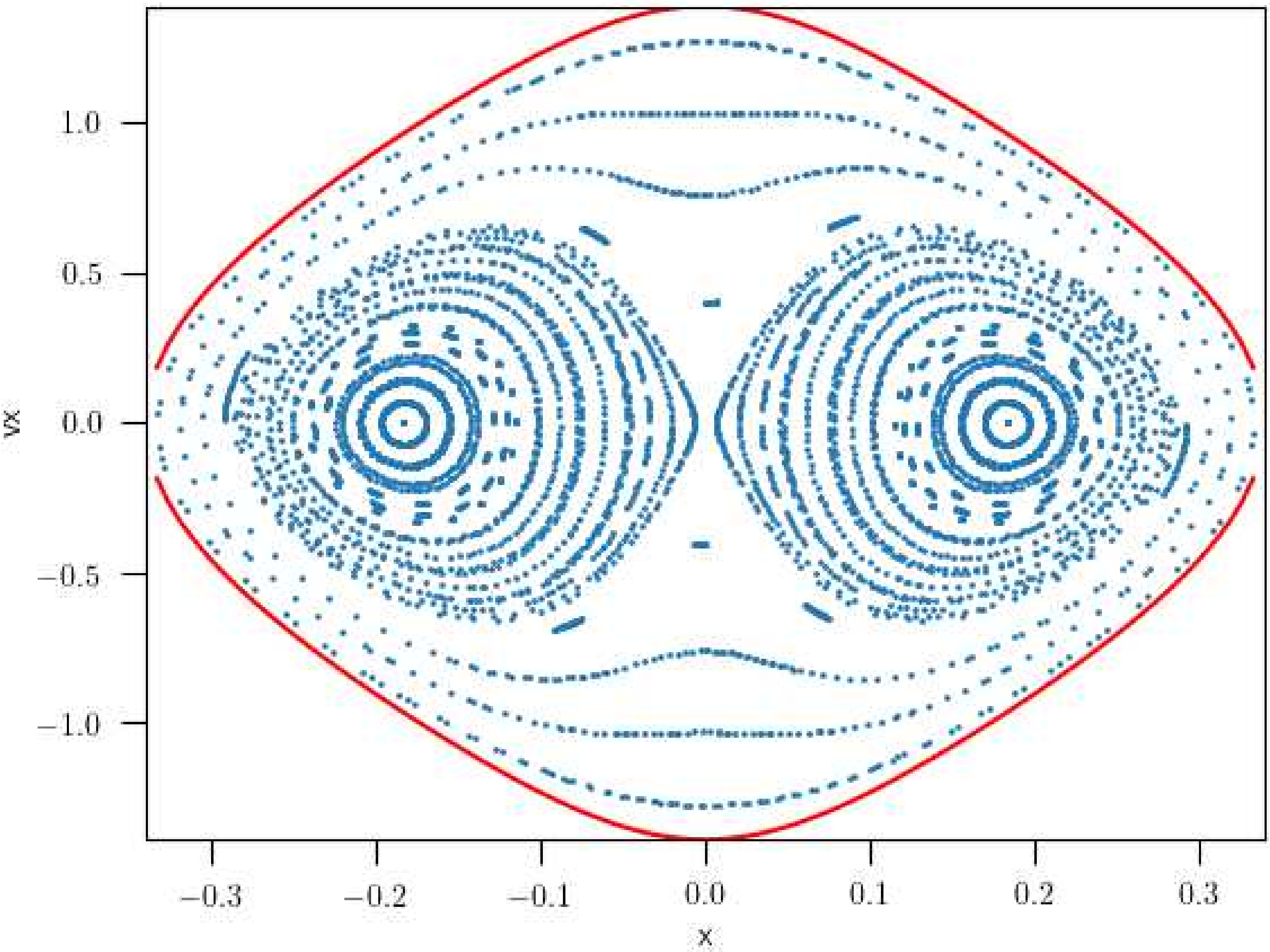

```
./mapping.py --V0 1. --Rc 0.14 --q 0.9  -E -1.75  --norbits 50
```

bifurcation $E = -1.70$



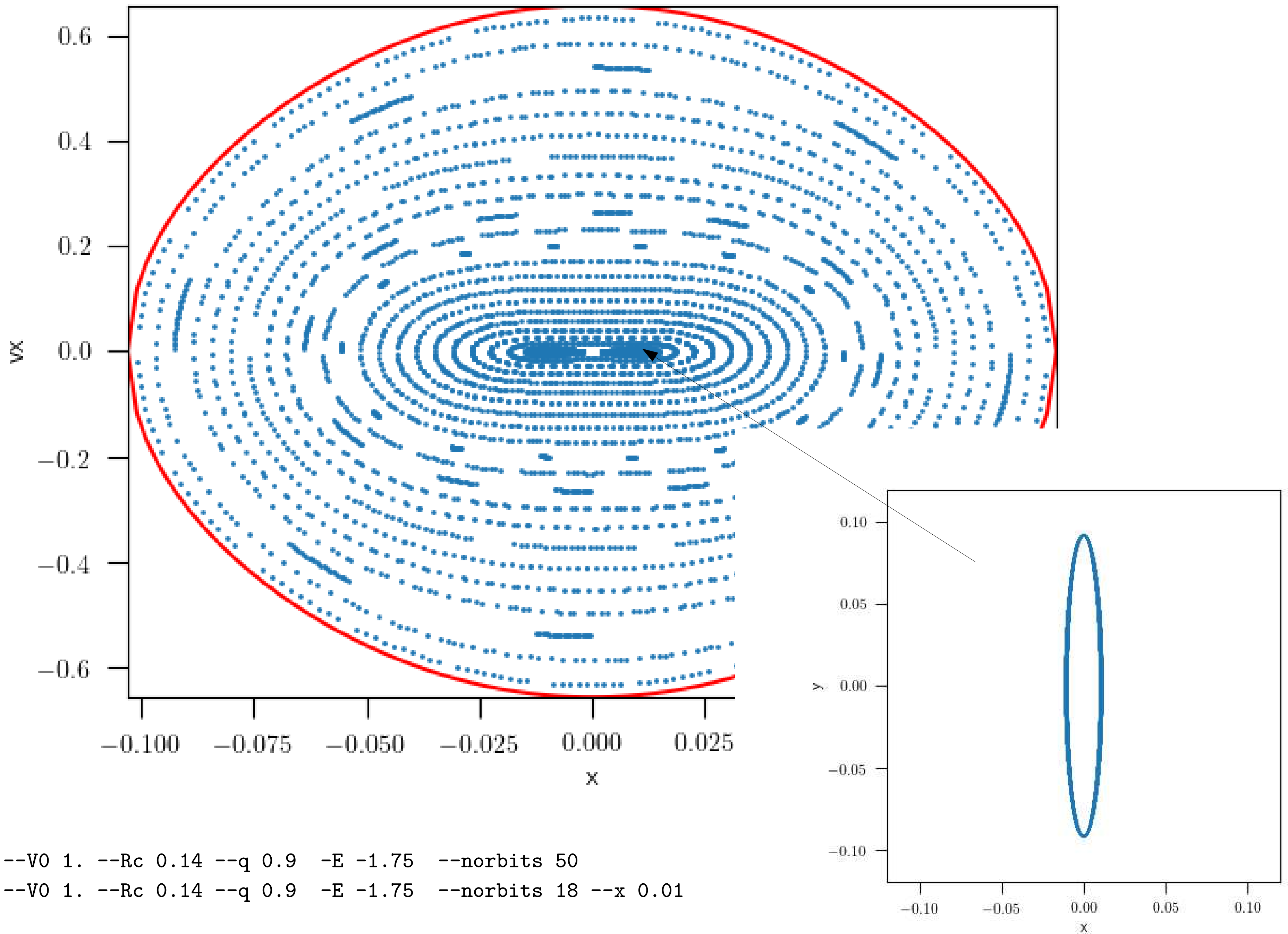

```
./mapping.py --V0 1. --Rc 0.14 --q 0.9  -E -1.70  --norbits 50
```

$$E = -1.$$




```
./mapping.py --V0 1. --Rc 0.14 --q 0.9  -E -1     --norbits 50
```

$$E = -1.75$$

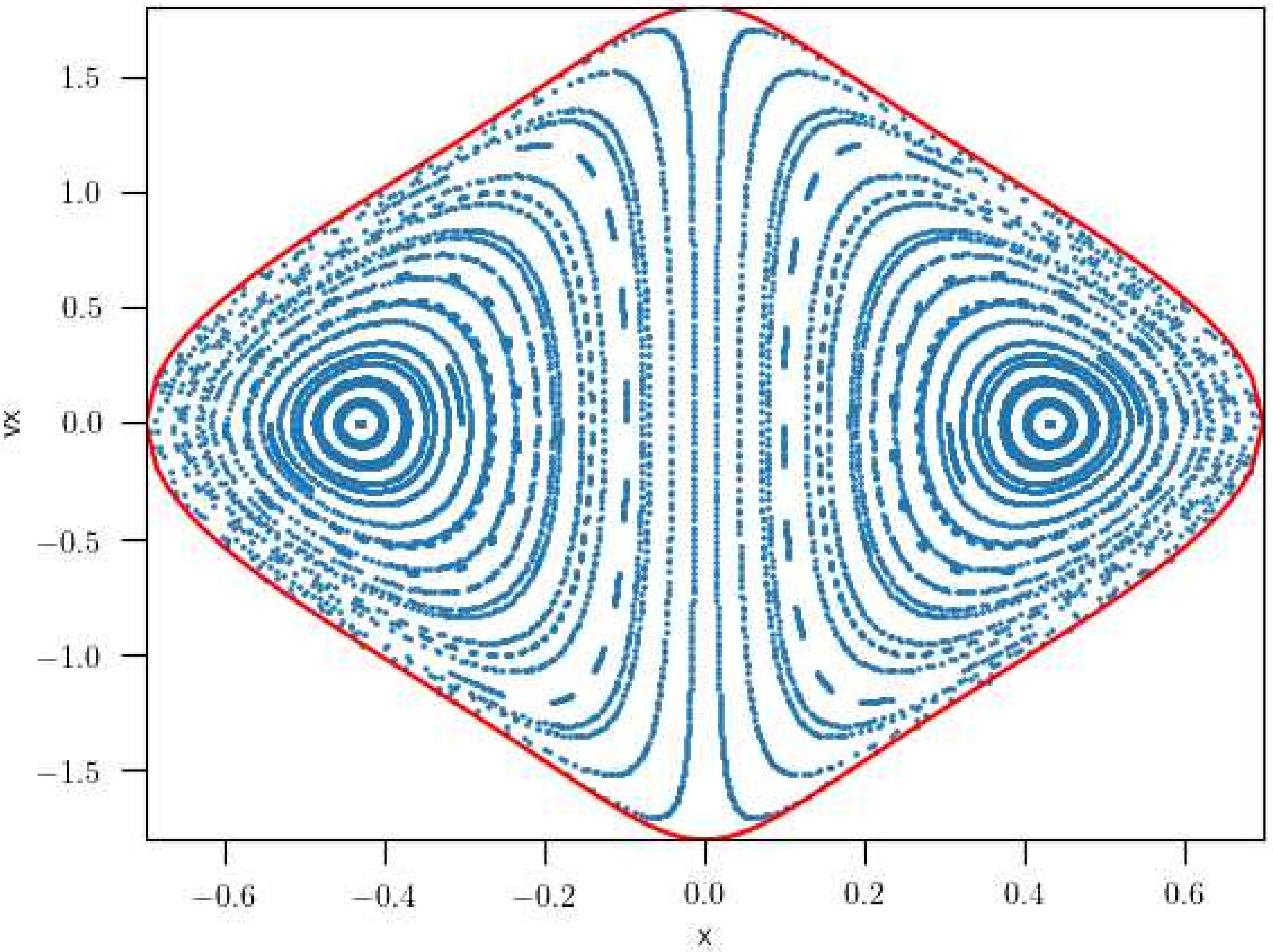



```
./mapping.py --V0 1. --Rc 0.14 --q 0.9  -E -1.75  --norbits 50
./mapping.py --V0 1. --Rc 0.14 --q 0.9  -E -1.75  --norbits 18 --x 0.01
```

Evolution with the flattening

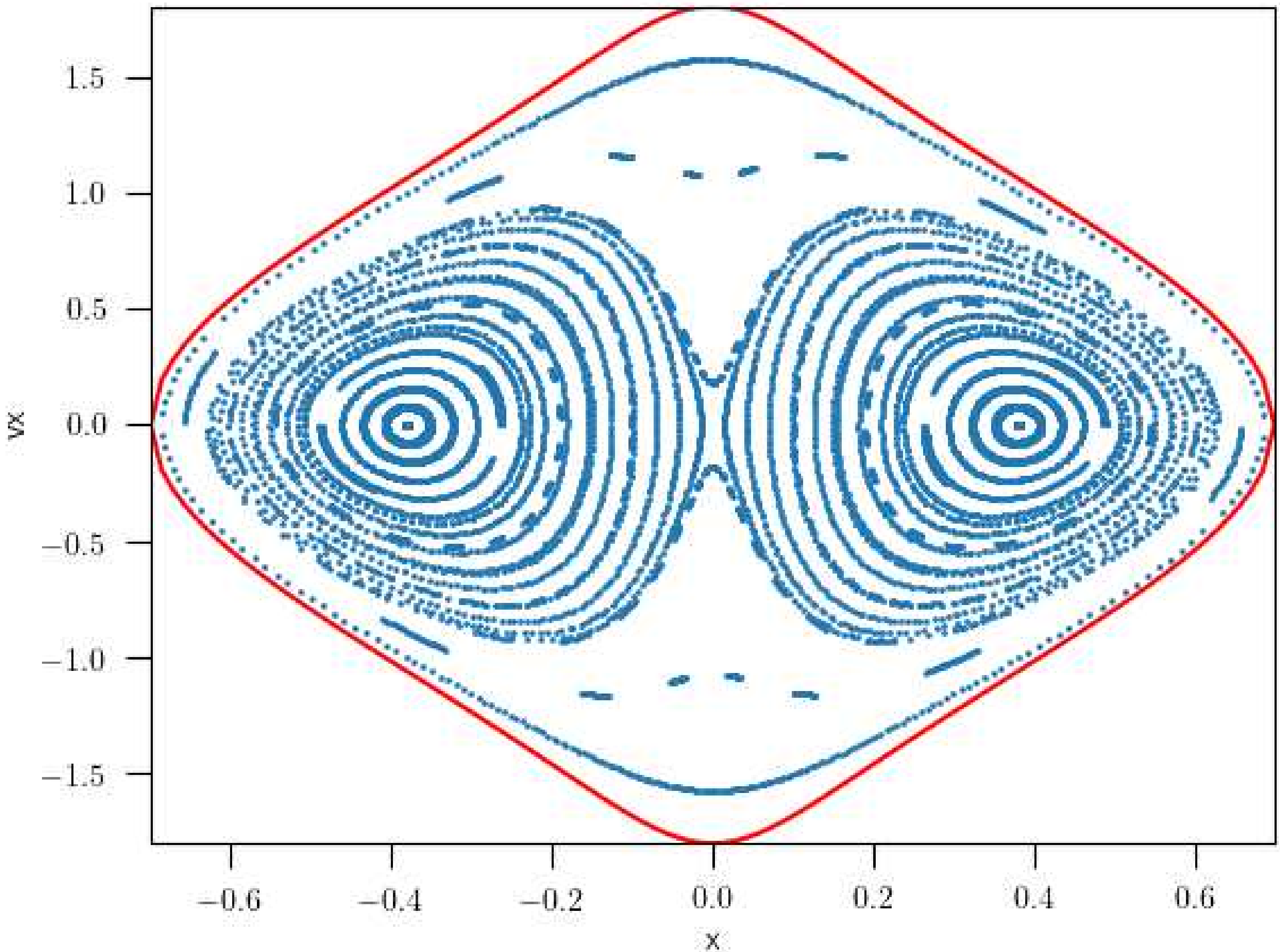
keeping the energy fixed

$q = 1.0$




```
./mapping.py --V0 1. --Rc 0.14 --q 1.0  -E -0.337  --norbits 50  --nlaps 200
```

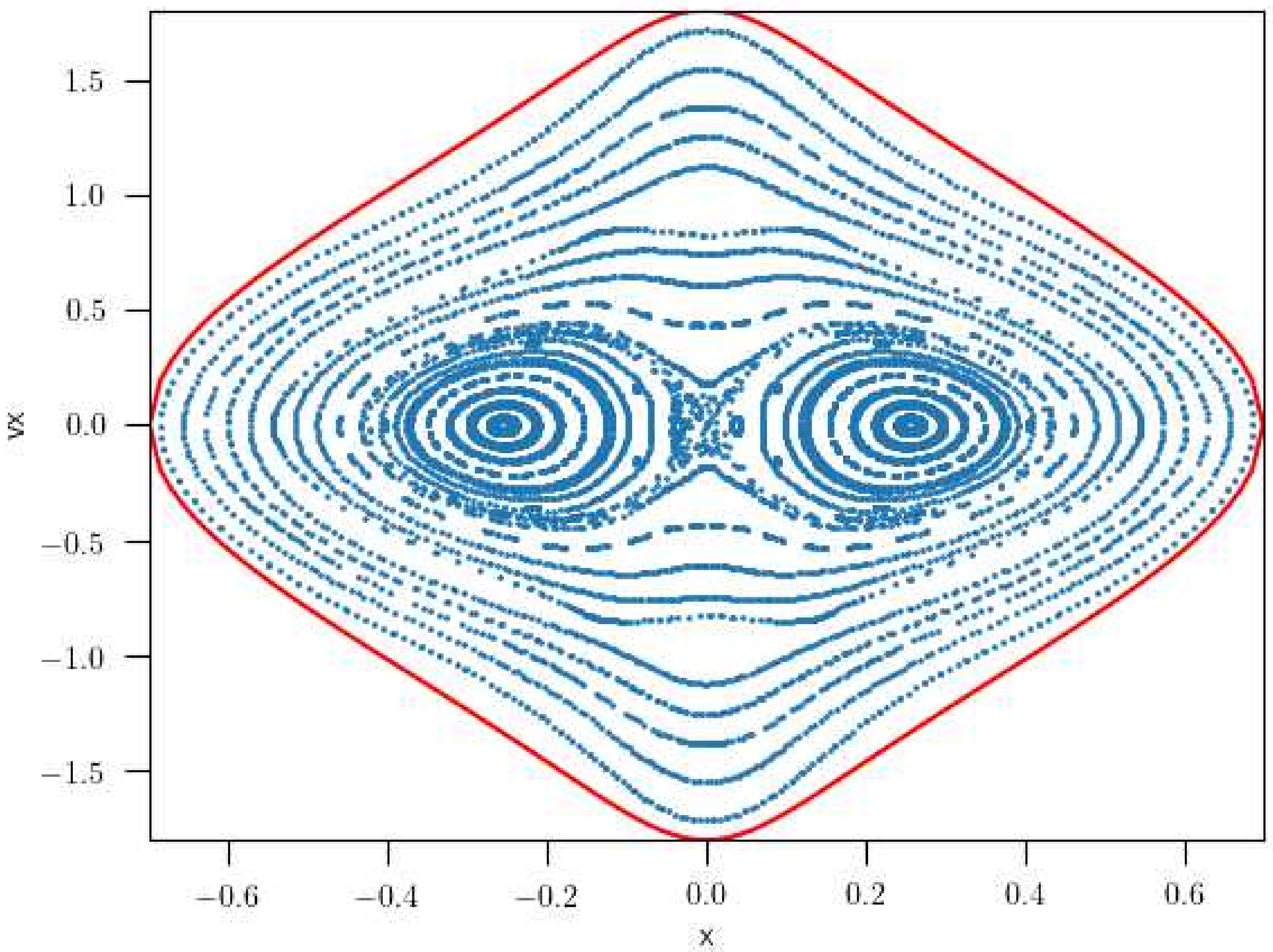
$q = 0.9$




```
./mapping.py --V0 1. --Rc 0.14 --q 0.9  -E -0.337  --norbits 50  --nlaps 200
```

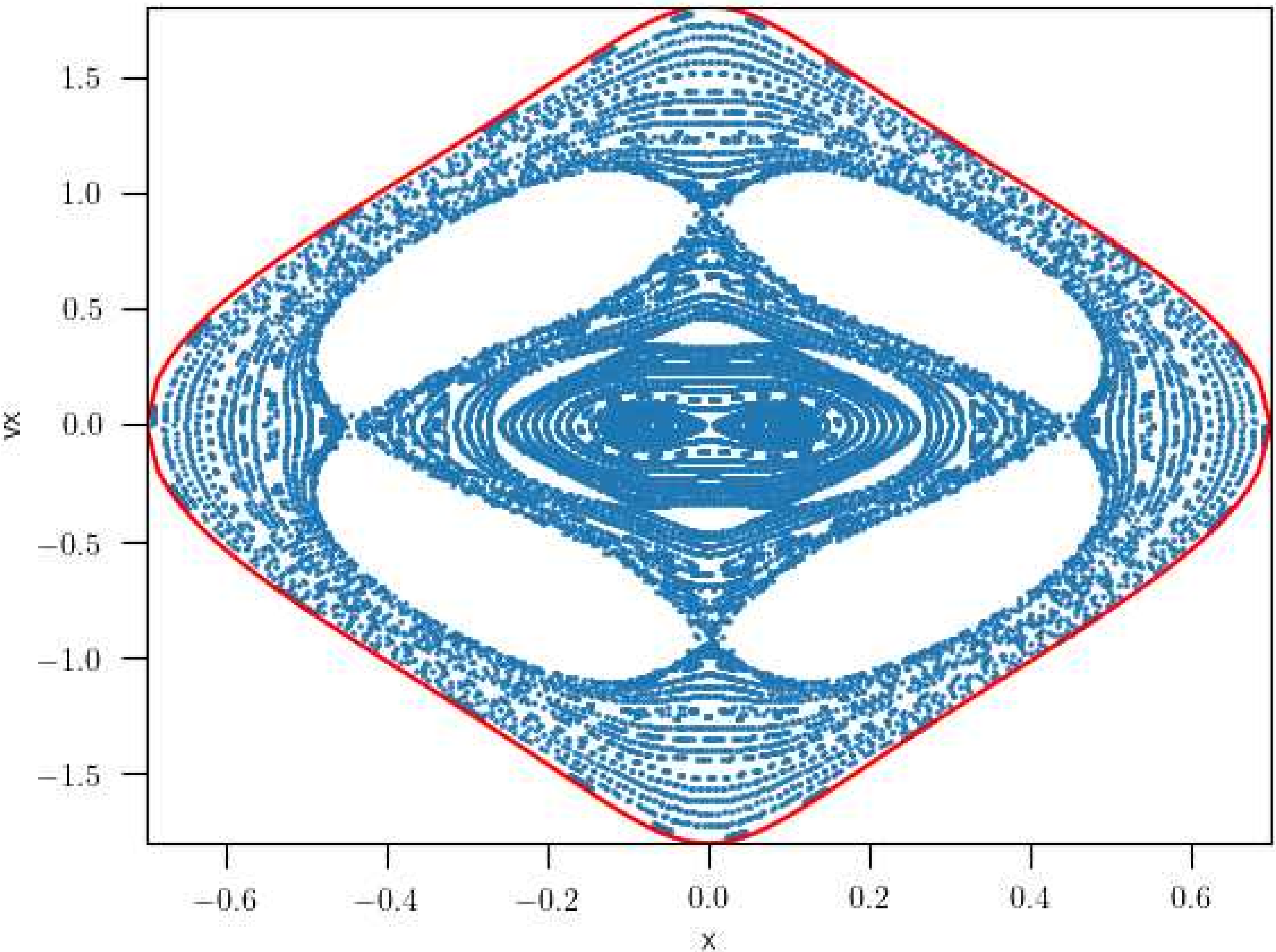
$q = 0.7$

Box orbits dominate the
phase space !



