

Astrophysics IV: Stellar and galactic dynamics

Solutions

Problem 1:

Write the definition of the specific angular momentum in cylindrical coordinates:

$$\vec{L} = \vec{x} \times \vec{v} = (R\vec{e}_R + z\vec{e}_z) \times (\dot{R}\vec{e}_R + R\dot{\theta}\vec{e}_\theta + \dot{z}\vec{e}_z) \quad (1)$$

$$= (-zR\dot{\theta})\vec{e}_R + (z\dot{R} - R\dot{z})\vec{e}_\theta + (R^2\dot{\theta})\vec{e}_z. \quad (2)$$

So, we have:

$$L_R = -zR\dot{\theta} \quad (3)$$

$$L_\theta = z\dot{R} - R\dot{z} \quad (4)$$

$$L_z = R^2\dot{\theta}. \quad (5)$$

We can now write:

$$\dot{L}_R - L_\theta\dot{\theta} = \frac{d}{dt}(-zR\dot{\theta}) - (z\dot{R} - R\dot{z})\dot{\theta} = 0 \quad (6)$$

$$= -\dot{z}R\dot{\theta} - z\dot{R}\dot{\theta} - zR\ddot{\theta} - z\dot{R}\dot{\theta} + \dot{z}R\dot{\theta}. \quad (7)$$

So, this implies that:

$$2\dot{R}\dot{\theta} + R\ddot{\theta} = 0, \quad (8)$$

which is equivalent to:

$$\frac{d}{dt}(R^2\dot{\theta}) = 0 = \frac{d}{dt}L_z, \quad (9)$$

and thus L_z being constant.

Problem 2:

For a given potential $\Phi(\vec{r})$, the acceleration reads:

$$\vec{a}(\vec{r}) = -\vec{\nabla}\Phi(\vec{r}). \quad (10)$$

In Cartesian coordinates, we have:

$$\Phi(\vec{r}) = \Phi(x, y, z) \quad \vec{\nabla}\Phi(\vec{r}) = \frac{\partial\Phi(x, y, z)}{\partial x}\vec{e}_x + \frac{\partial\Phi(x, y, z)}{\partial y}\vec{e}_y + \frac{\partial\Phi(x, y, z)}{\partial z}\vec{e}_z$$

In spherical symmetry, we can write:

$$\Phi(\vec{r}) = \Phi(r) \quad \vec{\nabla}\Phi(\vec{r}) = \frac{d\Phi(r)}{dr} \vec{e}_r \quad \text{and} \quad \vec{e}_r = \frac{x}{r} \vec{e}_x + \frac{y}{r} \vec{e}_y + \frac{z}{r} \vec{e}_z .$$

So the acceleration reads:

$$a_x = -\frac{d\Phi(r)}{dr} \times \frac{x}{r} \tag{11}$$

$$a_y = -\frac{d\Phi(r)}{dr} \times \frac{y}{r} \tag{12}$$

$$a_z = -\frac{d\Phi(r)}{dr} \times \frac{z}{r} . \tag{13}$$

Similarly, in cylindrical symmetry, we have:

$$\Phi(\vec{r}) = \Phi(R, z) \quad \vec{\nabla}\Phi(R, z) = \frac{\partial\Phi(R, z)}{\partial R} \vec{e}_r + \frac{\partial\Phi(R, z)}{\partial z} \vec{e}_z \quad \text{and} \quad \vec{e}_r = \frac{x}{R} \vec{e}_x + \frac{y}{R} \vec{e}_y .$$