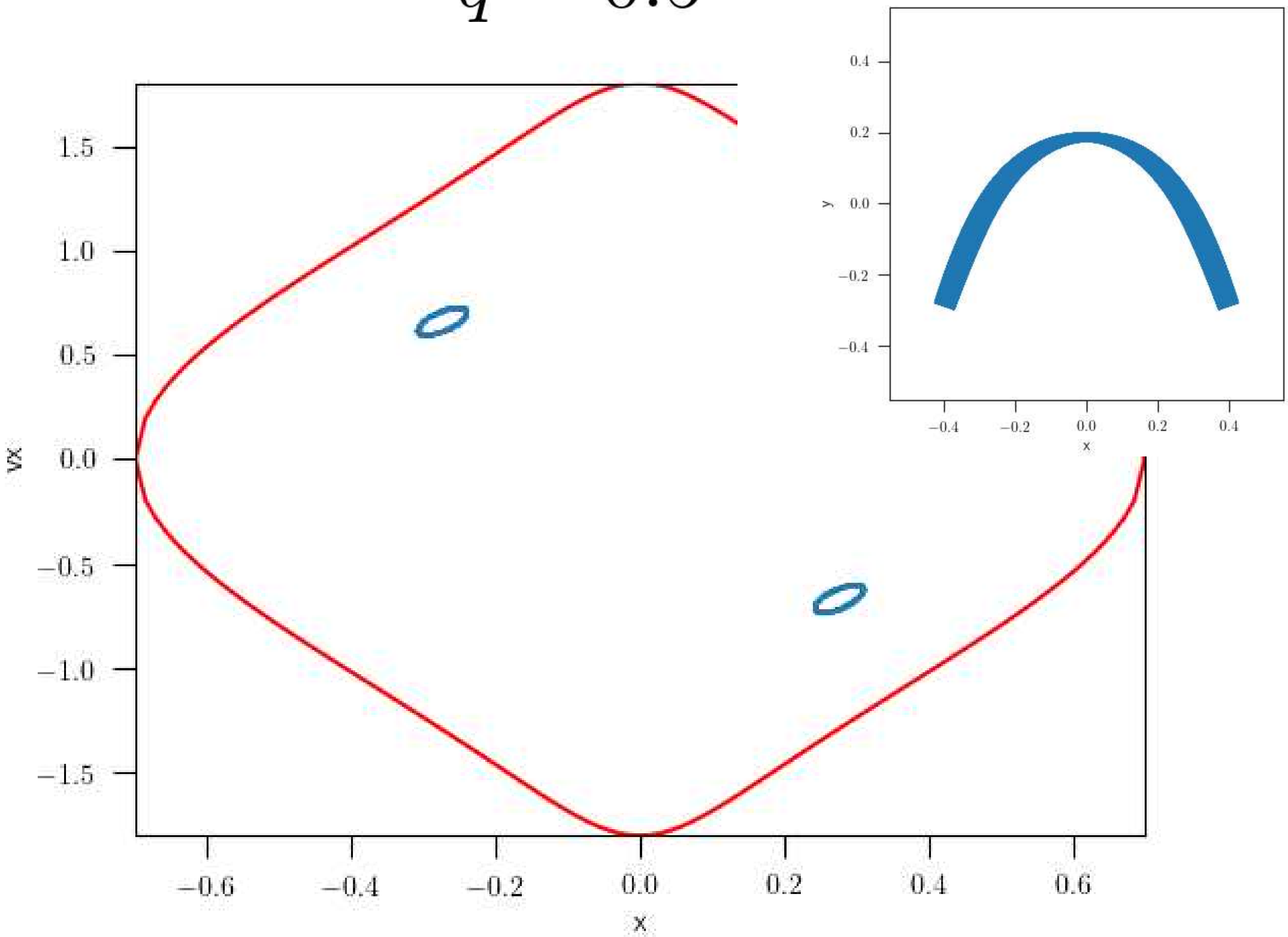

```
./mapping.py --V0 1. --Rc 0.14 --q 0.7  -E -0.337  --norbits 50  --nlaps 200
```

$q = 0.5$



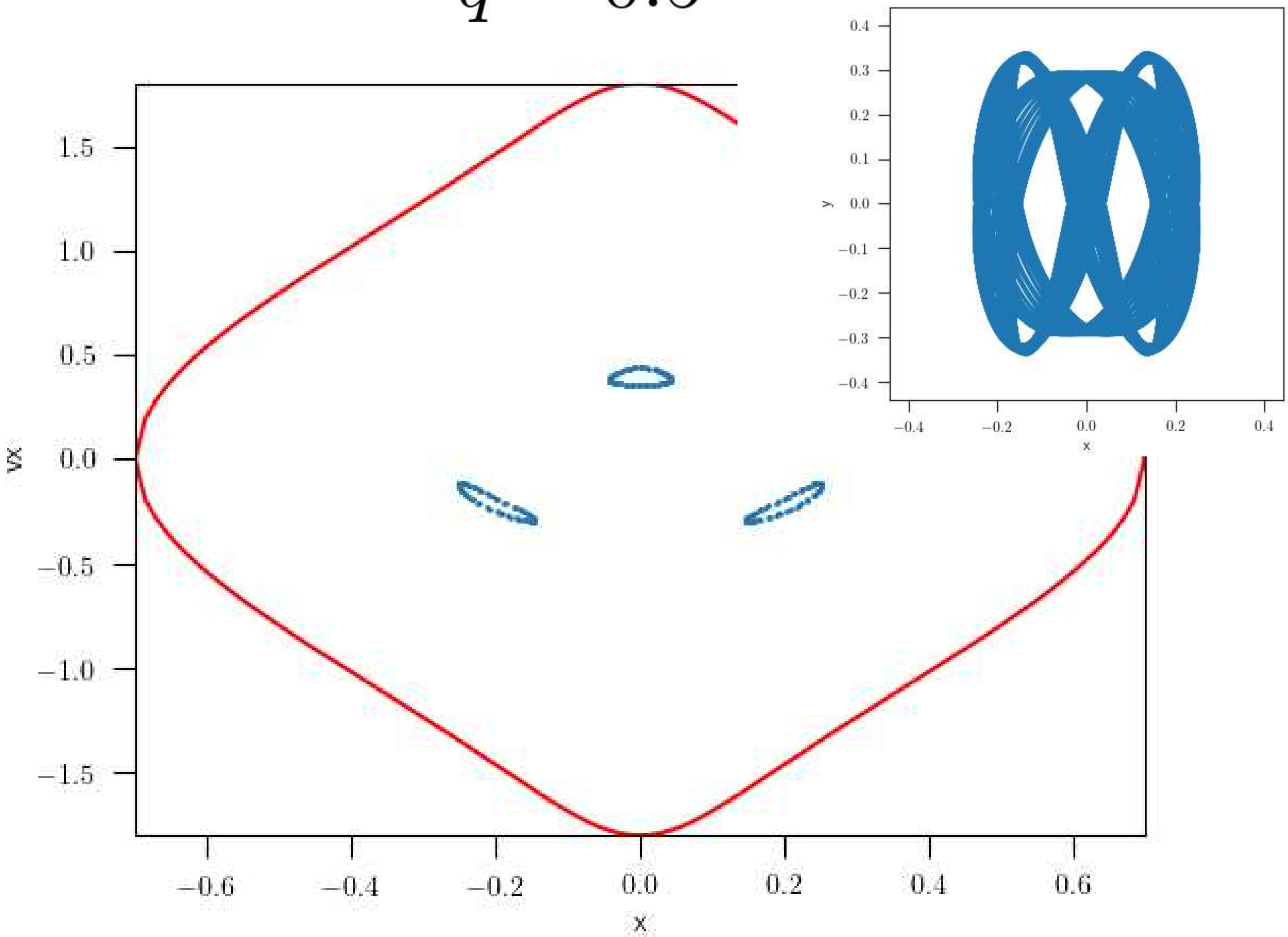

```
./mapping.py --V0 1. --Rc 0.14 --q 0.5  -E -0.337  --norbits 100  --nlaps 200
```

$q = 0.5$




```
./mapping.py --V0 1. --Rc 0.14 --q 0.5  -E -0.337  --norbits 50  --nlaps 200 --x 0.3 --vx -0.6
```

$q = 0.5$




```
./mapping.py --V0 1. --Rc 0.14 --q 0.5  -E -0.337  --norbits 50  --nlaps 200 --x 0.3 --vx -0.6
```

Conclusions

Many 2D bared potential have orbital structures like the logarithmic potential:

- Most orbits respect a 2nd integral (L_z or H_x)
- 2 types of orbits:
  - Loop : - fixed sense of rotation
    - never reach the centre
  - Box : - no fixed sense of rotation
    - many reach the centre

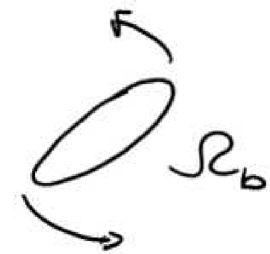
Loop orbits dominate when the axis ratio of of the potential is nearly unity.
Box orbits dominate instead.

Orbits
in planar non-axisymmetric rotating potentials

Two dimentional rotating potential

$$\phi(\theta, t) \quad \begin{cases} \theta \rightarrow & L_z \neq \text{cte} \\ t \rightarrow & E \neq \text{cte} \end{cases}$$

Assume a static rotation of the bar at constant angular frequency Ω_b

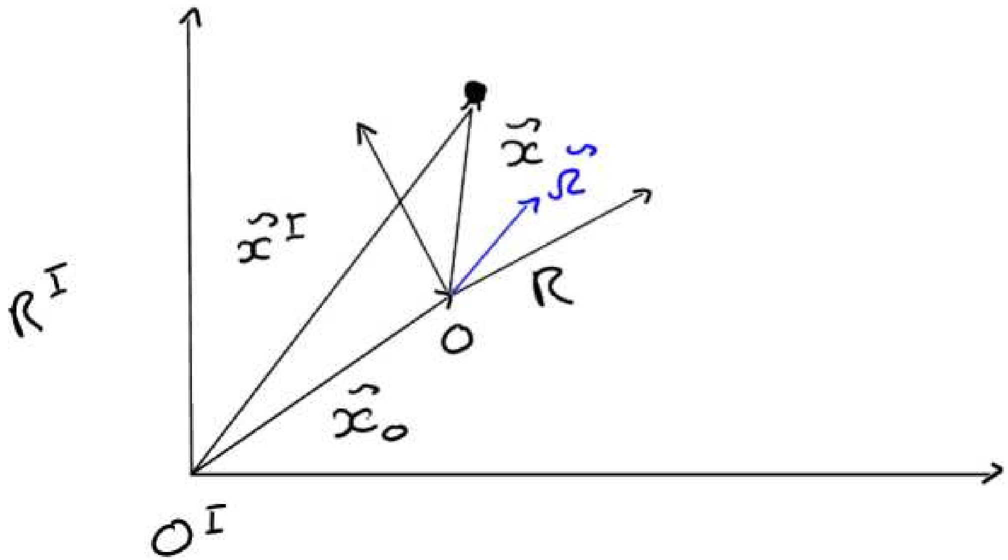


<u>Idea</u> : Describe the motion from the rotating frame where the bar is static

$$(\vec{x}^I, \dot{\vec{x}}^I) \rightarrow (\vec{x}, \dot{\vec{x}})$$

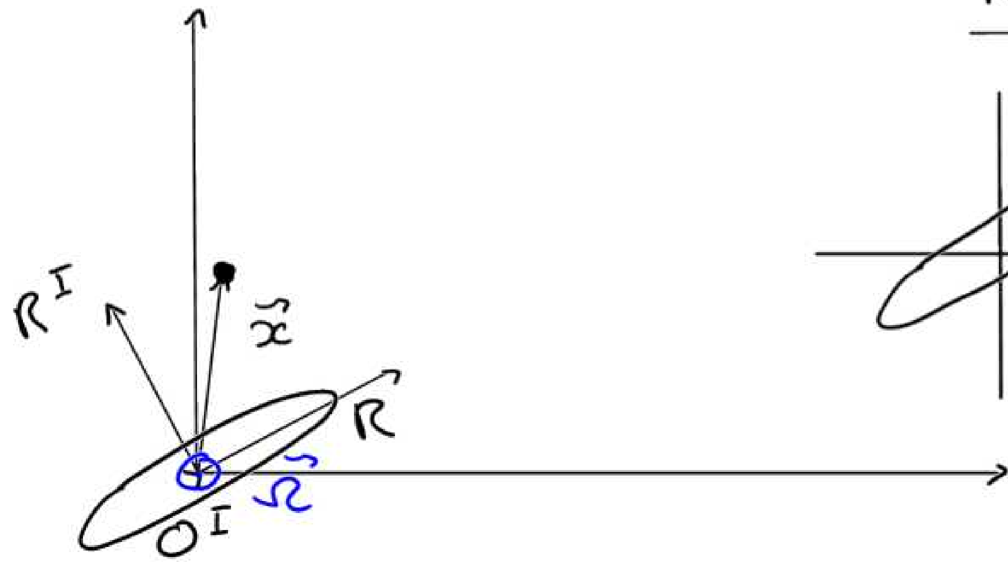
inertial frame R^I	rotating frame R	accelerated

Velocity in an accelerated rest frame

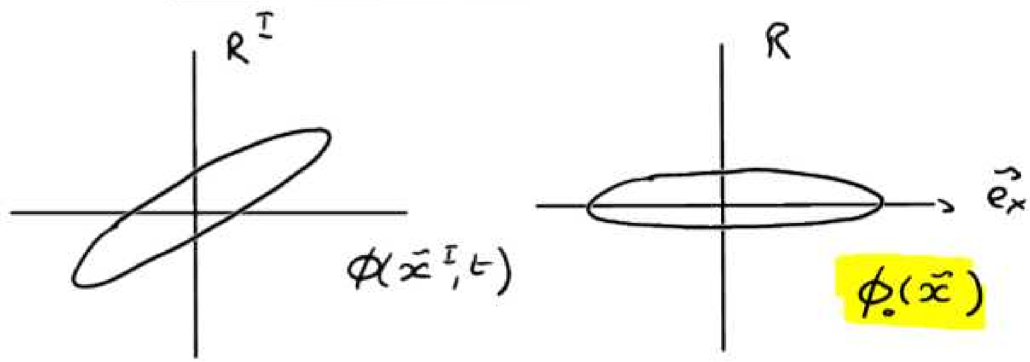


$$\boxed{\dot{\vec{x}}^I\Big)_{R_I} = \dot{\vec{x}}^I_0\Big)_{R_I} + \dot{\vec{x}}\Big)_R + \vec{\Omega} \times \vec{x}\Big)_R}$$

Velocity in a rotating rest frame



R : the bar is alignes with $\vec{e}_x$



$$\boxed{\dot{\vec{x}}^I\Big)_{R_I} = \cancel{\dot{\vec{x}}^I_O}\Big)_{R_I} + \dot{\vec{x}}\Big)_R + \vec{\Omega} \times \vec{x}\Big)_R}$$

$\vec{x}_0 = 0$

Lagrangian

In the inertial frame R^I

$$\boxed{\mathcal{L}(\vec{x}, \dot{\vec{x}}) = \frac{1}{2} \dot{\vec{x}}^{I\,2} - \phi^I(\vec{x}^I, t)}$$

In the rotating frame R

- $\frac{1}{2} \dot{\vec{x}}^{I\,2} \longrightarrow \frac{1}{2}\left(\dot{\vec{x}} + \vec{\Omega}_b \times \vec{x}\right)^2$
- $\phi^I(\vec{x}^I, t) \longrightarrow \phi_0(\vec{x})$

$$\boxed{\mathcal{L}(\vec{x}, \dot{\vec{x}}) = \frac{1}{2}\left(\dot{\vec{x}} + \vec{\Omega}_s \times \vec{x}\right)^2 - \phi_0(\vec{x})}$$

Hamiltonian

$$\boxed{H_J(\vec{x}, \vec{p}) = \frac{1}{2}\vec{p}^{\,2} - \vec{\Omega}\cdot(\vec{p}\times\vec{x}) + \phi(\vec{x})}$$

EXERCICE

H_J has no explicit time dependency

$\Rightarrow$ $H_J = E_J = \text{cte}$

Jacobi integral

Equations of motion from Hamilton's equations

$$H_J = \frac{1}{2}\vec{p}^{\,2} - \vec{\Omega}\cdot(\vec{p}\times\vec{x}) + \phi(\vec{x})$$

$$\boxed{\begin{aligned} \dot{\vec{x}} &= \frac{\partial H_J}{\partial \vec{p}} = \vec{p} - \vec{\Omega}\times\vec{x} \\ \dot{\vec{p}} &= -\frac{\partial H_J}{\partial \vec{x}} = -\vec{\nabla}\phi - \vec{\Omega}\times\vec{p} \end{aligned}}$$

EXERCICE

Effective potential

split the kinetic term in the Lagrangian

$$\mathcal{L}(\vec{x}, \dot{\vec{x}}) = \frac{1}{2}\left(\dot{\vec{x}} + \vec{\Omega}_s \times \vec{x}\right)^2 - \phi_0(\vec{x})$$

$$\equiv \frac{1}{2}\dot{\vec{x}}^2 + \dot{\vec{x}}\left(\vec{\Omega}_s \times \vec{x}\right) \underbrace{- \phi_0(\vec{x}) + \frac{1}{2}\left(\vec{\Omega}_s \times \vec{x}\right)^2}_{\text{depends only on } \vec{x}}$$

$$\boxed{\begin{aligned}\phi_{\text{eff}}(\vec{x}) &:= \phi(\vec{x}) - \frac{1}{2}\left(\vec{\Omega} \times \vec{x}\right)^2 \\ &= \phi(\vec{x}) \underbrace{- \frac{1}{2}\vec{\Omega}^2\vec{x}^2 + \frac{1}{2}\left(\vec{\Omega}\cdot\vec{x}\right)^2}_{\phi_{\text{centr}}(\vec{x}) \,:\, \text{repulsive centrifugal potential}}\end{aligned}}$$

Equations of motion from the Euler - Lagrange equation

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\vec{x}}} - \frac{\partial \mathcal{L}}{\partial \vec{x}} = 0$$

$$\ddot{\vec{x}} = - \vec{\nabla}\phi_{eff}(\vec{x}) - 2\left(\Omega \times \dot{\vec{x}}\right)$$

$$\boxed{\ddot{\vec{x}} = - \vec{\nabla}\phi(\vec{x}) \underbrace{+ \Omega^2 \vec{x} - \vec{\Omega}(\vec{\Omega}\cdot\vec{x})}_{\text{centrifugal force}} \underbrace{- 2\left(\Omega \times \dot{\vec{x}}\right)}_{\text{Coriolis force}}}$$

$$= \Omega^2 \vec{x} \quad \text{if } \vec{\Omega} = \begin{pmatrix} 0 \\ 0 \\ \Omega \end{pmatrix}$$ and z=0

Stationary points

$\dot{\vec{x}} = \ddot{\vec{x}} = 0$ (in the rotating frame)

1) The point co-rotate with the bar ($V_\perp = \Omega R$)

2) with $\ddot{\vec{x}} = -\vec{\nabla}\phi_{eff} - 2\,\vec{\Omega}\times\underbrace{\dot{\vec{x}}}_{=0} = 0$

$$\boxed{\vec{\nabla}\phi_{eff} = 0}$$

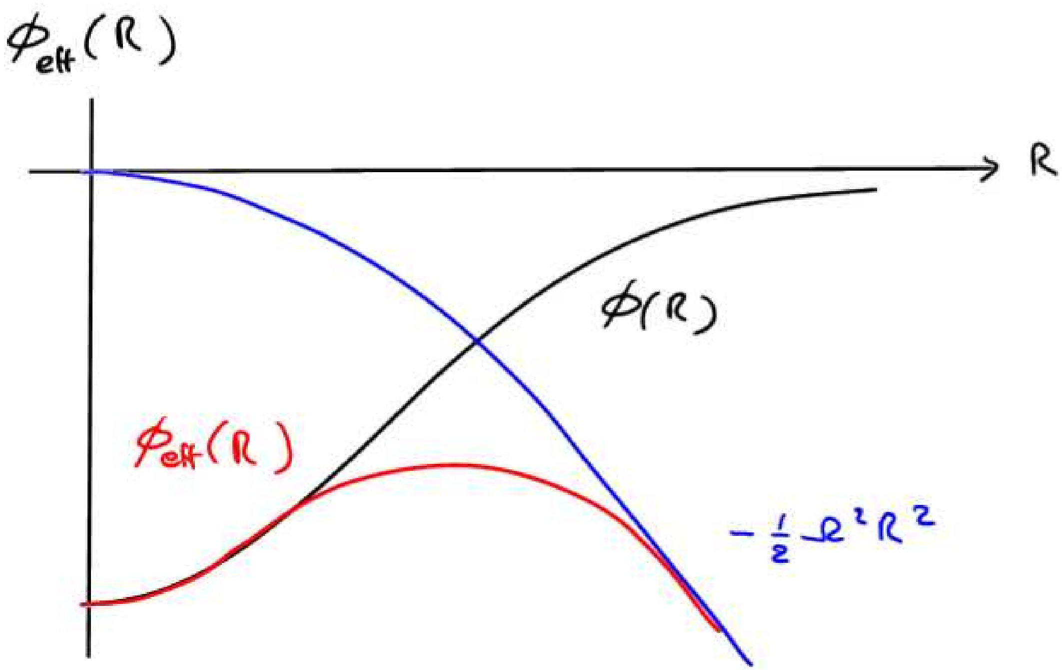
"gravity" counter balance the centrifugal force

Positions of the stationary points

1) assume $y = 0$ $\quad \vec{\Omega}_b = \begin{pmatrix} 0 \\ 0 \\ \Omega \end{pmatrix}$

2) assume an axi-symetric system

$$\phi_{eff}(\vec{x}) \rightarrow \quad \phi_{eff}(R) = \phi(R) - \frac{1}{2}\Omega^2 R^2$$

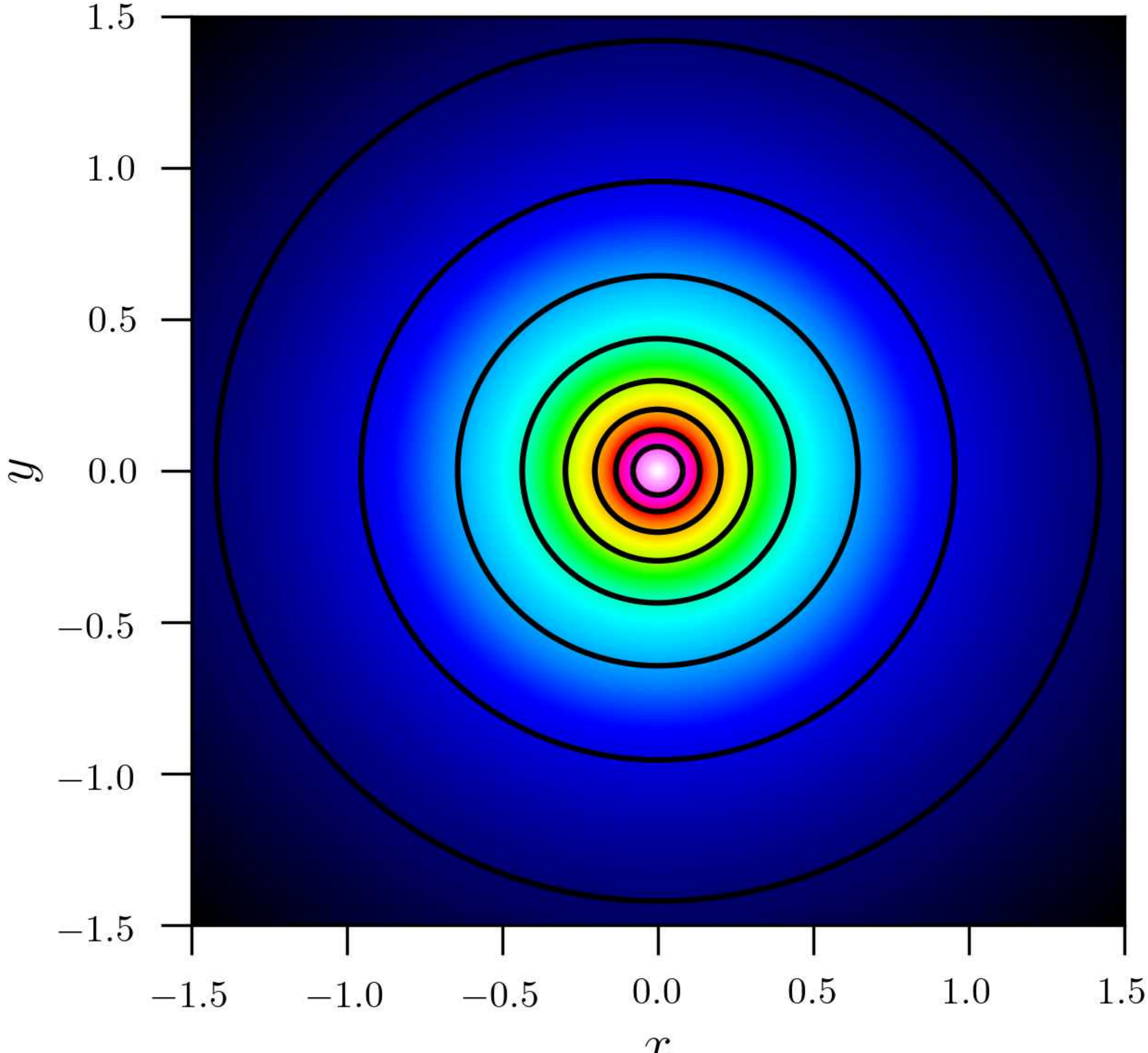


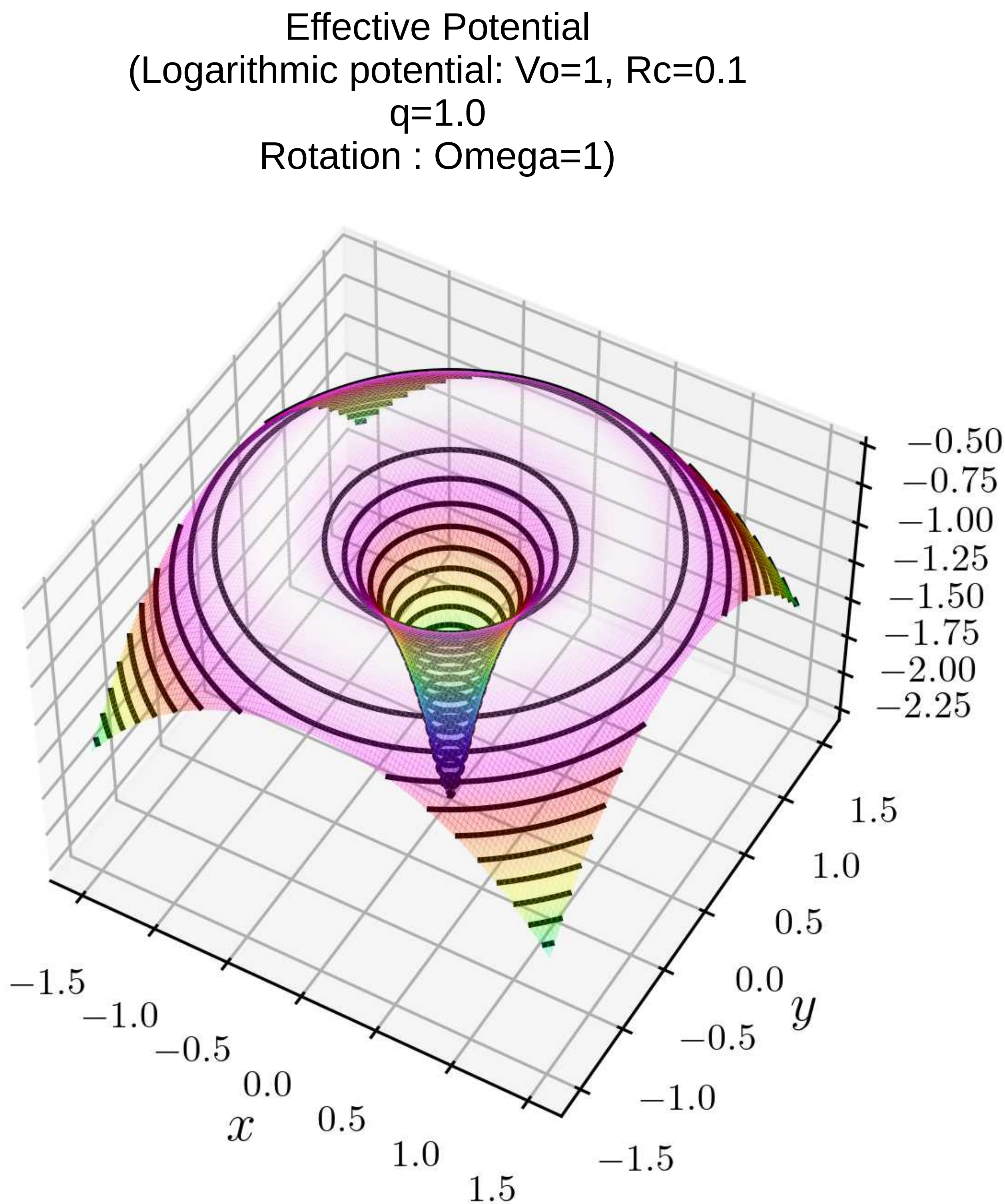
$$\vec{\nabla}\phi_{eff}(\vec{x}) = 0 \quad \equiv \quad \frac{\partial}{\partial R}\phi_{eff}(R) = 0$$

$$\frac{\partial}{\partial R}\phi(R) = \Omega^2 R$$

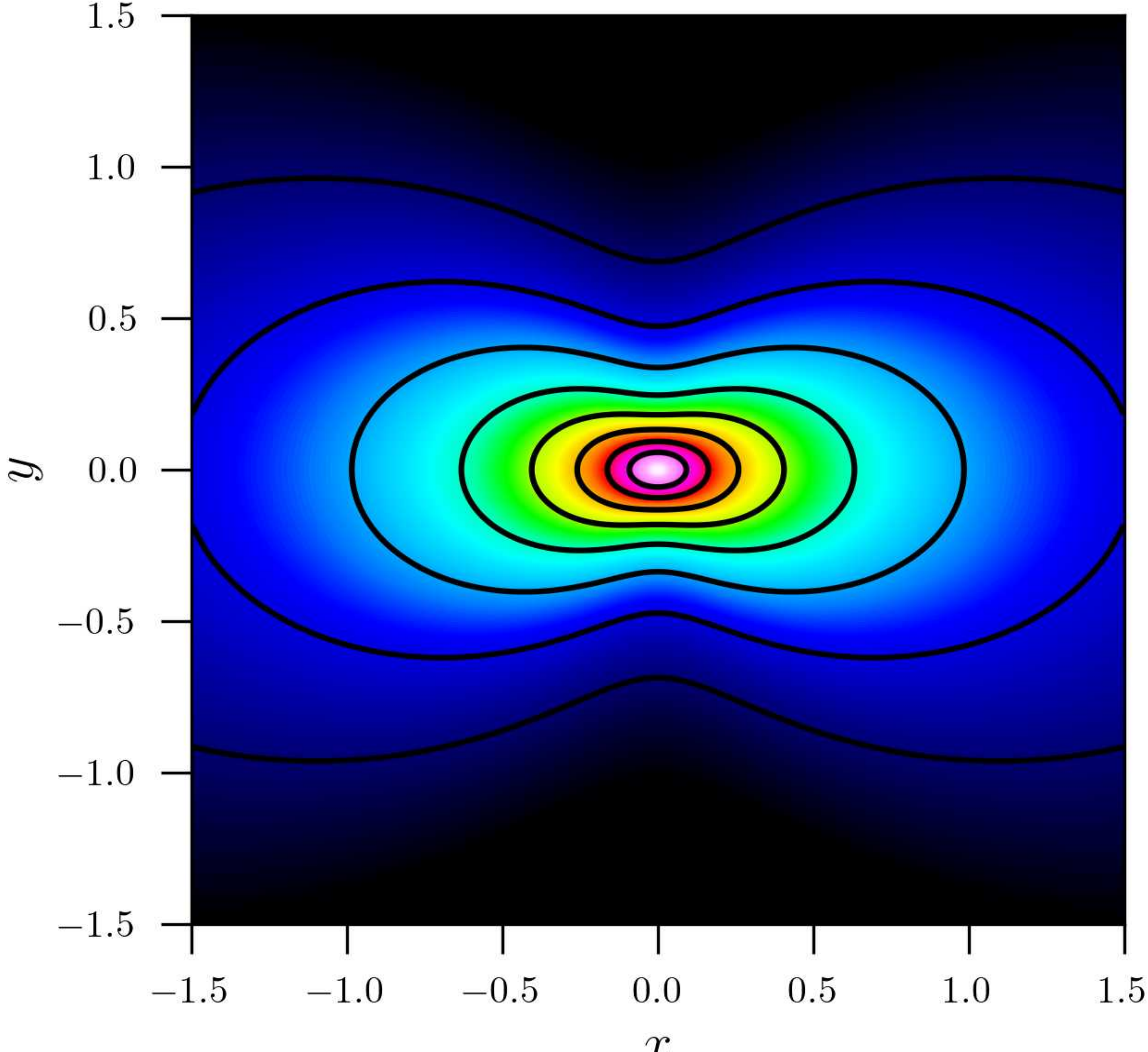
$\Rightarrow$ selection of a circular orbit in the inertial frame

Bar potential (density)
(Logarithmic potential: Vo=1, Rc=0.1
q=1.0)

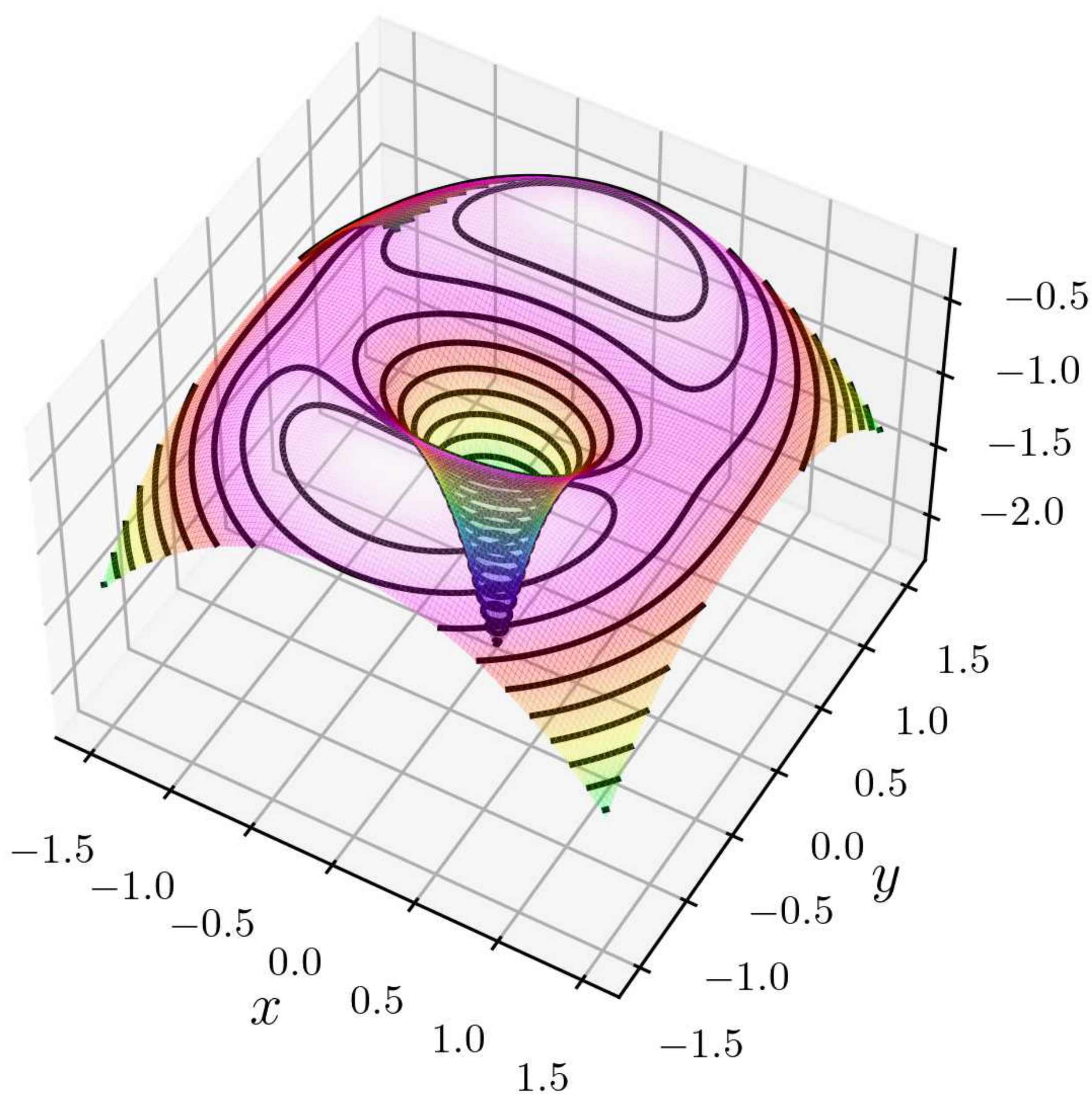


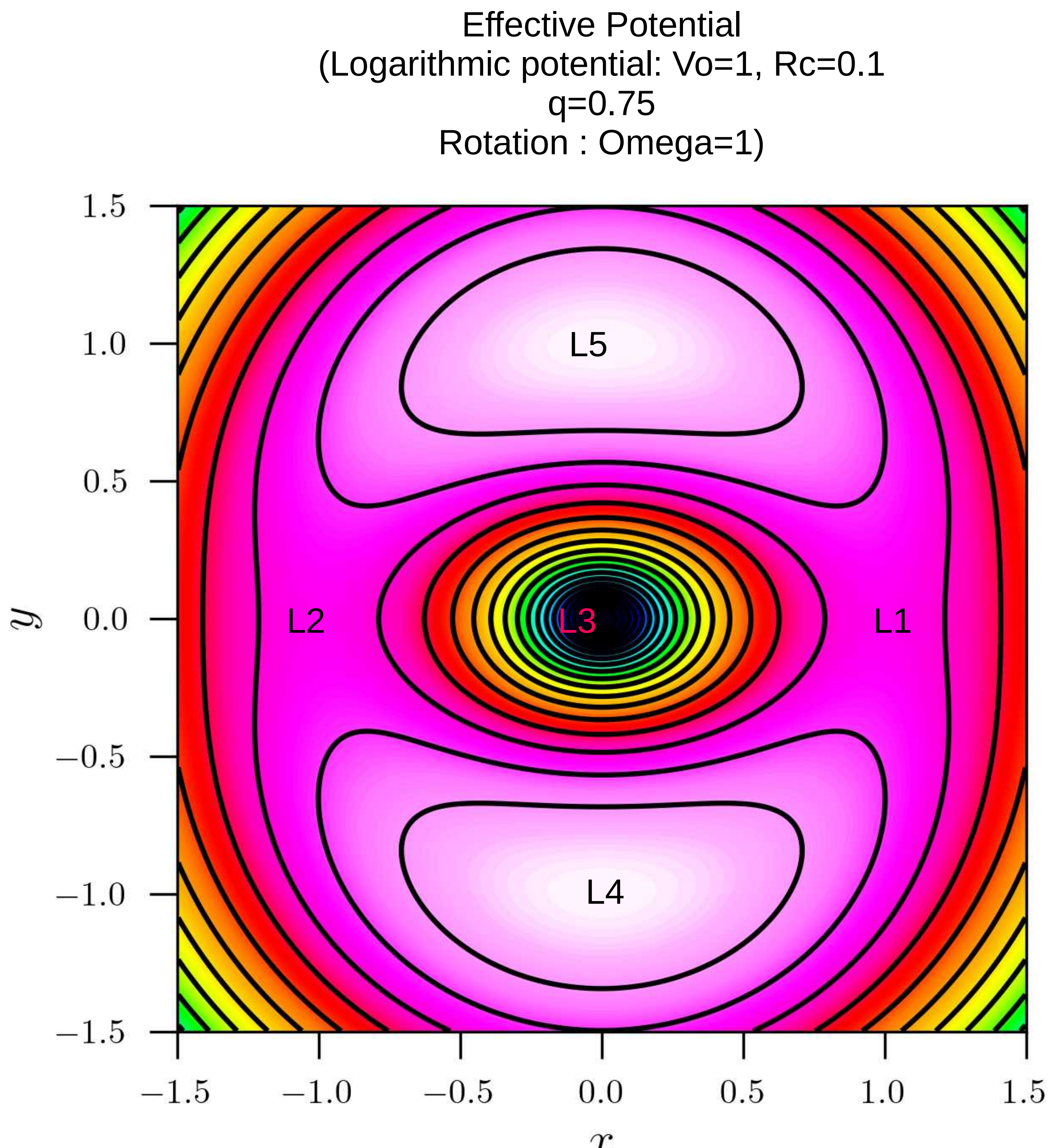

Effective Potential
(Logarithmic potential: Vo=1, Rc=0.1
q=1.0
Rotation : Omega=1)
−0.50
−0.75
−1.00
−1.25
−1.50
−1.75
−2.00
−2.25
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
y
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5
x

Bar potential
(Logarithmic potential: Vo=1, Rc=0.1
q=0.75)



Effective Potential
(Logarithmic potential: Vo=1, Rc=0.1
q=0.75
Rotation : Omega=1)




Effective Potential
(Logarithmic potential: Vo=1, Rc=0.1
q=0.75
Rotation : Omega=1)
L5
L2
L3
L1
L4
y
x
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5

Orbits around Lagrange points

Stability of orbits around Lagrange points

Expand the effective potential in Taylor serie around the Lagrange points (x_L, y_L)

$$\phi_{eff}(x,y) \cong \phi_{eff}(x_L, y_L) + \underbrace{\cancel{\frac{\partial \phi_{eff}}{\partial x}(x - x_L)}}_{=0} + \underbrace{\cancel{\frac{\partial \phi_{eff}}{\partial y}(y - y_L)}}_{=0}$$

$$+ \frac{1}{2}\frac{\partial^2 \phi_{eff}}{\partial x^2}(x - x_L)^2 + \frac{1}{2}\frac{\partial^2 \phi_{eff}}{\partial y^2}(y - y_L)^2 + \underbrace{\cancel{\frac{1}{2}\frac{\partial^2 \phi_{eff}}{\partial x \partial y}}}_{=0}(x - x_L)(y - y_L)$$

by symmetry of the bar, if it is aligned with $\bar{x}$

Now we define

$$\xi := x - x_L \qquad \phi_{xx} := \frac{\partial^2 \phi_{eff}}{\partial x^2}$$

$$\eta := y - y_L \qquad \phi_{yy} := \frac{\partial^2 \phi_{eff}}{\partial y^2}$$

$$\phi_{eff}(\xi, \eta) = \phi_{eff}(0,0) + \frac{1}{2}\phi_{xx}\xi^2 + \frac{1}{2}\phi_{yy}\eta^2$$

Equations of motions $\qquad \ddot{\vec{x}} = -\vec{\nabla}\phi_{\text{eff}} - 2(\vec{\Omega} \times \dot{\vec{x}})$

in the plane $z = 0$ assuming $\vec{\Omega} = \begin{pmatrix} 0 \\ 0 \\ \Omega \end{pmatrix}$

$$\begin{cases} \ddot{x} = -\dfrac{\partial \phi_{\text{eff}}}{\partial x} + 2\Omega \dot{y} \\[2ex] \ddot{y} = -\dfrac{\partial \phi_{\text{eff}}}{\partial y} - 2\Omega \dot{x} \end{cases}$$

$$\boxed{\begin{cases} \ddot{\xi} = +2\Omega\dot{\eta} - \phi_{xx}\xi \\[1ex] \ddot{\eta} = -2\Omega\dot{\xi} - \phi_{yy}\eta \end{cases}}$$

We assume solutions of the form

$$\begin{cases} \xi(t) = X e^{\lambda t} \\ \eta(t) = Y e^{\lambda t} \end{cases} \qquad X, Y, \lambda \in \mathbb{C}$$

The EoM become

$$\begin{cases} (\lambda^2 + \phi_{xx})\, X - (2\lambda\Omega)\, Y = 0 \\ (2\lambda\Omega)\, X + (\lambda^2 + \phi_{yy})\, Y = 0 \end{cases}$$

$$\underbrace{\begin{pmatrix} \lambda^2 + \phi_{xx} & -2\lambda\Omega \\ 2\lambda\Omega & \lambda^2 + \phi_{yy} \end{pmatrix}}_{M} \begin{pmatrix} X \\ Y \end{pmatrix} = 0$$

simple linear equation

Non trivial solutions (i.e $X \neq 0$, $Y \neq 0$) only if $\underline{\mathrm{Det}(M) = 0}$

$$\mathrm{Det}\, M = \boxed{\lambda^4 + \lambda^2 (\phi_{xx} + \phi_{yy} + 4\Omega^2) + \phi_{xx}\phi_{yy} = 0}$$

"characteristic equation"

Solutions (4 roots, two are coupled)

- if λ is a solution $\Rightarrow$ $-\lambda$ is a solution
- if λ is real

$$\begin{cases} \xi(t) = X e^{\lambda t} & \rightarrow \text{exponential growth} \\ \eta(t) = Y e^{\lambda t} & \rightarrow \text{exponential growth} \end{cases}$$

$\rightarrow$ the star leaves the Lagrange point

$$\boxed{\text{UNSTABLE}}$$

- if all λ are purely complex $\quad \lambda_1 = \alpha i \quad \lambda_2 = -\alpha i \quad \lambda_3 = \beta i \quad \lambda_4 = -\beta i \quad \alpha, \beta \in \mathbb{R}$

$$\begin{aligned} \xi(t) &= \mathrm{Re}\left(X_1' e^{i\alpha t} + X_2' e^{-i\alpha t} + X_3' e^{i\beta t} + X_4' e^{-i\beta t} \right) \\ &= X_1' \cos(\alpha t) + X_2' \cos(-\alpha t) + X_3' \cos(\beta t) + X_4' \cos(-\beta t) \\ &= X_1 \cos(\alpha t) + X_2 \cos(\beta t) \end{aligned}$$

idem for $\eta(t)$, so we get

$$\begin{cases} \xi(t) = X_1 \cos(\alpha t) + X_2 \cos(\beta t) \\ \eta(t) = Y_1 \cos(\alpha t) + Y_2 \cos(\beta t) \end{cases}$$

STABLE

with

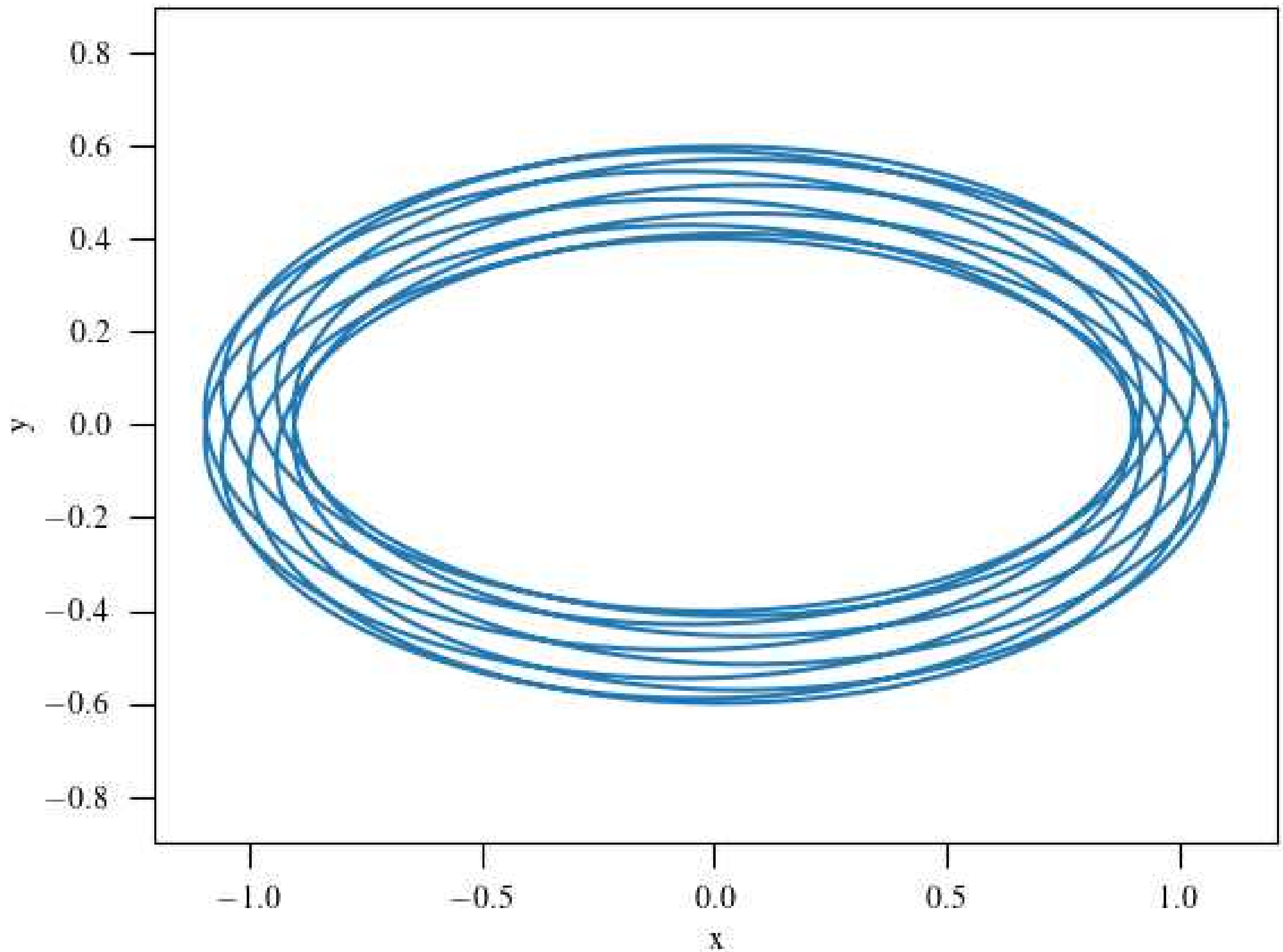
$$\begin{cases} Y_1 = \dfrac{\phi_{xx} - \alpha^2}{2 \Omega \alpha} X_1 = \dfrac{2 \Omega \alpha}{\phi_{yy} - \alpha^2} X_1 \\[2ex] Y_2 = \dfrac{\phi_{xx} - \beta^2}{2 \Omega \beta} X_1 = \dfrac{2 \Omega \beta}{\phi_{yy} - \beta^2} X_2 \end{cases}$$

It is possible to demonstrate that :