Then the acceleration reads:

$$a_x = -\frac{\partial\Phi(R, z)}{\partial R} \times \frac{x}{R} \tag{14}$$

$$a_y = -\frac{\partial\Phi(R, z)}{\partial R} \times \frac{y}{R} \tag{15}$$

$$a_z = -\frac{\partial\Phi(R, z)}{\partial z} . \tag{16}$$

The derivatives of the potentials are the following:

a) Point mass:

$$\frac{d\Phi(r)}{dr} = \frac{GM}{r^2} \tag{17}$$

b) Plummer-Schuster potential:

$$\frac{d\Phi(r)}{dr} = \frac{GMr}{(r^2 + e^2)^{3/2}} \tag{18}$$

c) Miyamoto-Nagai potential:

$$\frac{\partial\Phi(R, z)}{\partial R} = \frac{GMR}{\left(R^2 + [a + \sqrt{z^2 + b^2}]^2\right)^{3/2}} \quad (19)$$

$$\frac{\partial\Phi(R, z)}{\partial z} = \frac{GMz}{\left(R^2 + [a + \sqrt{z^2 + b^2}]^2\right)^{3/2}} \left(1 + \frac{a}{\sqrt{z^2 + b^2}}\right) \quad (20)$$

$$(21)$$

d) Harmonic potential:

$$\frac{\partial\Phi(x, y, z)}{\partial x} = \omega_x^2 x \quad (22)$$

$$\frac{\partial\Phi(x, y, z)}{\partial y} = \omega_y^2 y \quad (23)$$

$$\frac{\partial\Phi(x, y, z)}{\partial z} = \omega_z^2 z \quad (24)$$

The circular velocity follows trivially :

$$V_c = \sqrt{R \frac{\partial}{\partial R} \Phi(R)} \quad (25)$$

So does the period :

$$T = 2\pi \frac{R}{V_c(R)} \quad (26)$$

Problem 3:

For a given potential in cylindrical coordinates $\Phi(R, \theta, z)$, the vertical and epicyclic frequencies read:

$$\Omega^2(R_g) = \frac{1}{R} \left(\frac{\partial\Phi}{\partial R} \right)_{(R_g, z=0)} \quad (27)$$

$$\nu^2(R_g) = \left(\frac{\partial^2\Phi}{\partial z^2} \right)_{(R_g, z=0)} \quad (28)$$

$$\kappa^2(R_g) = \left(\frac{\partial^2\Phi_{\text{eff}}}{\partial R^2} \right)_{(R_g, z=0)} = \left(R \frac{\partial(\Omega^2)}{\partial R} + 4\Omega^2 \right)_{(R_g, z=0)} \quad (29)$$

For the point mass, one obtains (dropping the g subscript for short):

$$\Omega^2(R) = \frac{GM}{R^3} \quad (30)$$

$$\nu^2(R) = \frac{\partial}{\partial z} \left(\frac{\partial \Phi}{\partial z} \right)_{(R_g, z=0)} = \frac{\partial}{\partial z} \left(\frac{GMz}{(x^2 + y^2 + z^2)^{3/2}} \right)_{(R_g, z=0)} \quad (31)$$

$$= \frac{GM}{R^3} - 3 \left(\frac{GMz^2}{R^5} \right)_{z=0} = \frac{GM}{R^3} \quad (32)$$

$$\kappa^2(R) = \left[\frac{\partial^2 \Phi}{\partial R^2} - \frac{1}{R} \frac{\partial \Phi}{\partial R} + \frac{4}{R} \frac{\partial \Phi}{\partial R} \right]_{(R_g, z=0)} = \frac{GM}{R^3} \quad (33)$$

For the Plummer-Schuster potential:

$$\Omega^2(R) = \frac{GM}{(e^2 + R^2)^{3/2}} \quad (34)$$

$$\nu^2(R) = \frac{GM}{(e^2 + R^2)^{3/2}} \quad (35)$$

$$\kappa^2(R) = 4 \frac{GM}{(e^2 + R^2)^{3/2}} - 3R^2 \frac{GM}{(e^2 + R^2)^{5/2}} \quad (36)$$

For the Miyamoto-Nagai potential:

$$\Omega^2(R) = \frac{GM}{(R^2 + (a+b)^2)^{3/2}} \quad (37)$$

$$\nu^2(R) = \frac{GM(a+b)}{b(R^2 + (a+b)^2)^{3/2}} \quad (38)$$

$$\kappa^2(R) = 4 \frac{GM}{(R^2 + (a+b)^2)^{3/2}} - 3R^2 \frac{GM}{(R^2 + (a+b)^2)^{5/2}} \quad (39)$$