- At L_3 i.e $\min(\phi_{eff})$ — always stable
- At L_2, L_3 i.e the saddles points — always unstable
- At L_4, L_5 i.e $\max(\phi_{eff})$ — stable or unstable

depends on the detail of the potential

Note : The stability comes from the Coriolis force (see Padmanabhan)



lways stable

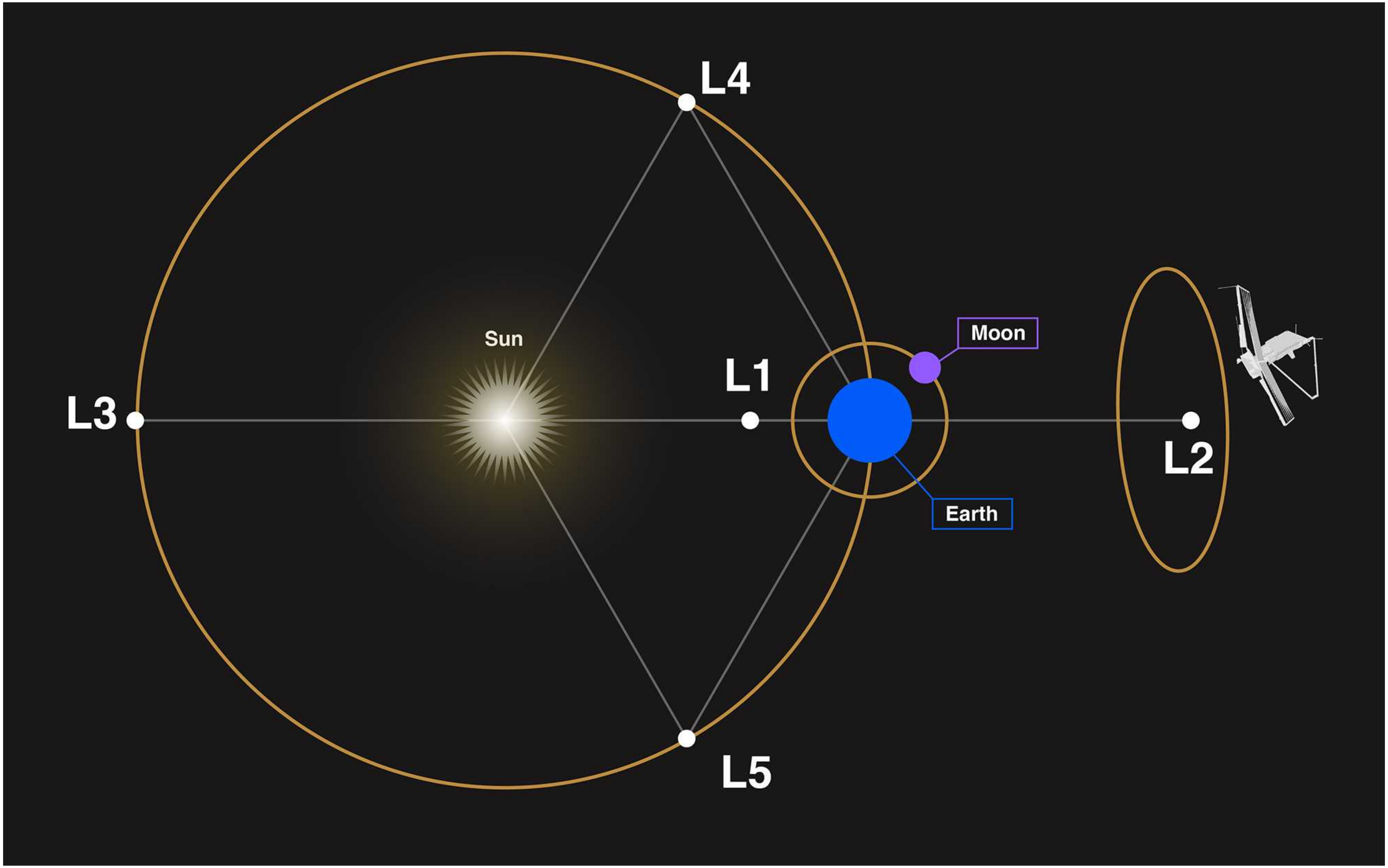
lways unstable

able or unstable

. the detail of
the potential

<u>Note</u>: The stability comes from
the Coriolis force (see Padmanabhan)

Lagrange points in celestial mechanics

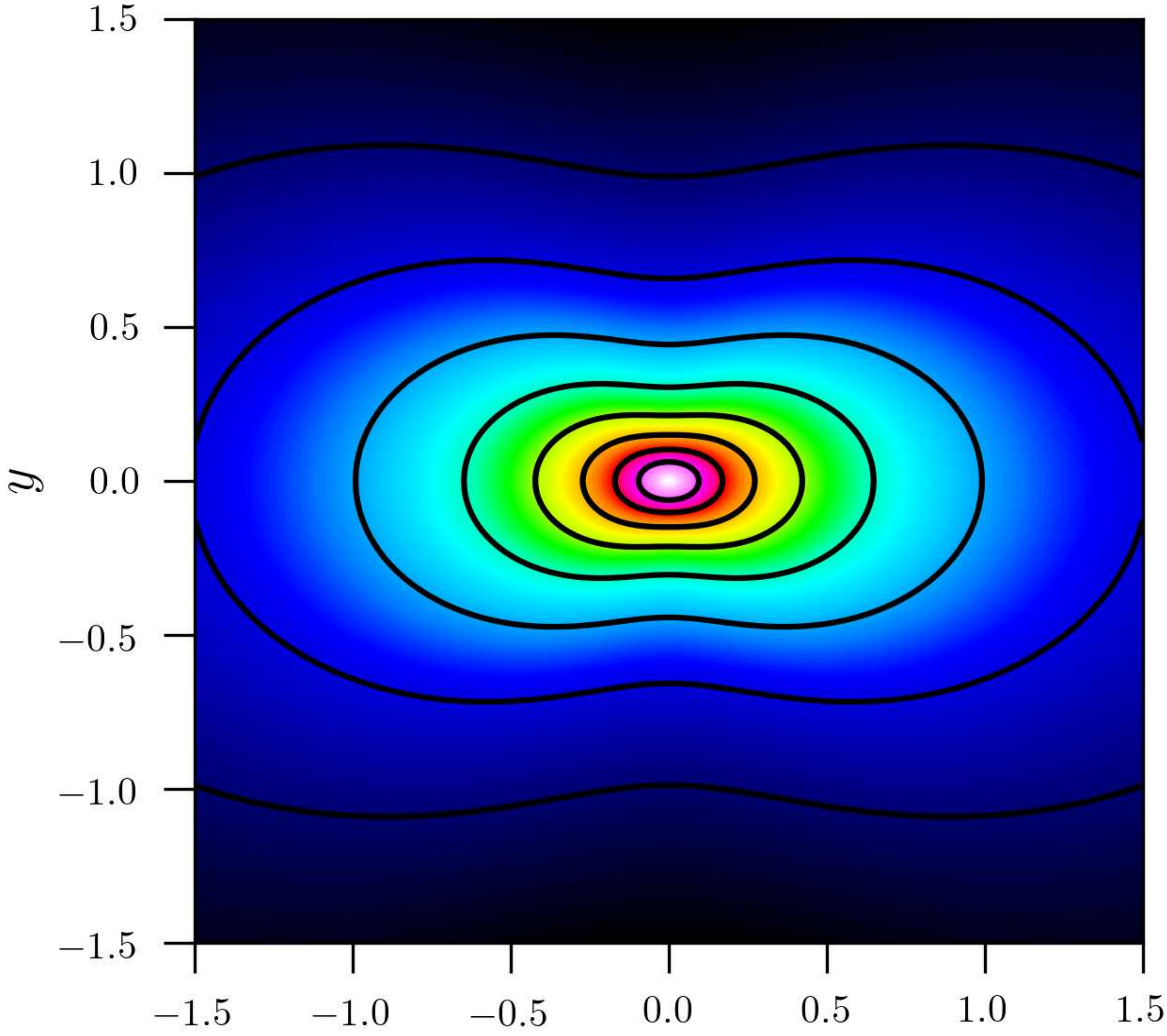


Stellar Orbits

Orbits not confined to Lagrange points

Bar model : Logarithmic potential:
Vo=01 Rc=0.1 q=0.8)

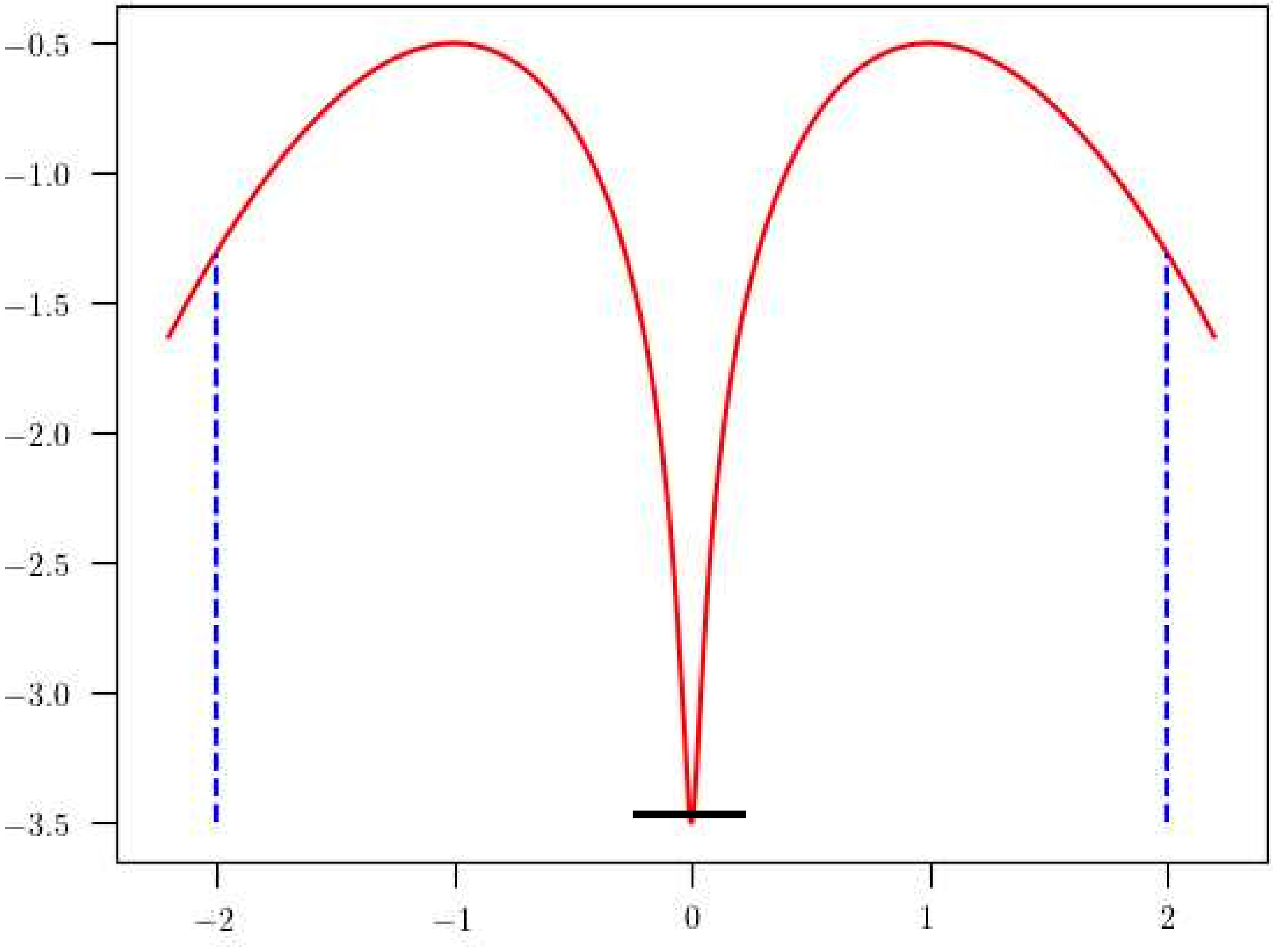
$$\Phi_{\log}(x,y) = \frac{1}{2}V_0^2 \ln\left(R_c^2 + x^2 + \left(\frac{y}{q}\right)^2\right) \qquad \Omega_{\mathrm{p}} \neq 0$$



Low energy orbits

$$R \ll R_{\mathrm{corot}}$$

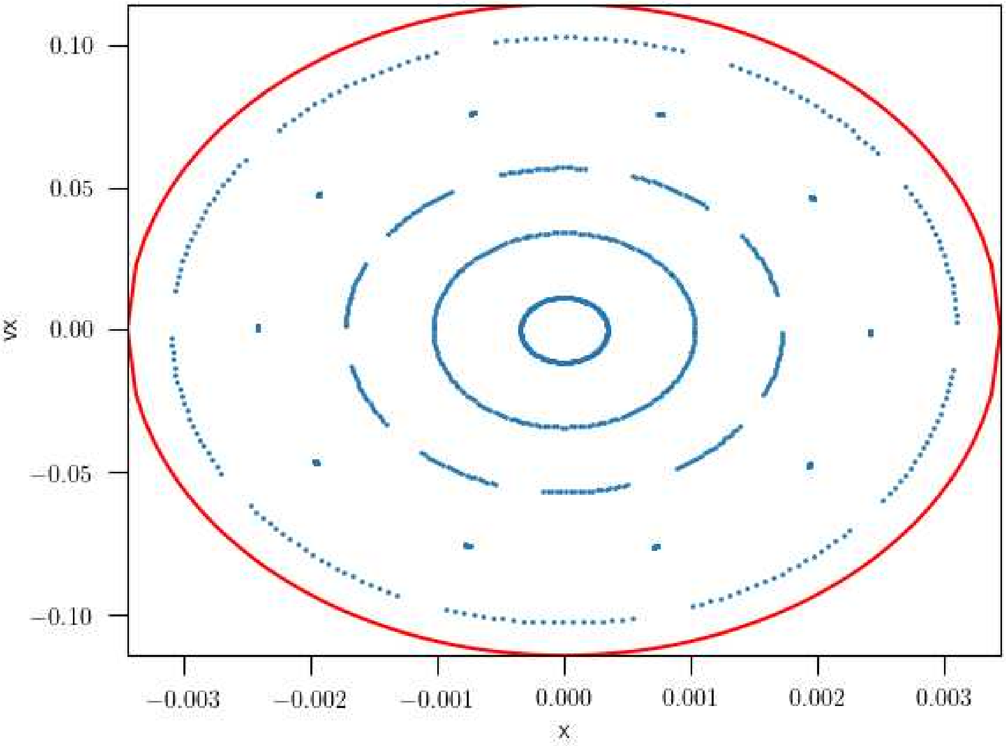
Potential and energy




```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 --plotpotential
```

Orbits around L_3

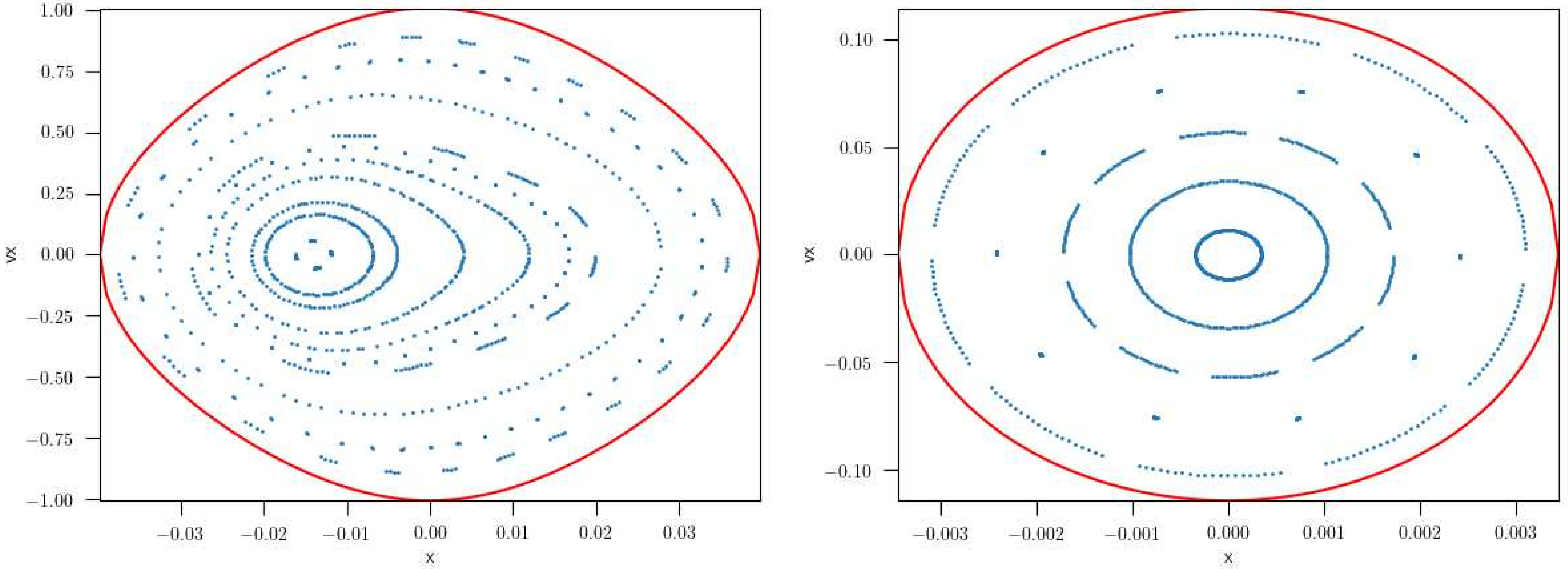
$$\Omega = 0$$




```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 0 -E -3.5
```

Orbits around L_3

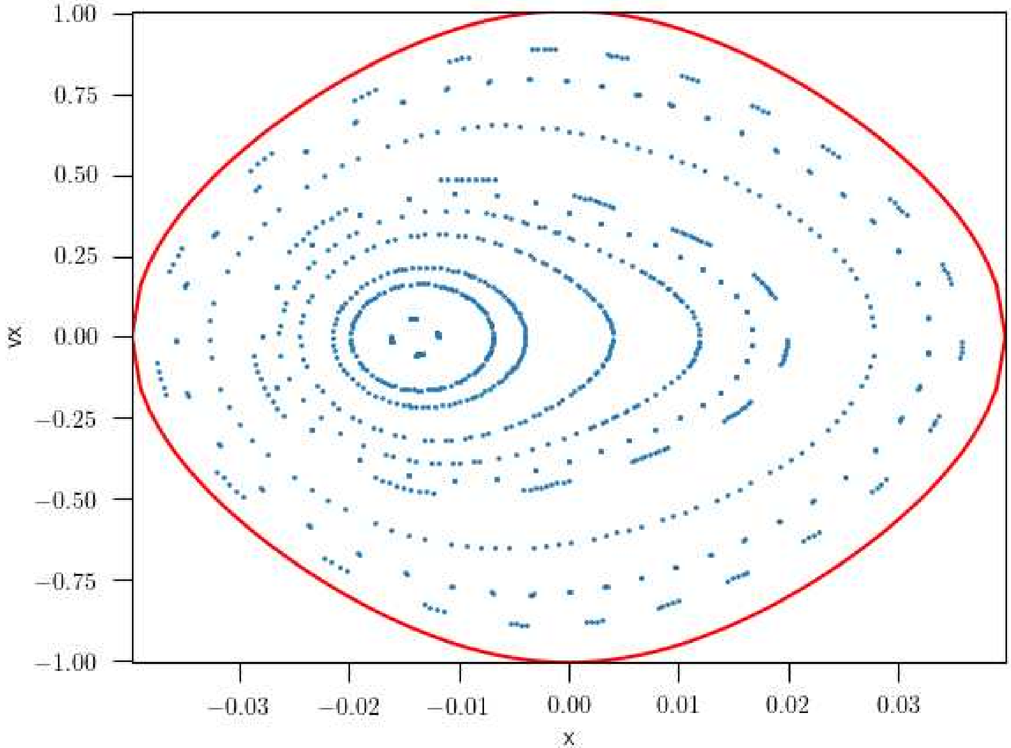
$\Omega = 1$ $\qquad$ $\Omega = 0$



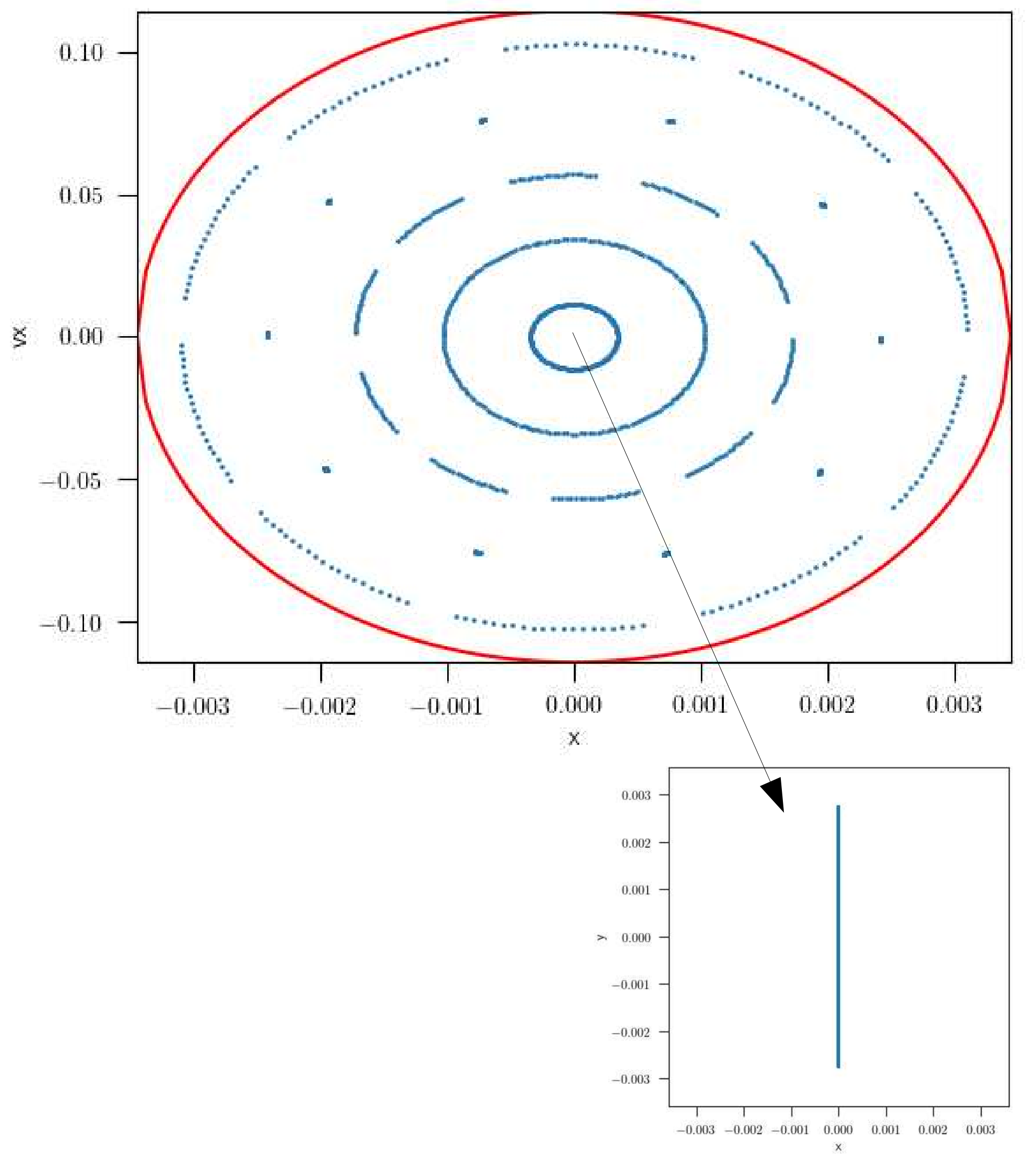

```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -3
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 0 -E -3.5
```

Orbits around L_3

$\Omega = 1$



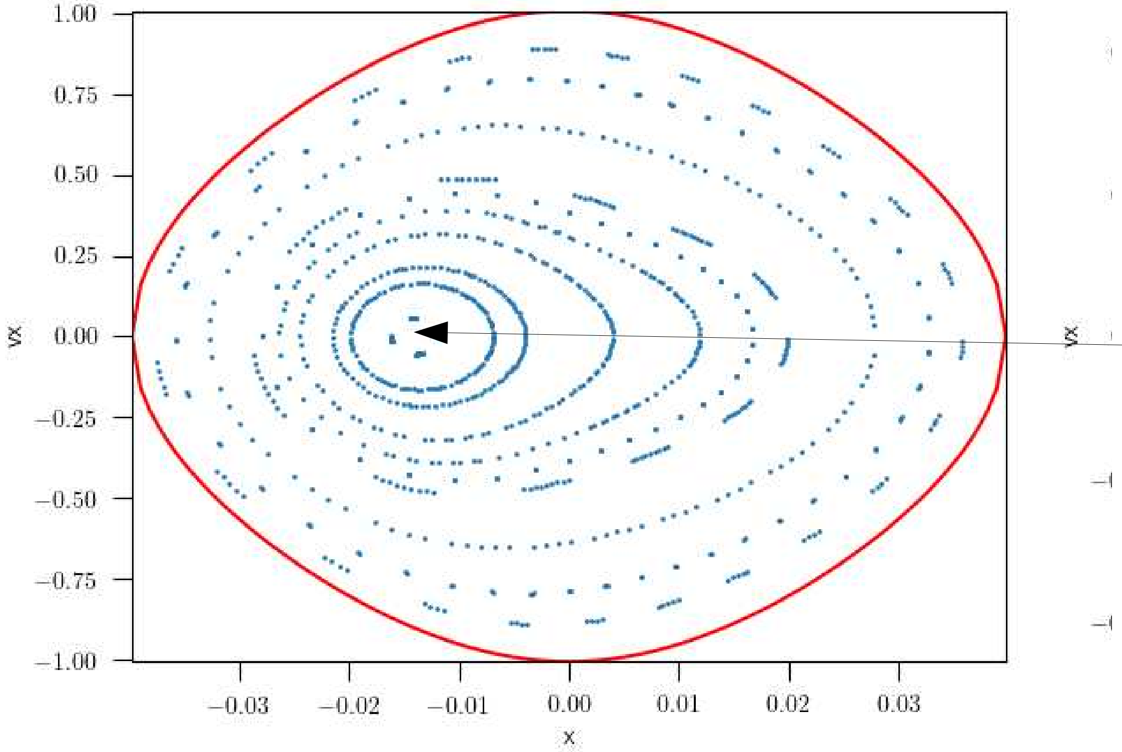
$\Omega = 0$

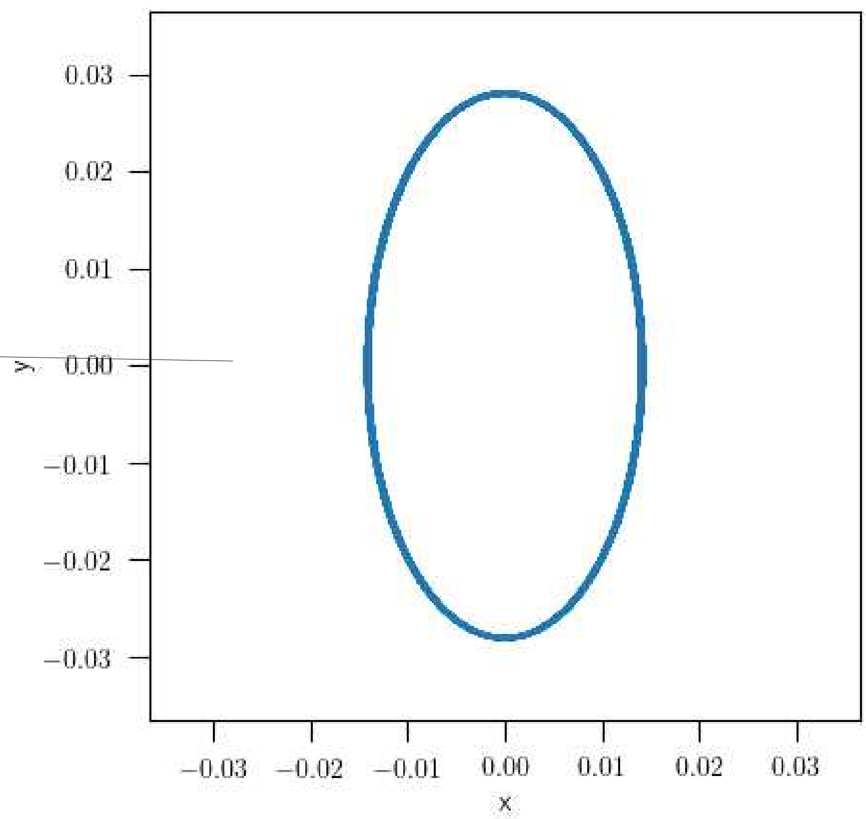



```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -3
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 0 -E -3.5
```

Short axis (Y) orbits (periodic)

$$\Omega = 1$$





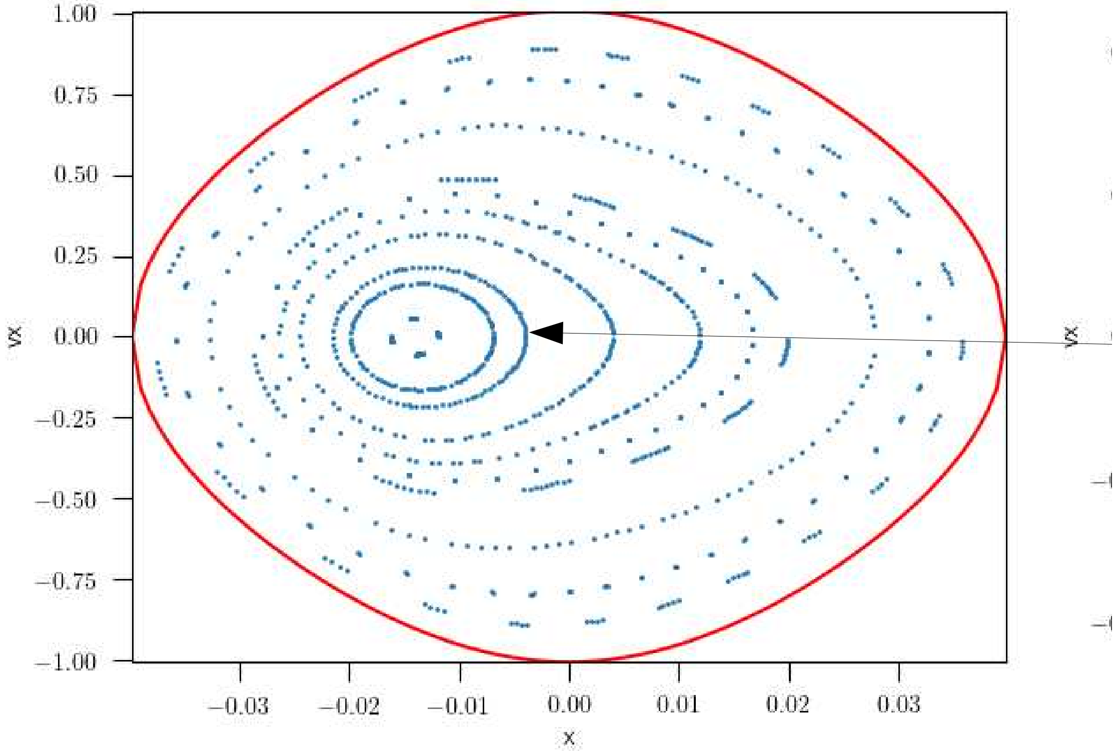
X4

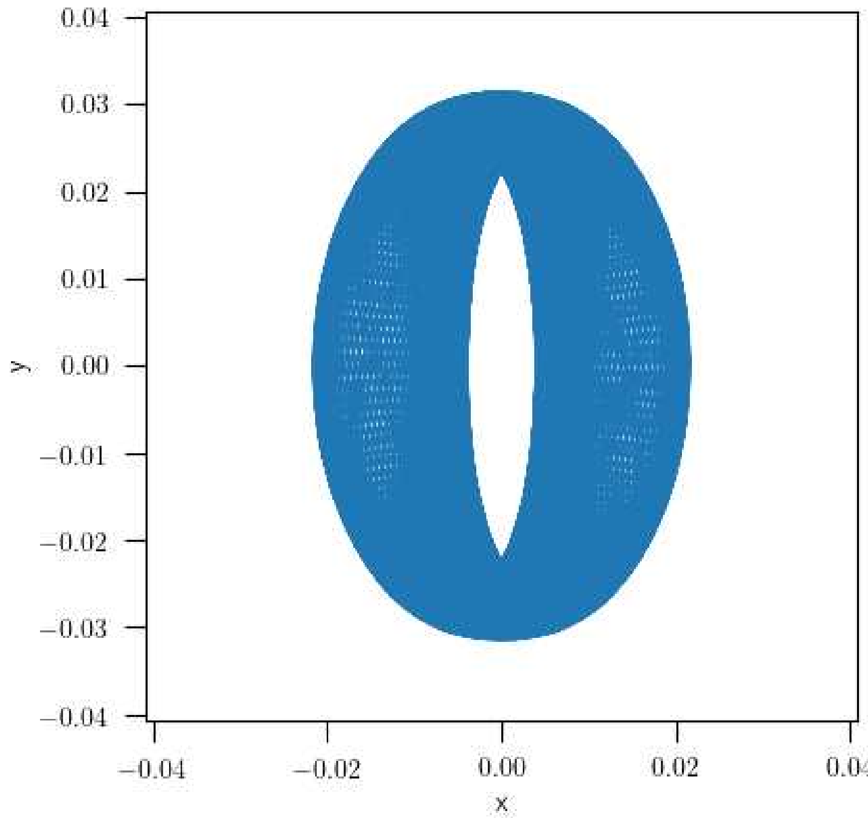
```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -3
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -3  --x -0.014
```

Apparition of a periodic loop orbit (replace the radial orbit, perpandicular to the bar), clockwise rotation

Short axis (Y) orbits (periodic)

$$\Omega = 1$$



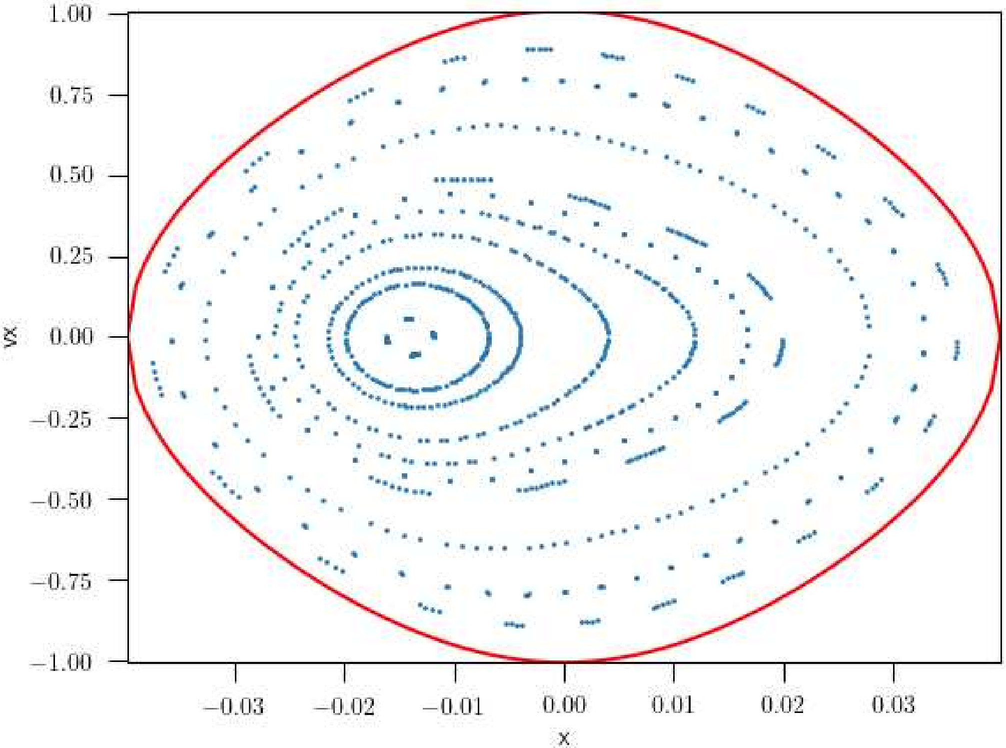


X4

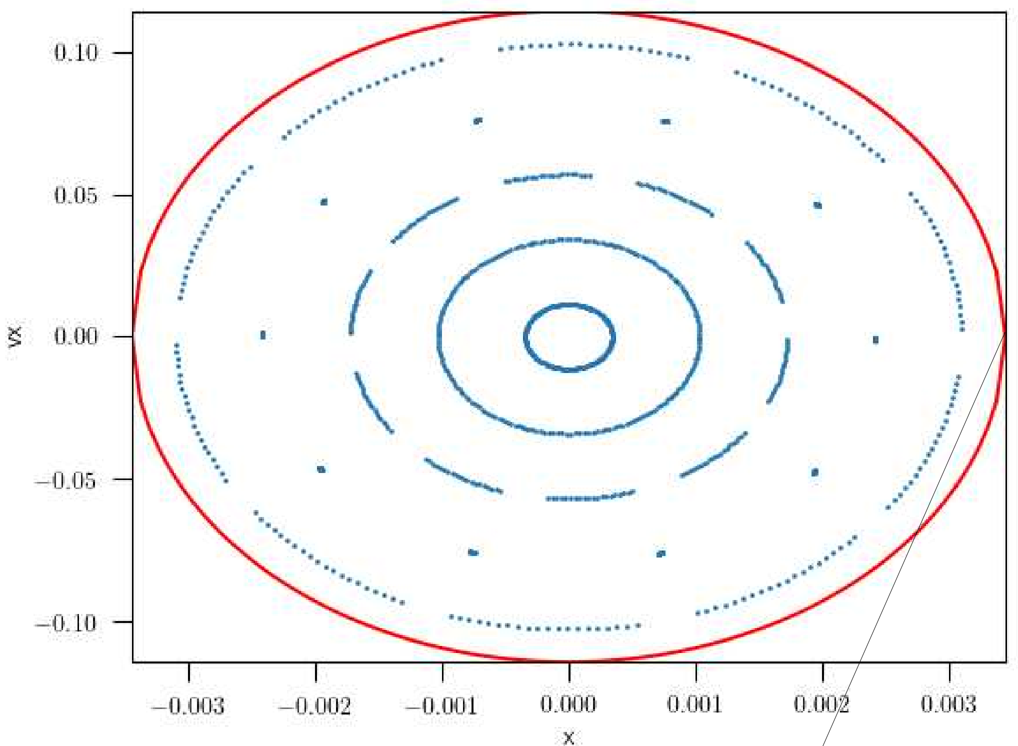
```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -3
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -3  --x -0.004
```

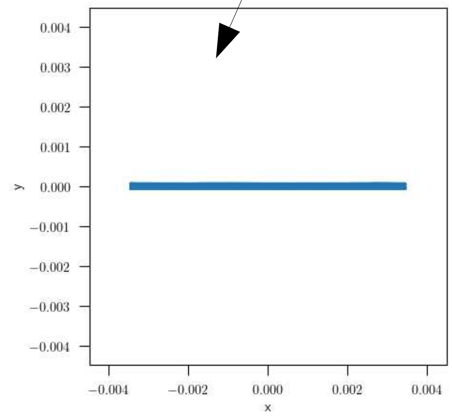
Orbits around L_3

$\Omega = 1$



$\Omega = 0$

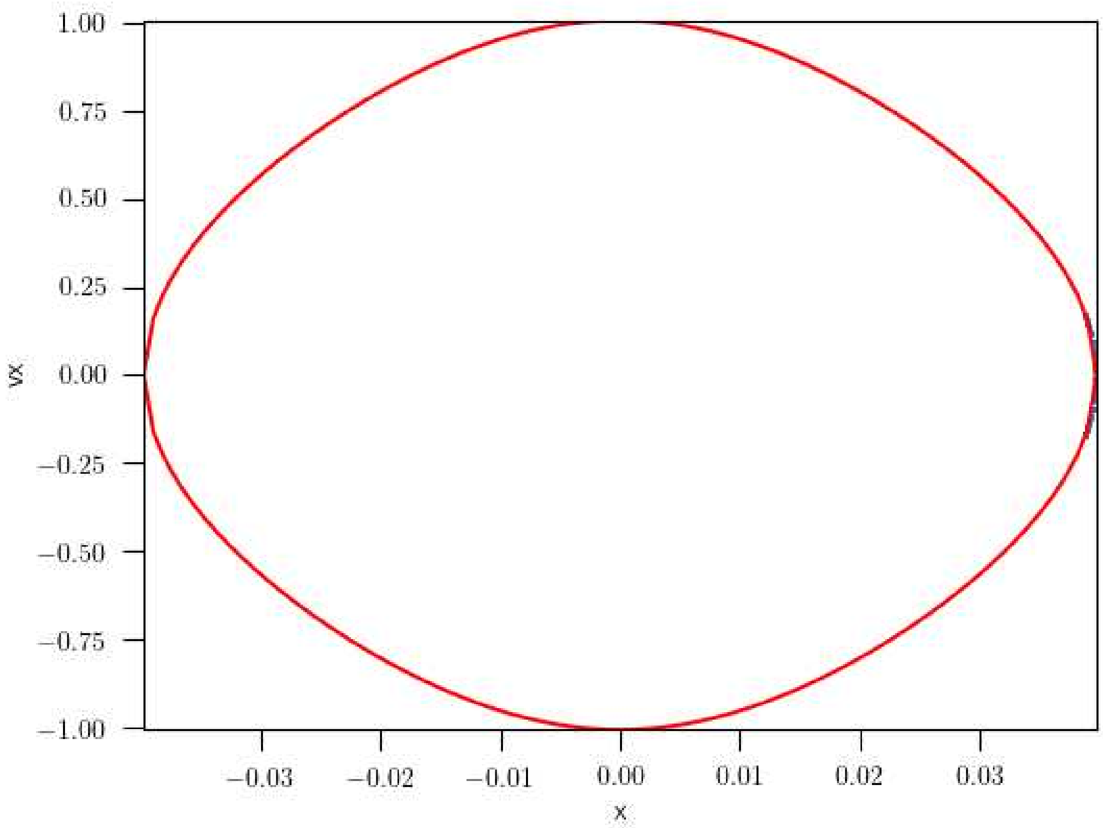


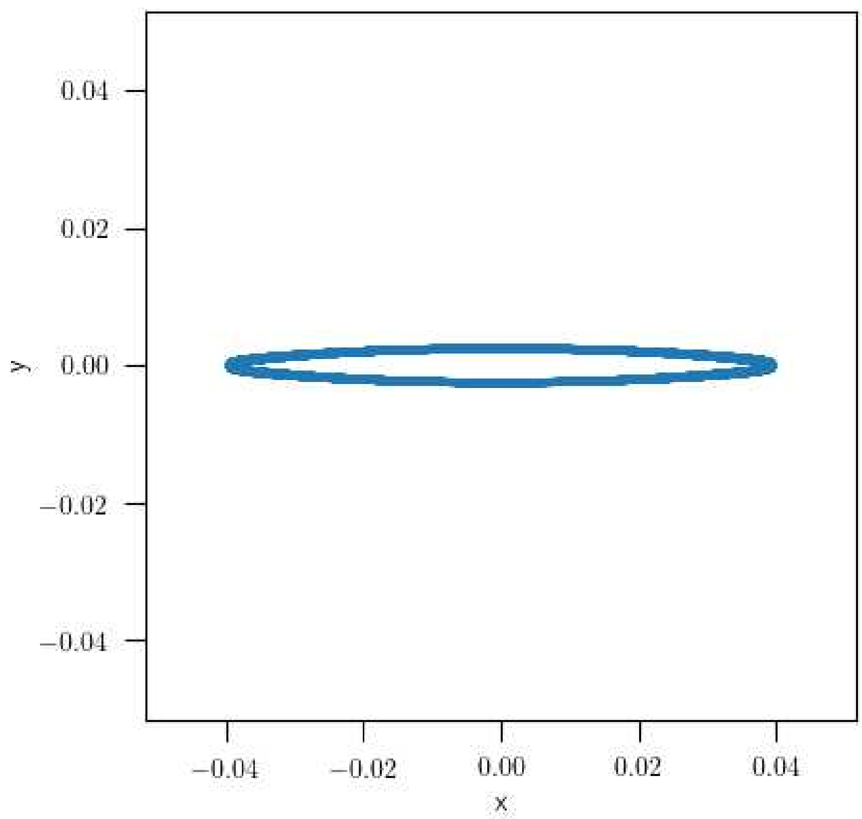



```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -3
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 0 -E -3.5
```

Long axis (X) orbits (periodic)

$$\Omega = 1$$





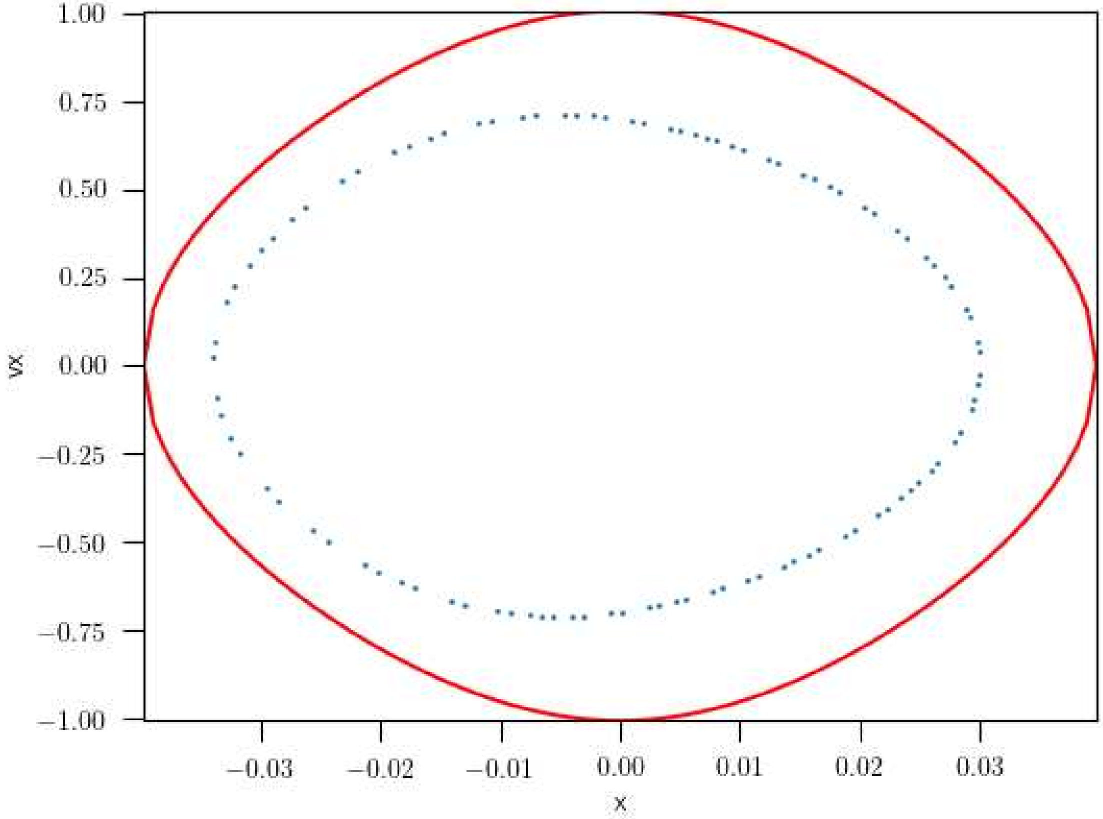
x1

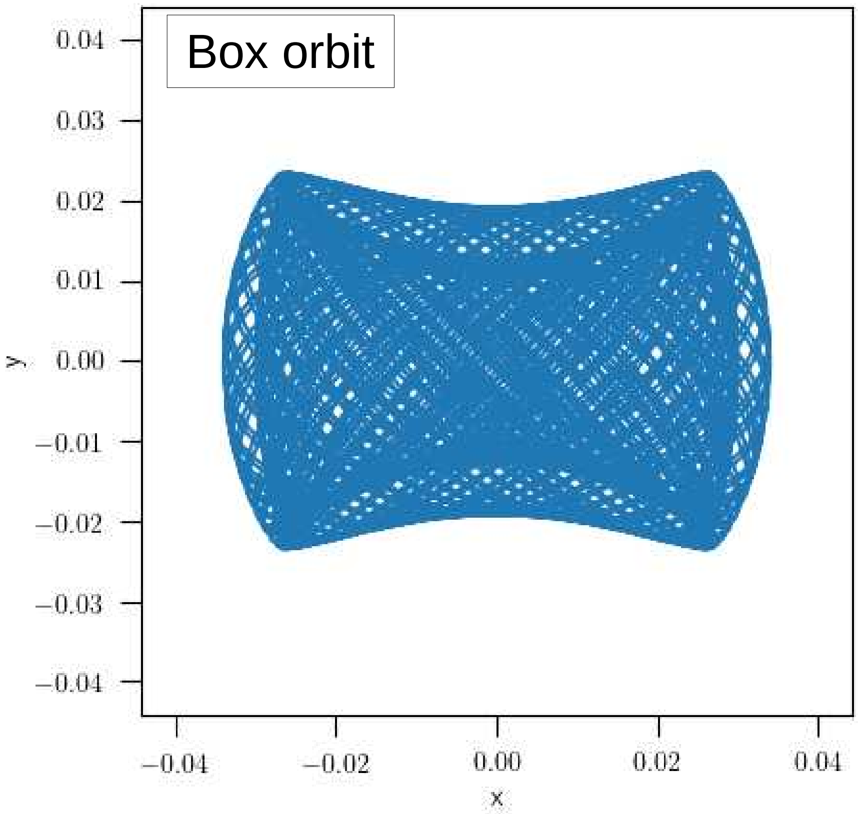
```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -3  --x 0.03975
```

Apparition of a periodic loop orbit (replace the radial orbit, parallel to the bar), anti-clockwise rotation

Long axis (X) orbits (non periodic)

$$\Omega = 1$$



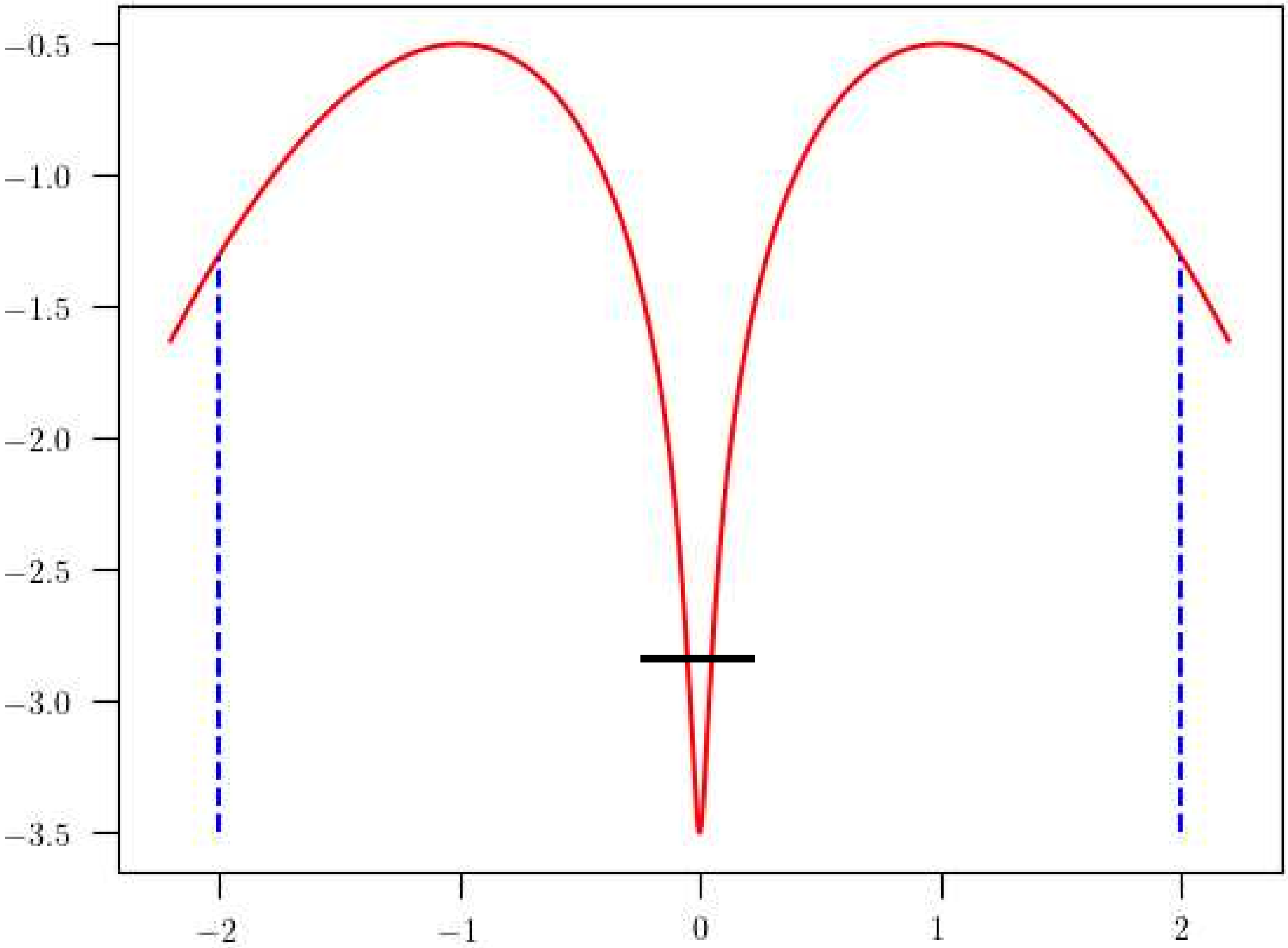


x1

```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -3  --x 0.03
```

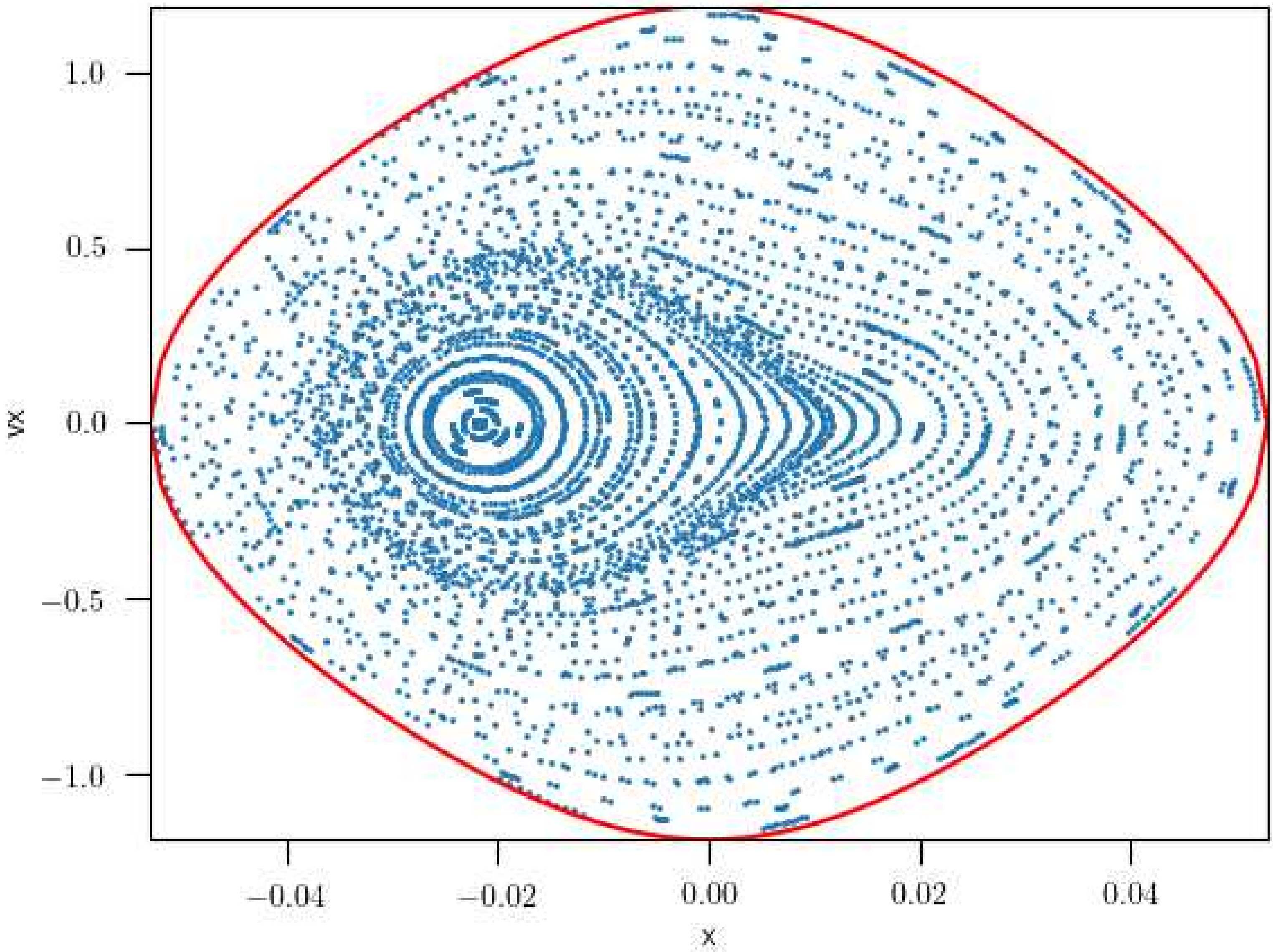
Increasing the energy

$$E = -2.8$$



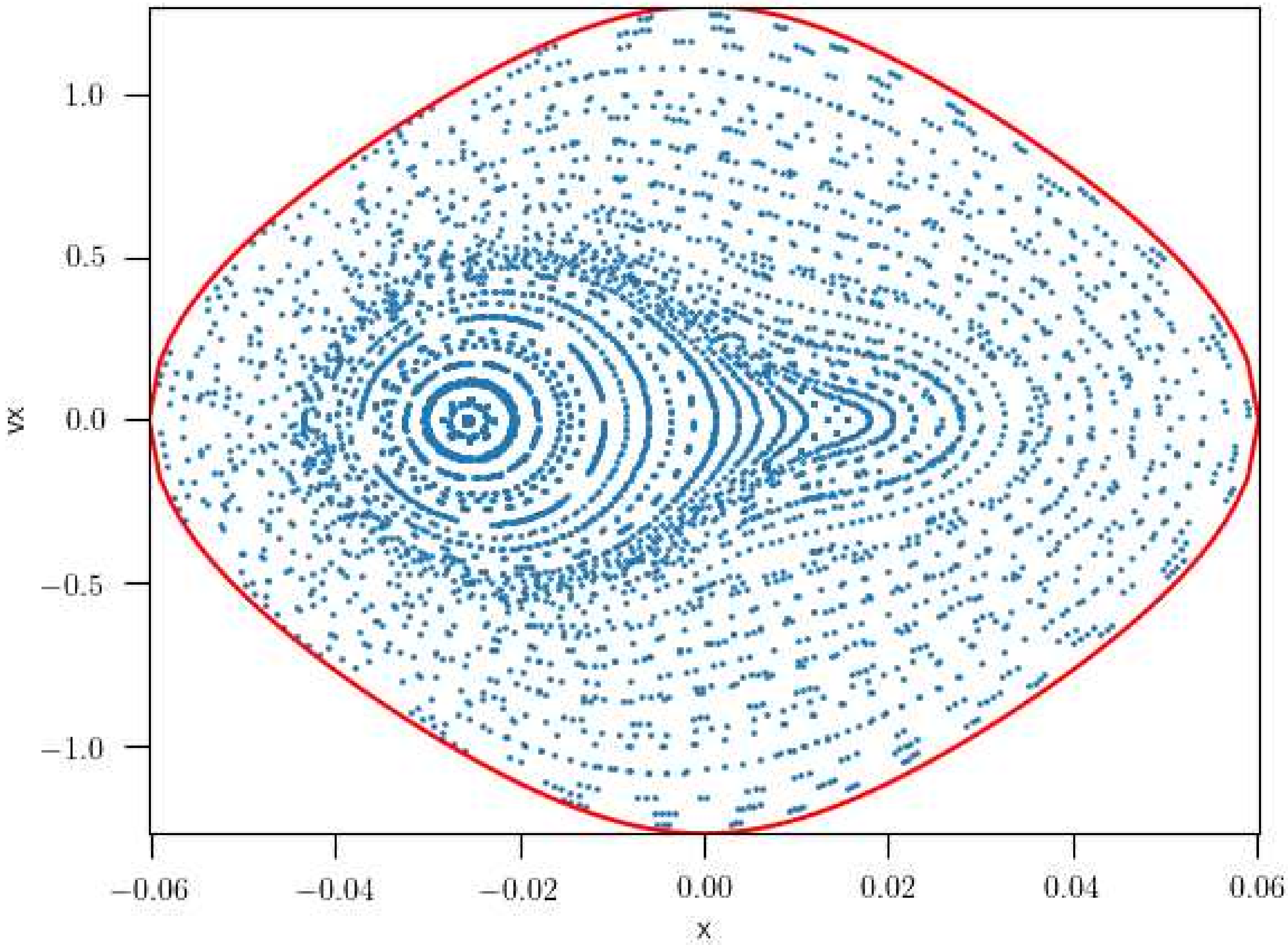

```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 --plotpotential
```

$$E = -2.8$$



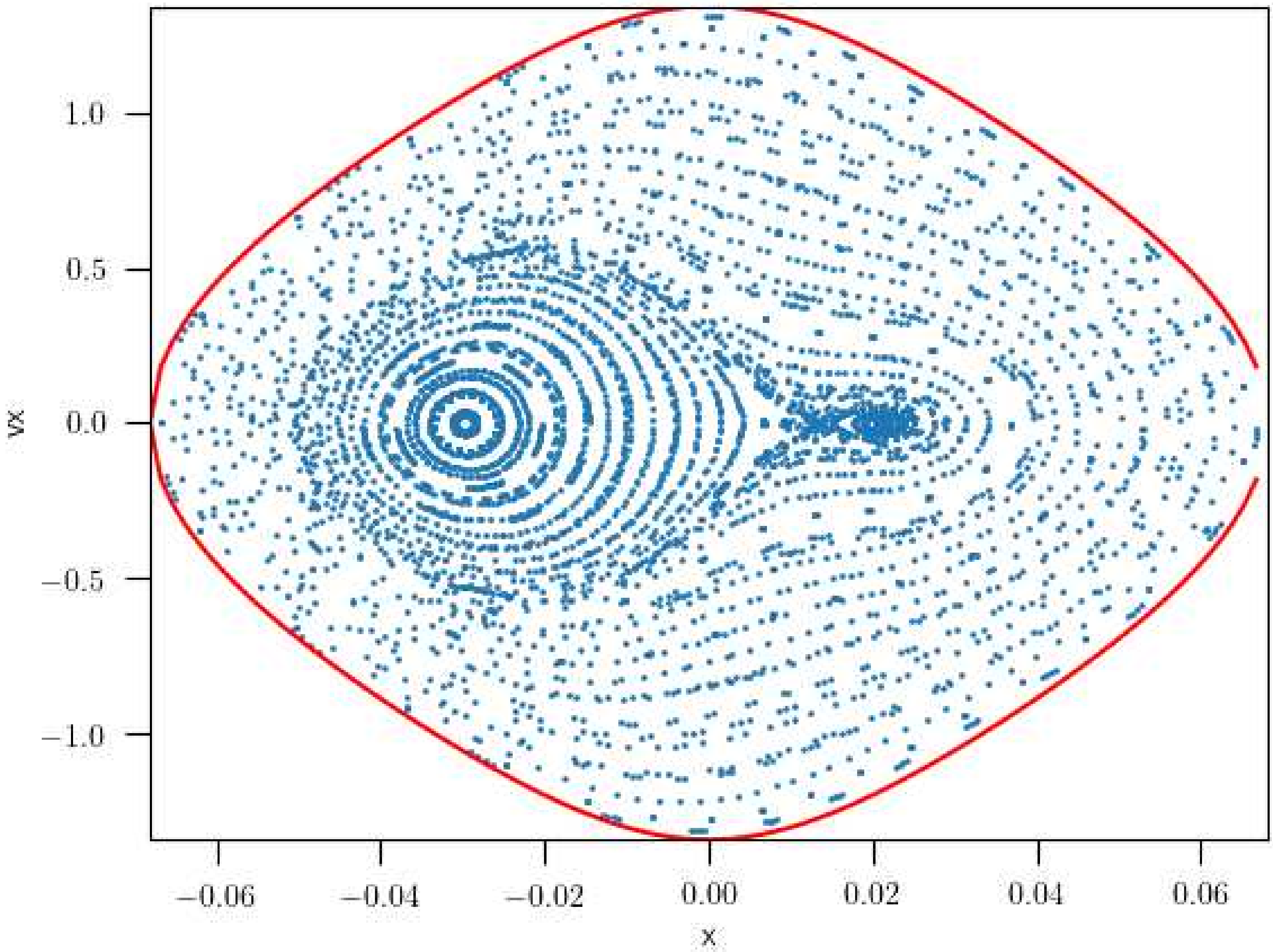

```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -2.8 --norbits 50
```

$$E = -2.7$$



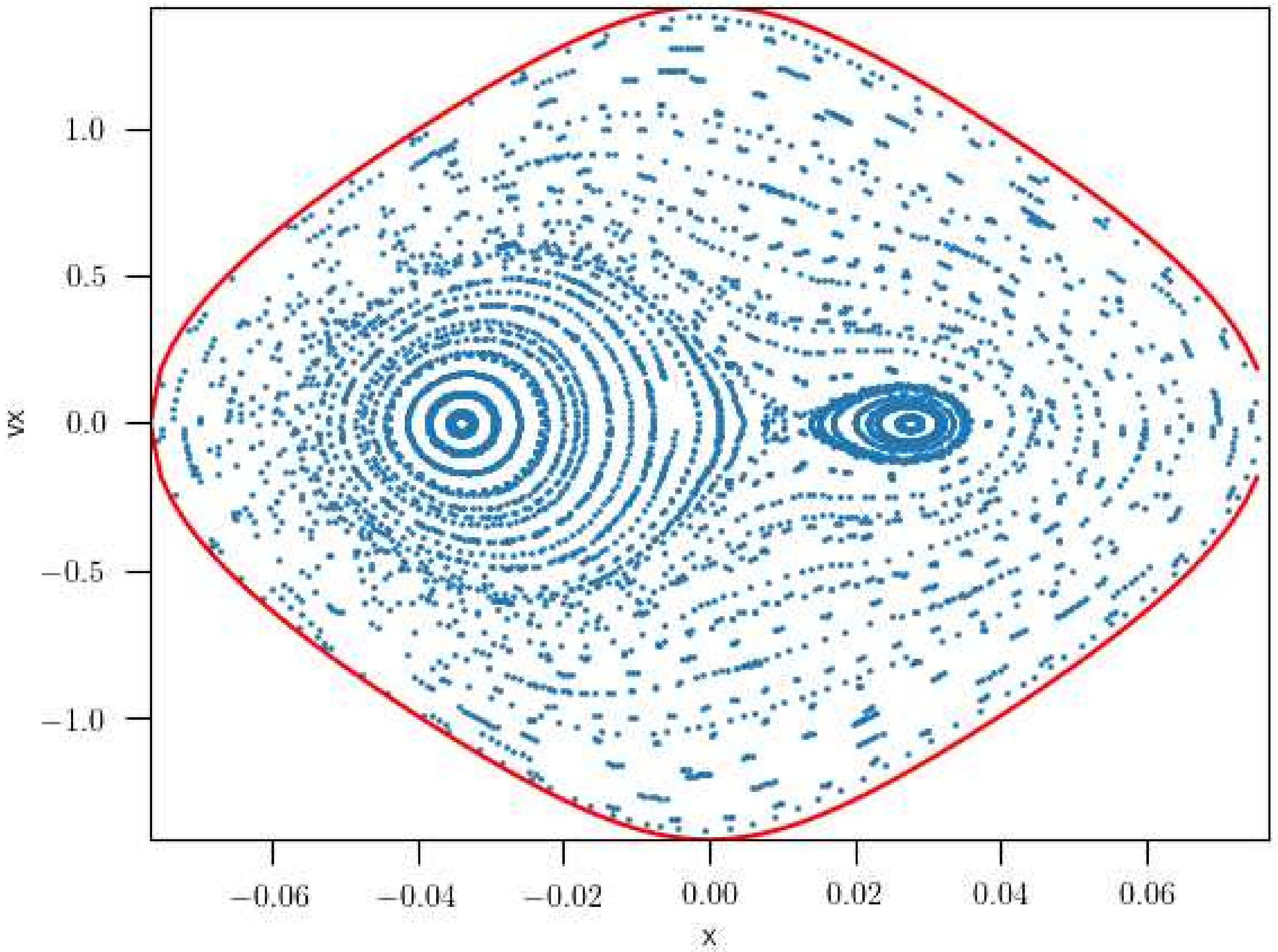

```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -2.7 --norbits 50
```

$$E = -2.6$$



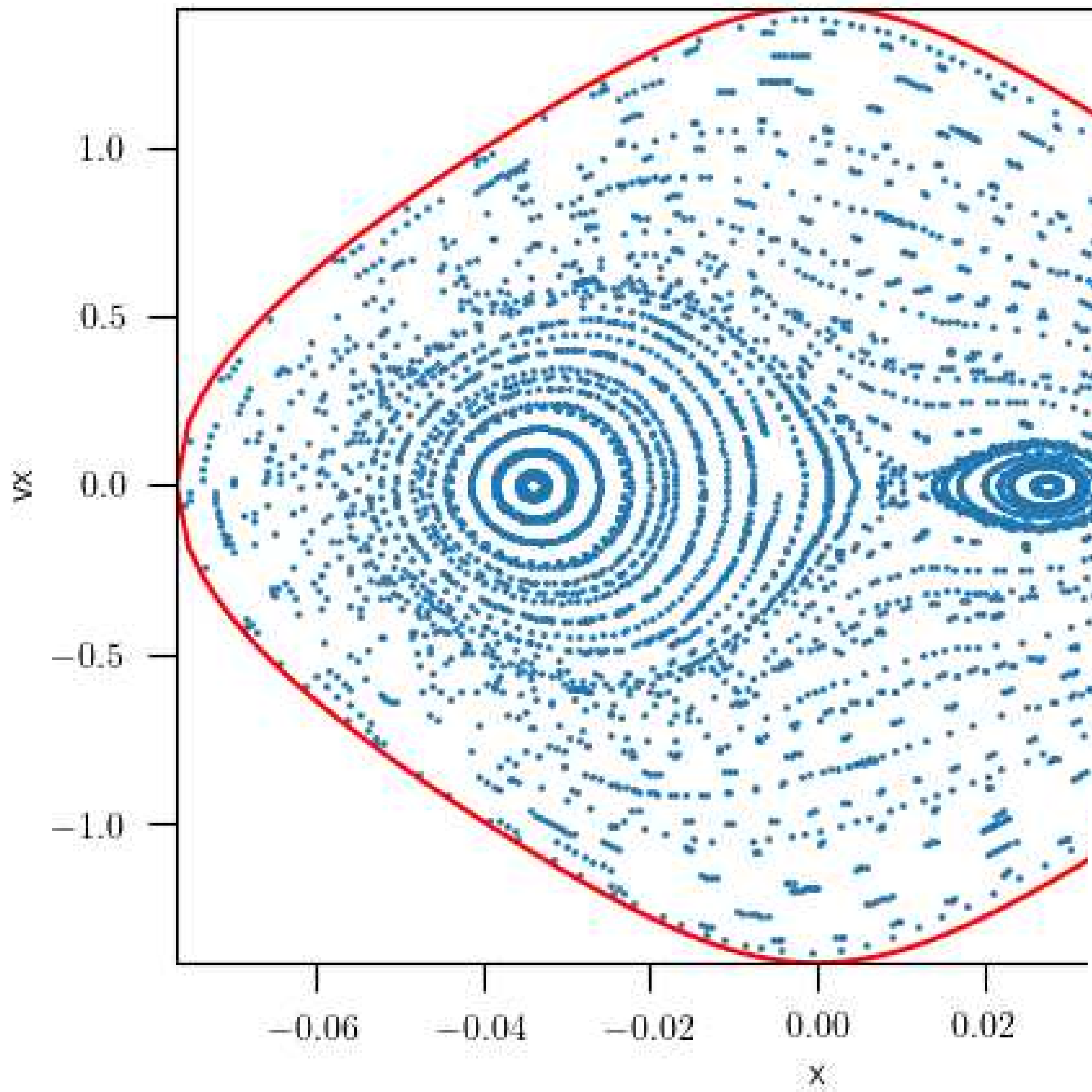

```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -2.6 --norbits 50
```

$$E = -2.5$$

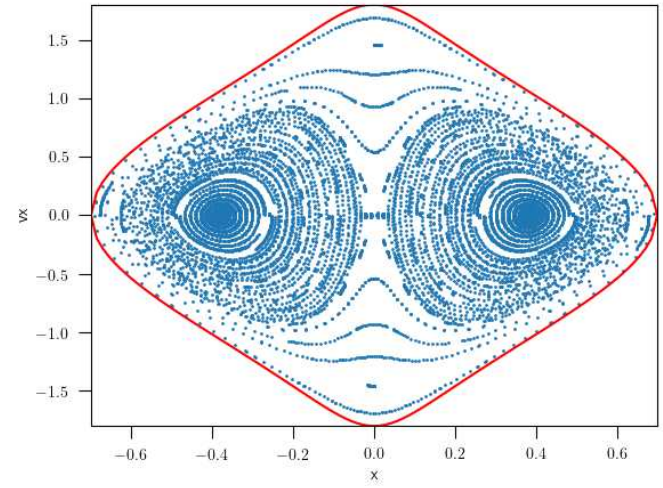



```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -2.5 --norbits 50
```

$$E = -2.5$$

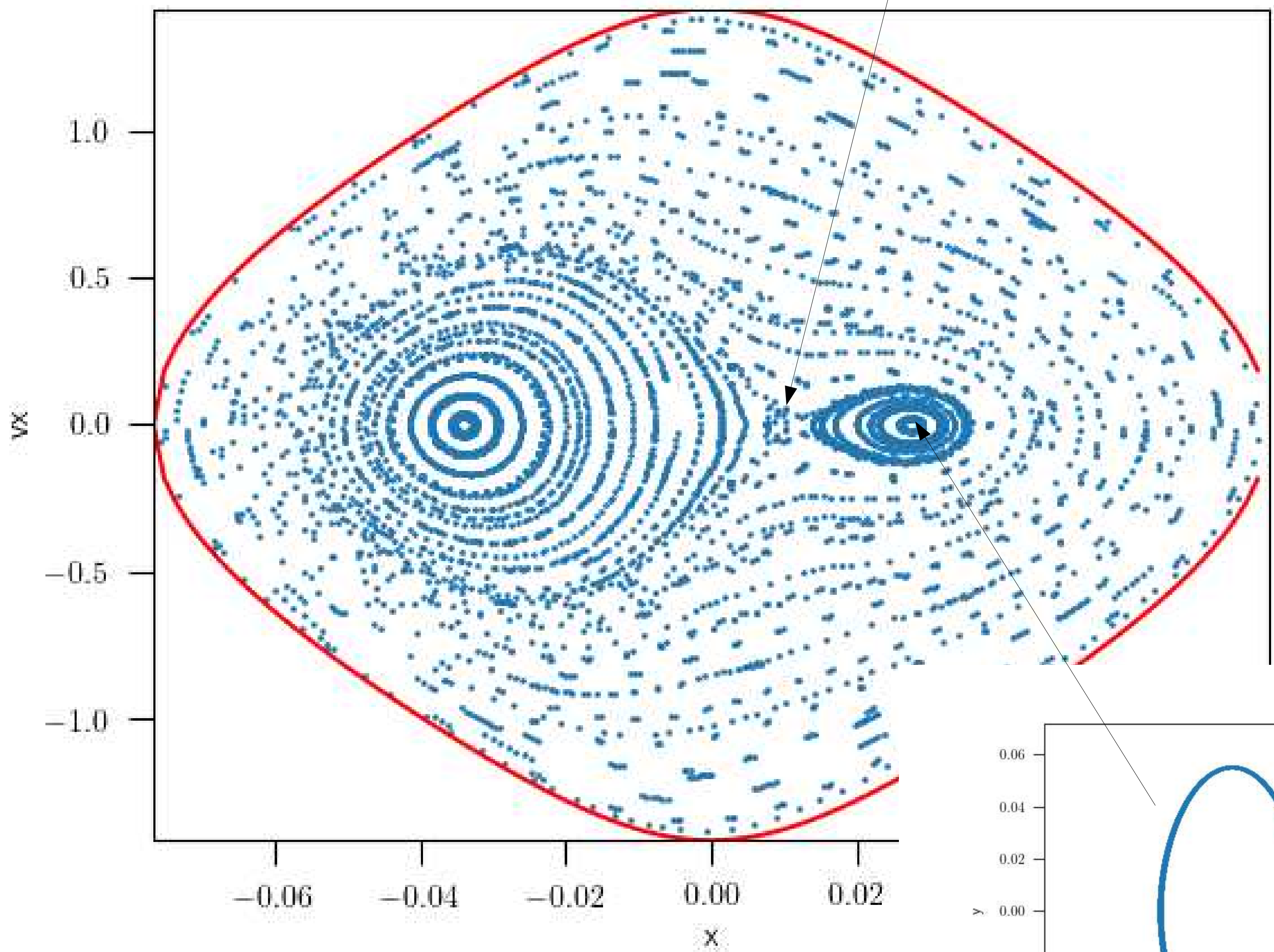


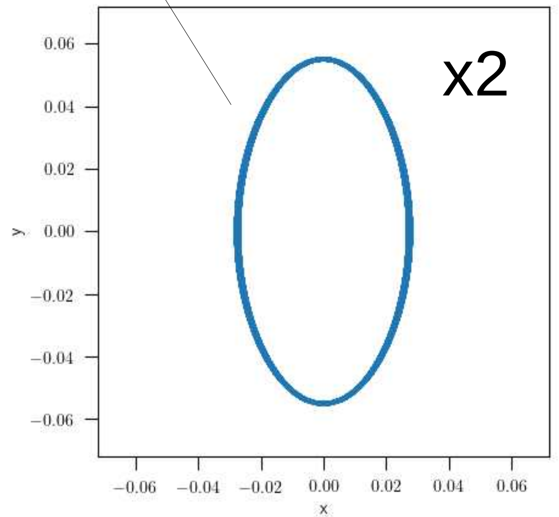
non-rotating case




```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -2.5 --norbits 50
```

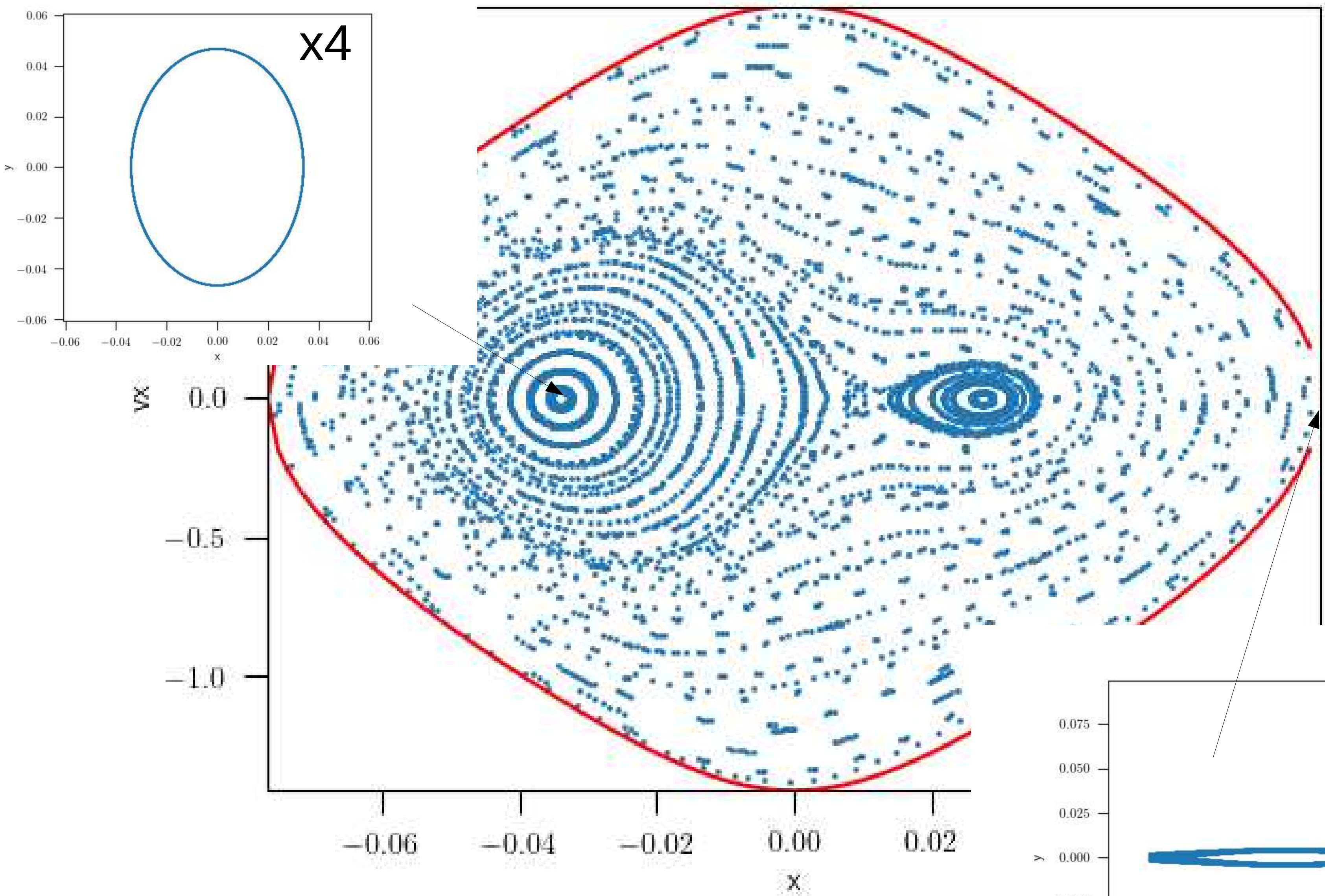
Bifurcation : apparition of x2 (stabe)/x3 (unstable) orbits

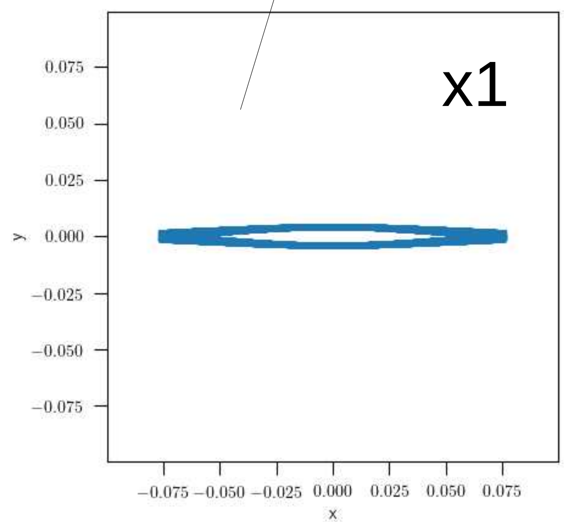





```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -2.5  --x 0.0268
```

x1 : prograde x4 : retrograde

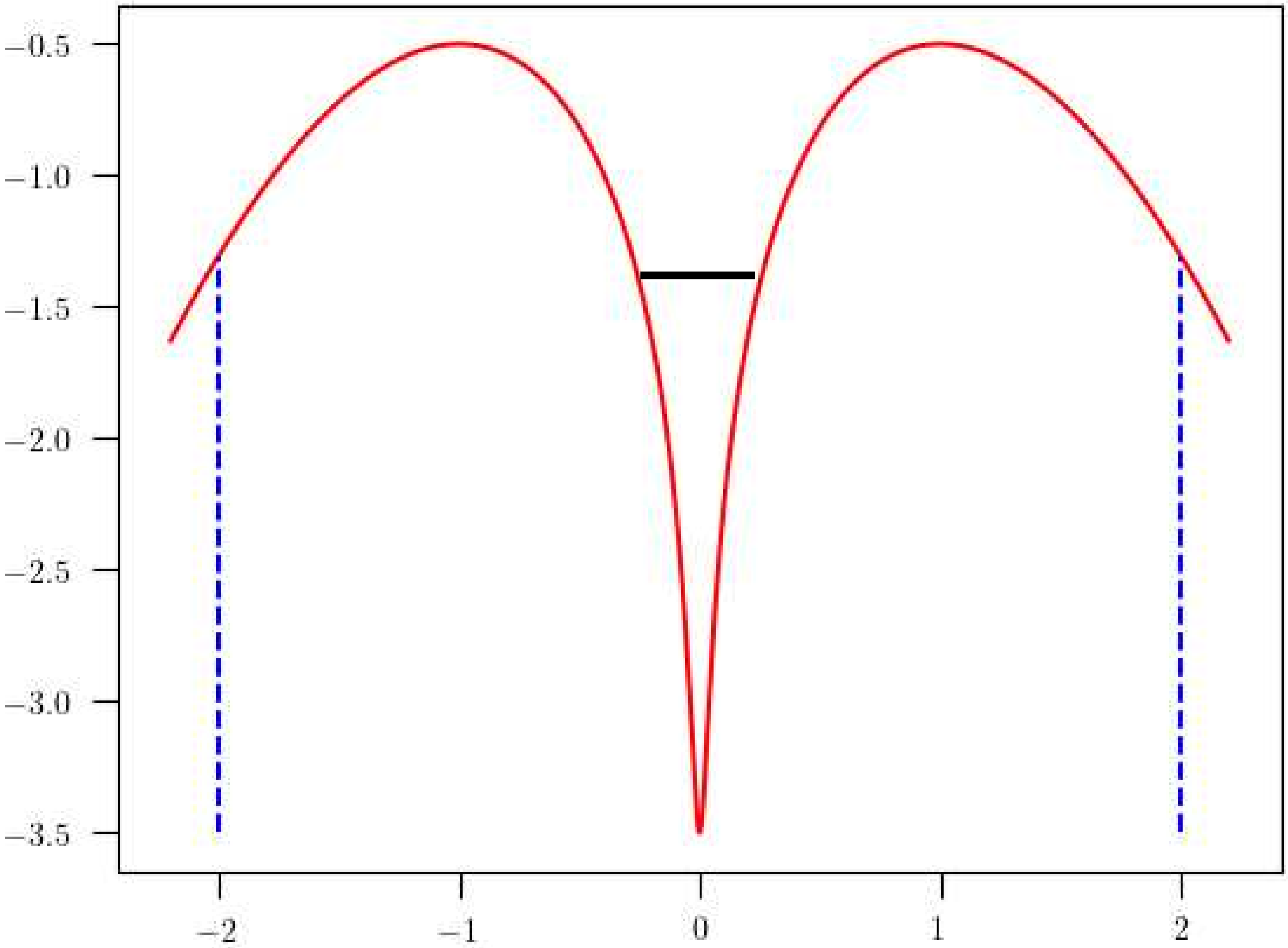





```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -2.5  --x 0.0766659
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -2.5  --x -0.034
```

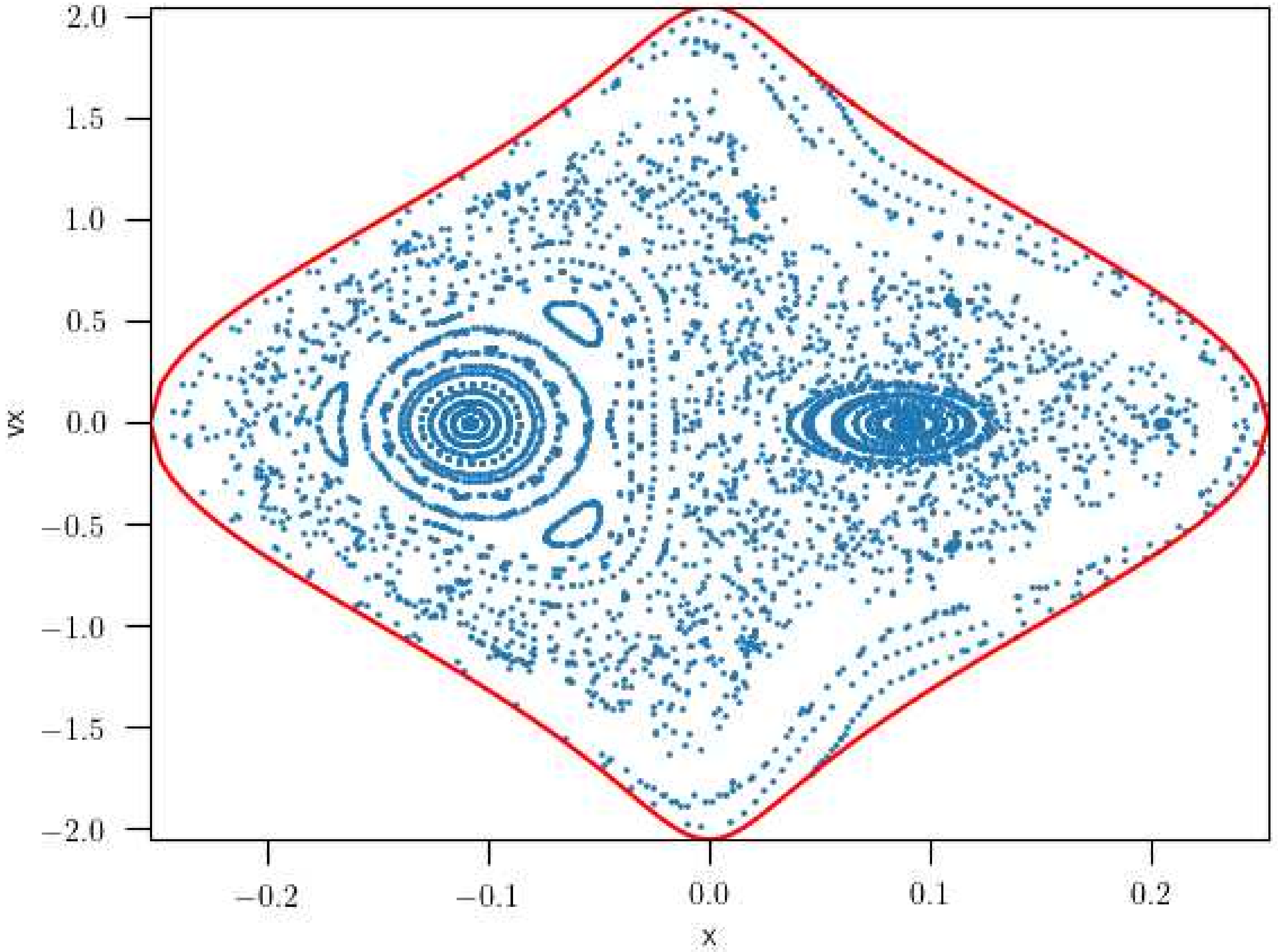
Increasing the energy

$$E = -1.4$$



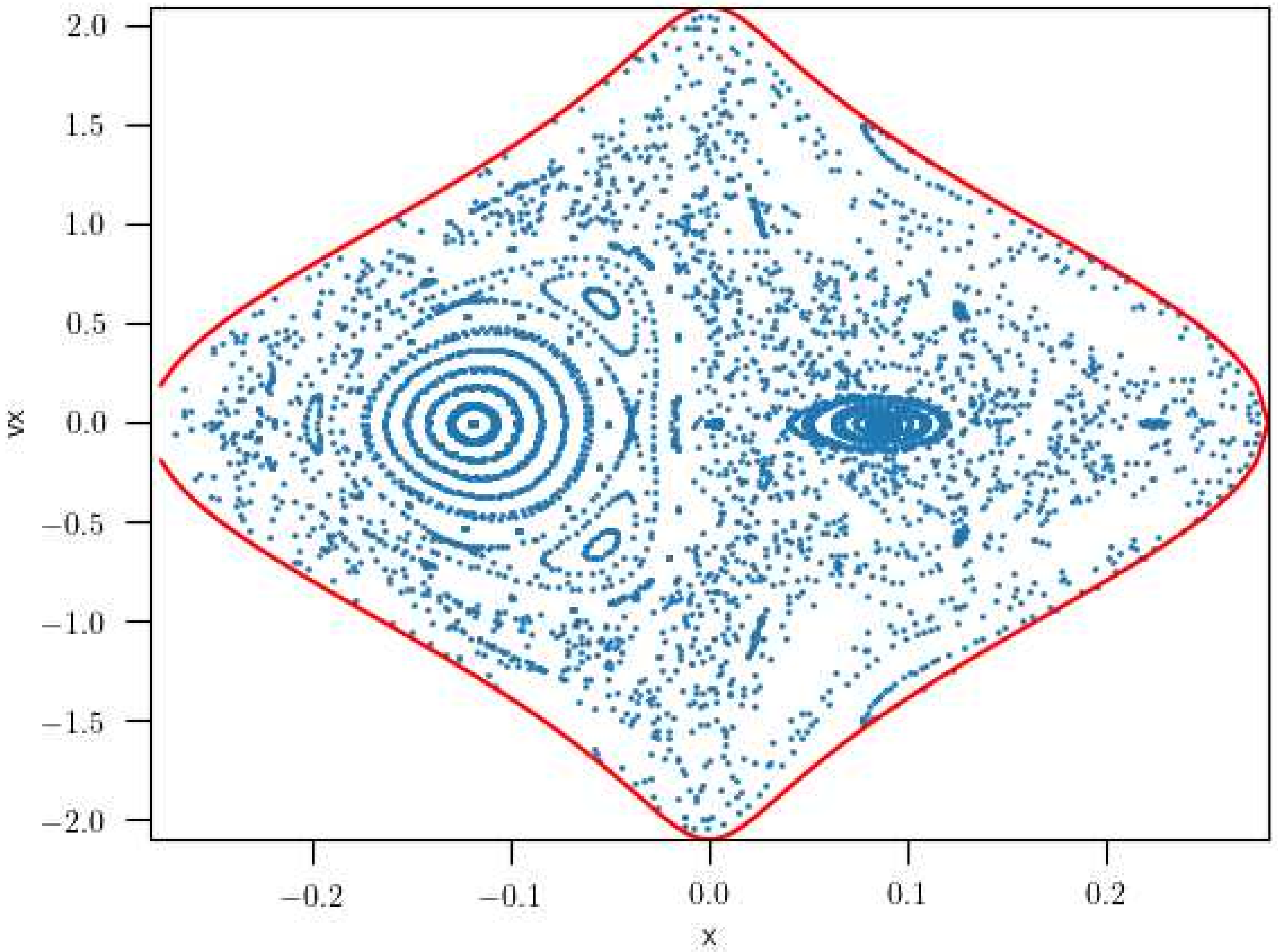

```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 --plotpotential
```

$E = -1.4$




```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -1.4 --norbits 50
```

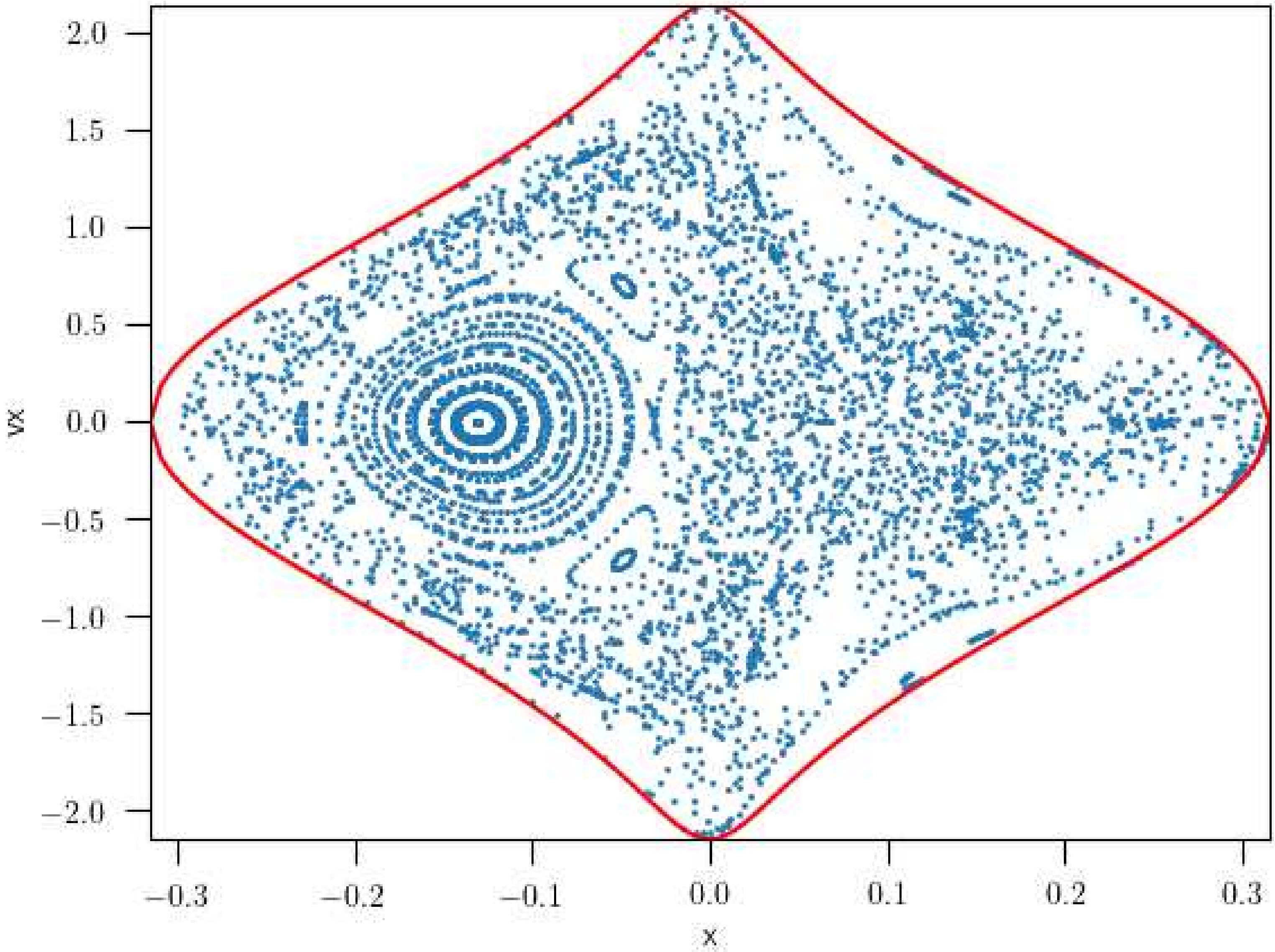
$$E = -1.3$$




```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -1.3 --norbits 50
```

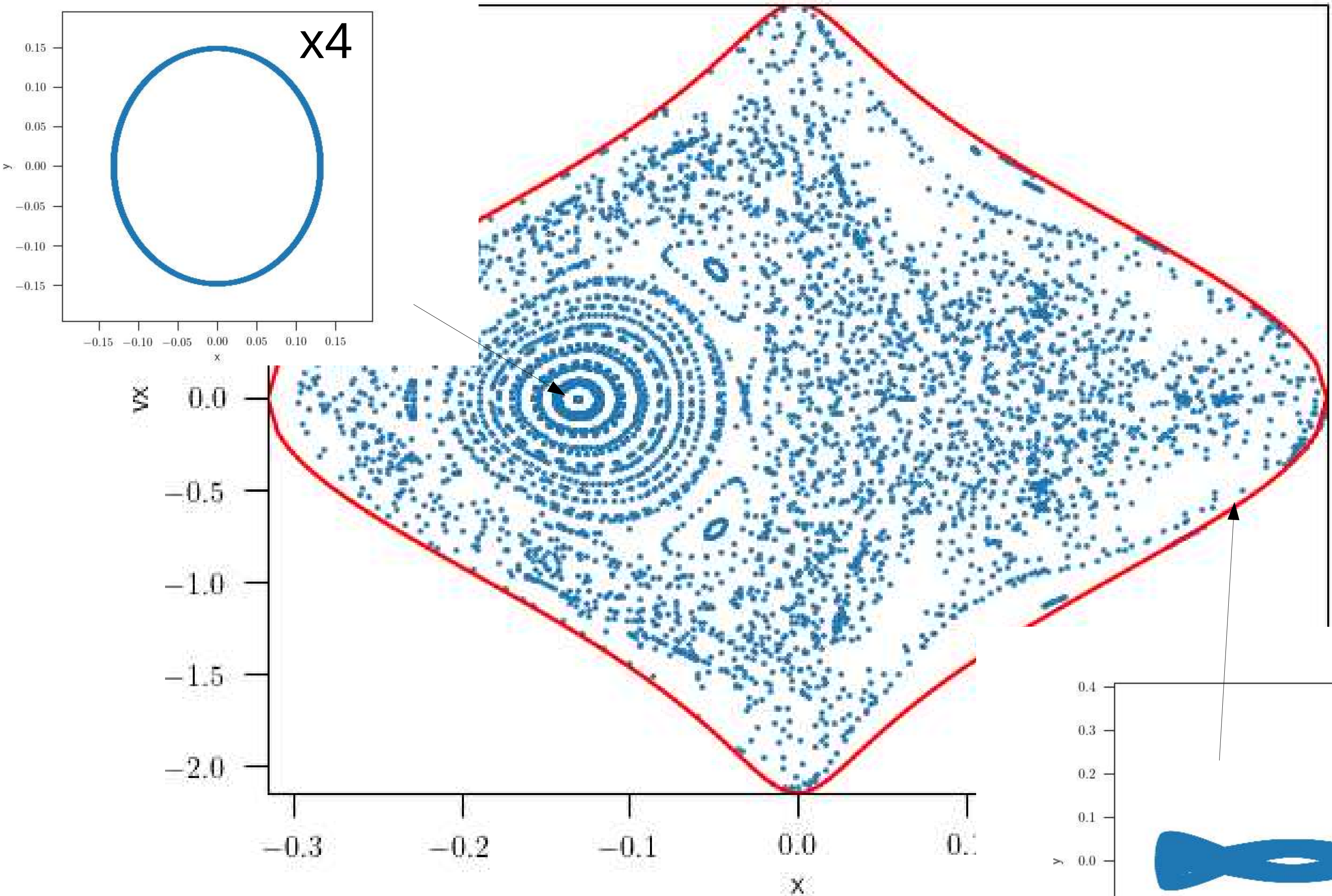
$$E = -1.2$$

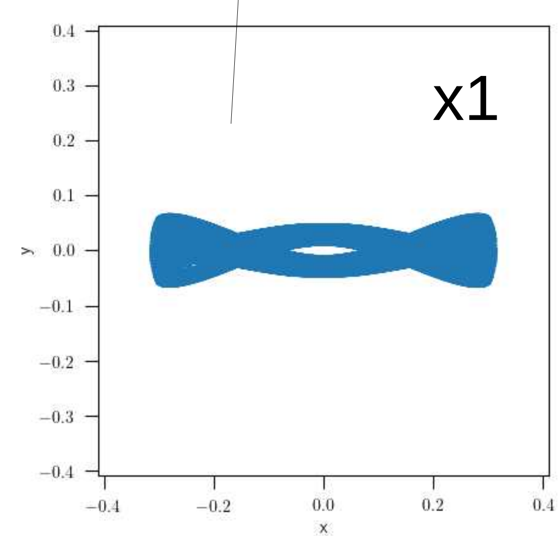
Bifurcation : x2/x3 disappeared




```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -1.2 --norbits 50
```

$$E = -1.2$$

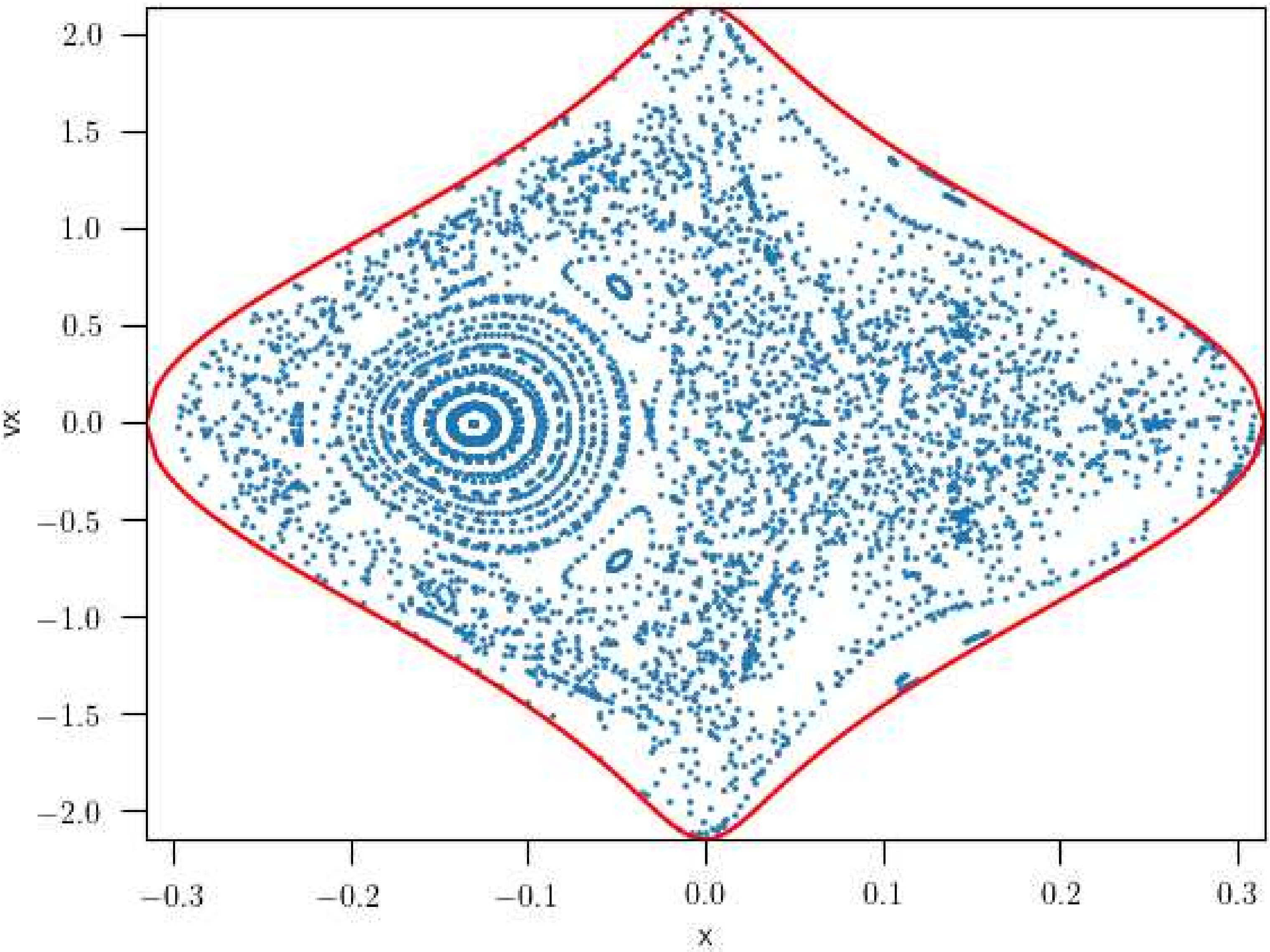




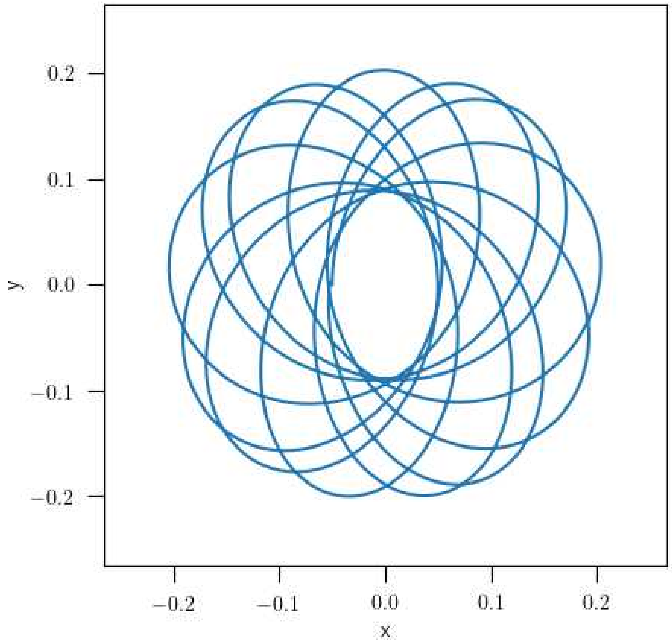

```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -1.2 --x 0.315099
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -1.2 --x -0.1283
```

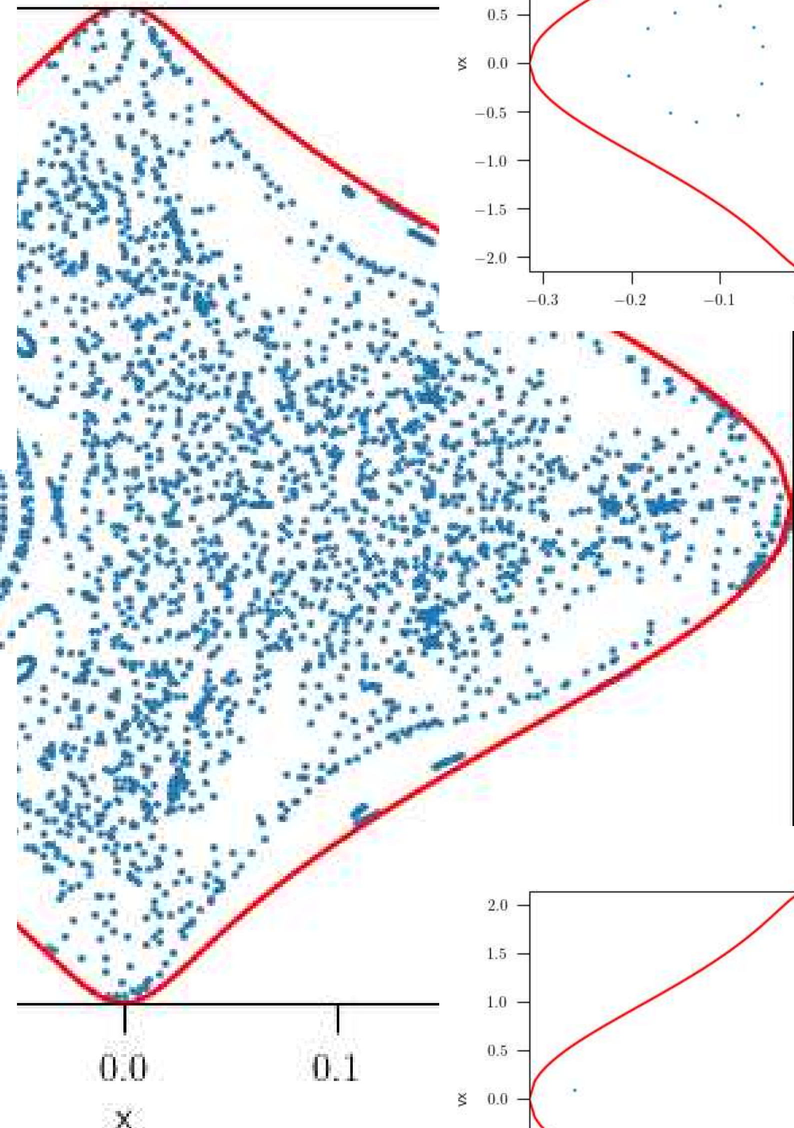
Chaotic orbits

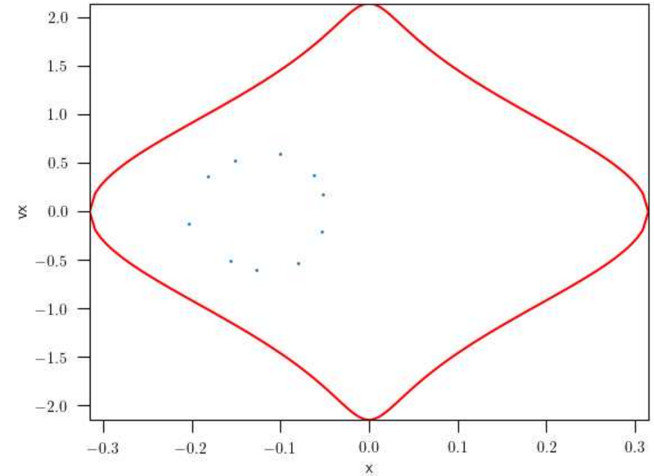
Chaotic orbits

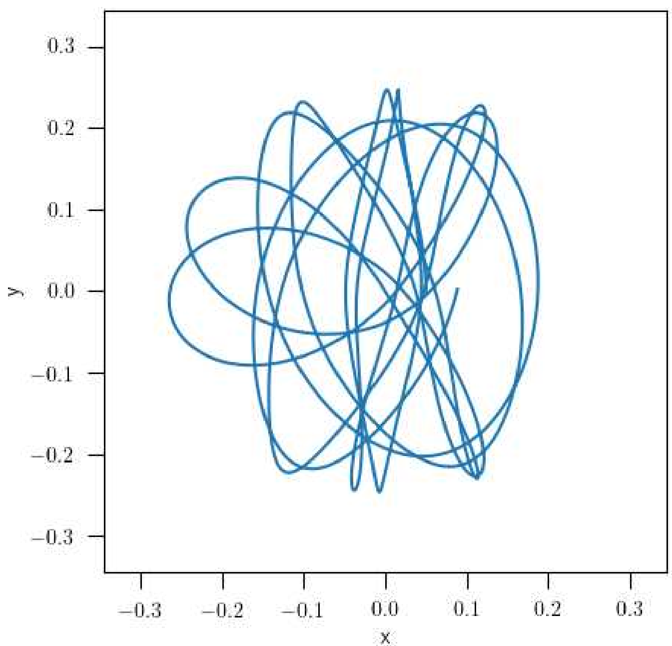


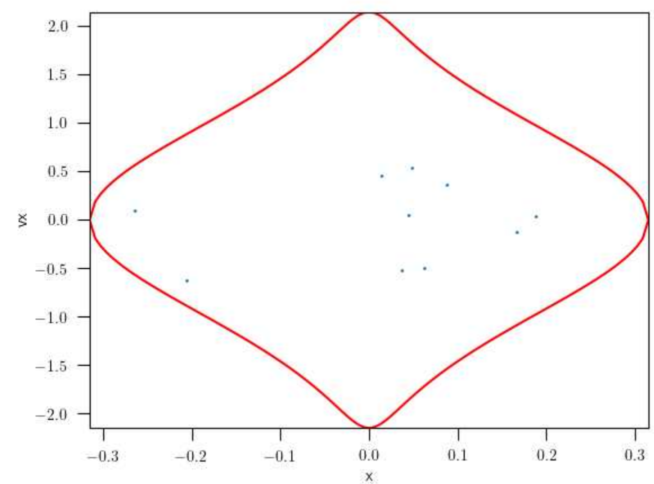
aotic orbits



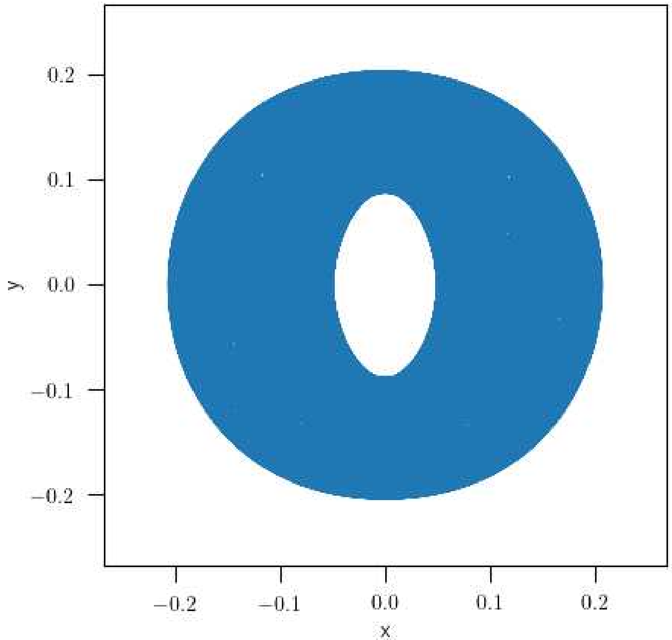


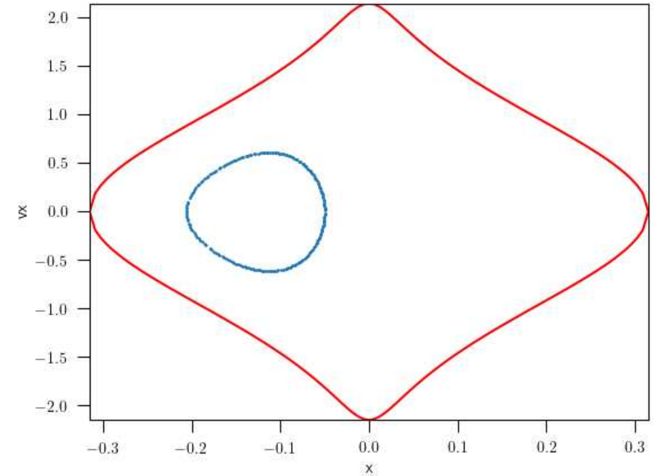


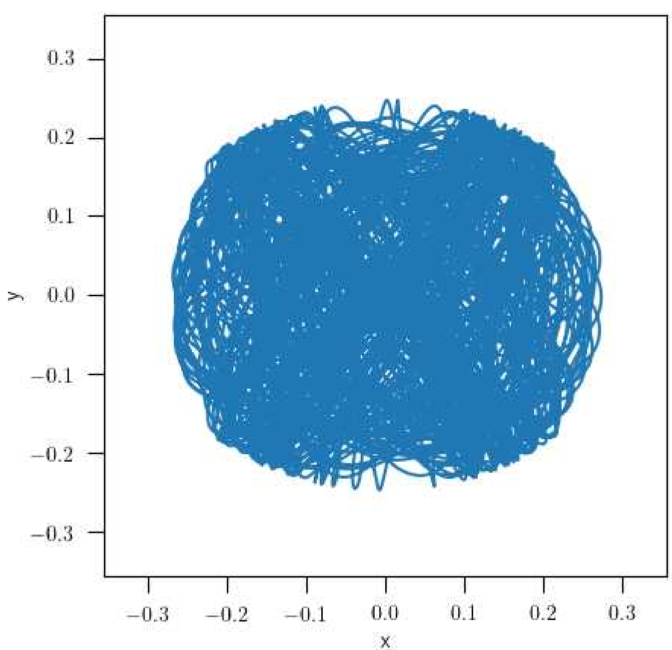


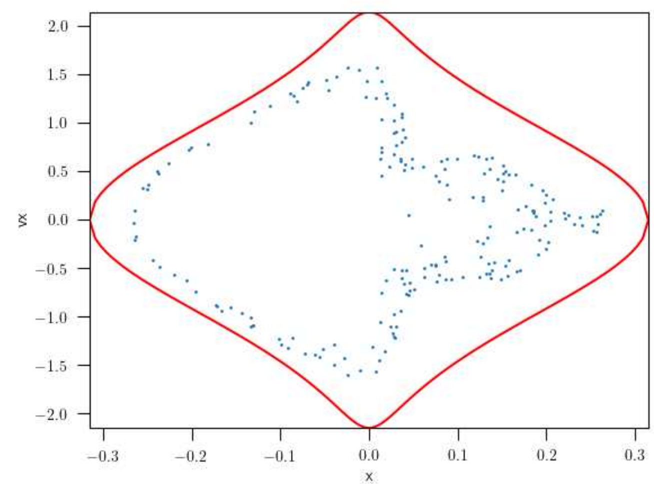


aotic orbits

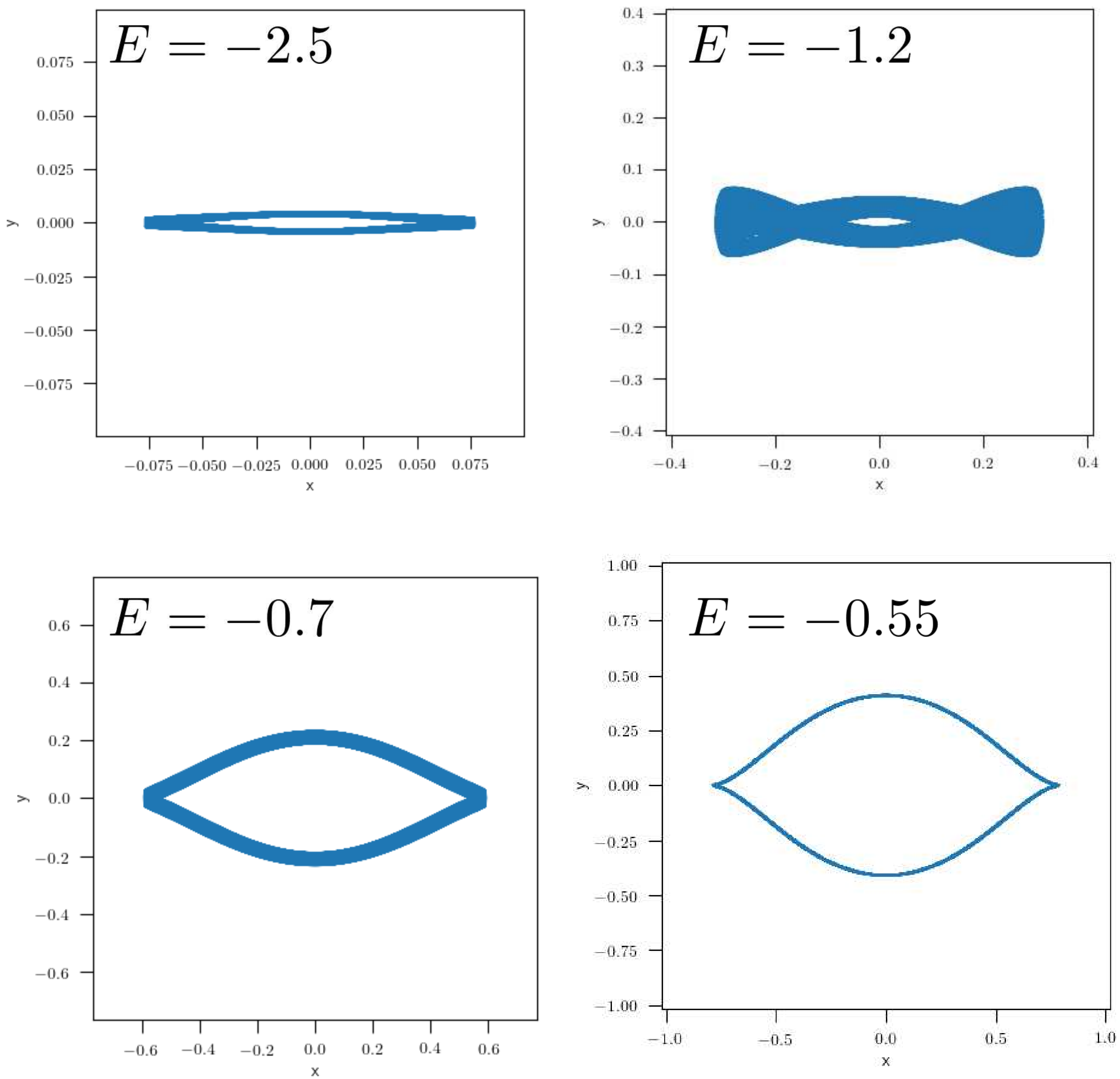








Evolution of the x1 orbit with increasing energy




```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -2.5 --x 0.0766659
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -1.2 --x 0.315099
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -0.7 --x 0.590356
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -0.55 --x 0.783882
```

The X-orbit families
(characteristics curves)

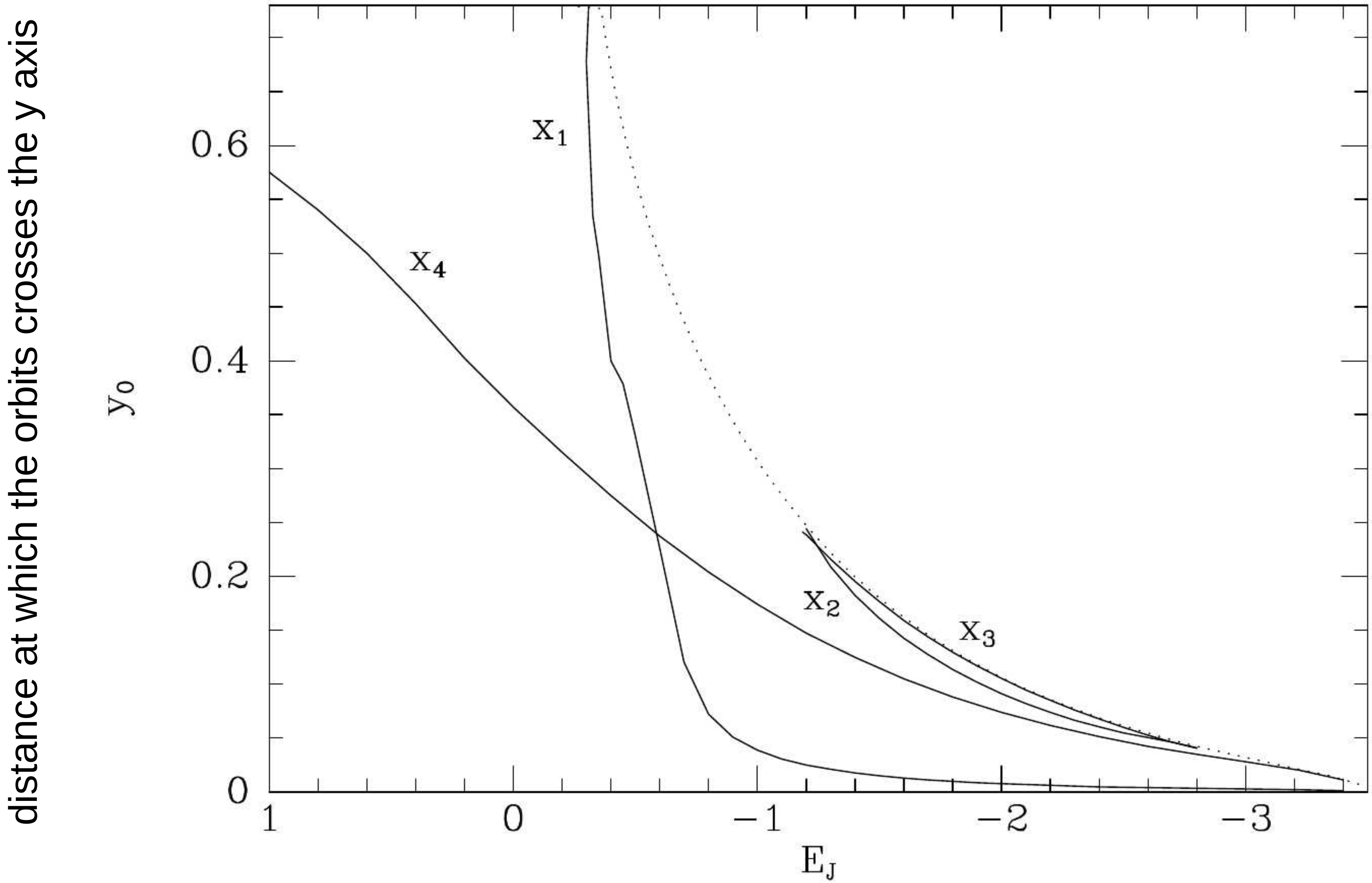


Figure 3.18 A plot of the Jacobi constant E_J of closed orbits in $\Phi_\mathrm{L}(q = 0.8, R_\mathrm{c} = 0.03, \Omega_\mathrm{b} = 1)$ against the value of y at which the orbit cuts the potential's short axis. The dotted curve shows the relation $\Phi_\mathrm{eff}(0, y) = E_\mathrm{J}$. The families of orbits $\mathrm{x_1}$–$\mathrm{x_4}$ are marked.

The End