Stellar orbits

2^{nd} part

Outlines

Examples of orbits in spherical potentials

- Keplerian orbits
- orbits in an homogeneous sphere
- important remarks

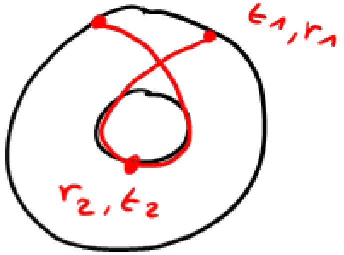
Orbits in axisymmetric potentials

- orbits in the equatorial plane
- orbits outside the equatorial plane
- equations of motion
- orbits in the meridian plane
- examples

Nearly circular orbits

- Epicycle frequences

Radial period



Time to travel from the apocenter to the pericenter

$$T_r = 2\int_{t_1}^{t_2} dt \underset{dr = \frac{dr}{dt}dt}{\overset{r(t)}{=}} 2\int_{r_1}^{r_2} \frac{1}{\dot r}\, dr \qquad \begin{cases} r(t_1) = r_1 \\ r(t_2) = r_2 \end{cases}$$

From

$$E = \frac{1}{2}\left(\dot r^2 + (r\dot\varphi)^2\right) + \phi(r) = \frac{1}{2}\dot r^2 + \frac{L}{2r^2} + \phi(r)$$

$$\dot r^2 = 2(E - \phi(r)) - \frac{L^2}{r^2}$$

$$\frac{dr}{dt} = \sqrt{2(E - \phi(r)) - \frac{L^2}{r^2}}$$

$$\frac{dt}{dr} = \frac{1}{\sqrt{2(E - \phi(r)) - \frac{L^2}{r^2}}}$$

$$\boxed{T_r = 2\int_{r_1}^{r_2} \frac{dr}{\sqrt{2(E - \phi(r)) - \frac{L^2}{r^2}}}}$$

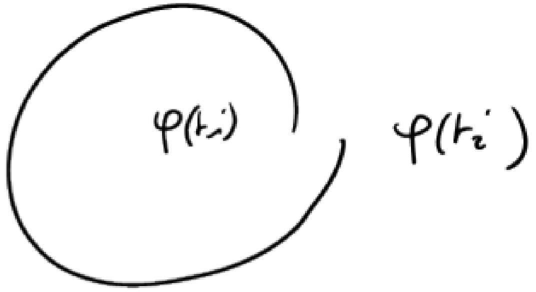
Increase of azimuth in a radial period

$$\Delta\varphi = 2\int_{\varphi_1}^{\varphi_2} d\varphi = 2\int_{t_1}^{t_2} \frac{d\varphi}{dt}\, dt = 2\int_{r_1}^{r_2} \underbrace{\frac{d\varphi}{dt}} \frac{dt}{dr}\, dr$$

$$= 2\int_{r_1}^{r_2} \frac{L}{r^2} \frac{1}{\sqrt{2(\epsilon - \Phi(r)) - \frac{L^2}{r^2}}}\, dr$$

Azimuthal period

$$T_\varphi = \int_{t'_1}^{t'_2} dt \qquad \begin{cases} \varphi(t'_1) = 0 \\ \varphi(t'_2) = 2\pi \end{cases}$$



Using the radial period T_r and the azimuthal increase :

$$\boxed{T_\varphi = \frac{2\pi}{\Delta\varphi} \; T_r}$$

As in general $\frac{2\pi}{\Delta\varphi}$ is not a rational number

the orbit is not guarantee to be closed

Stellar orbits

Spherical Systems

Examples

Examples

① Kepler potential (potential of a mass point)

EXERCICE

$$\left\{ \begin{aligned} \phi(r) &= -\frac{GM}{r} \\ \frac{\partial \phi}{\partial r}(r) &= \frac{GM}{r^2} = GMu^2 \end{aligned} \right.$$

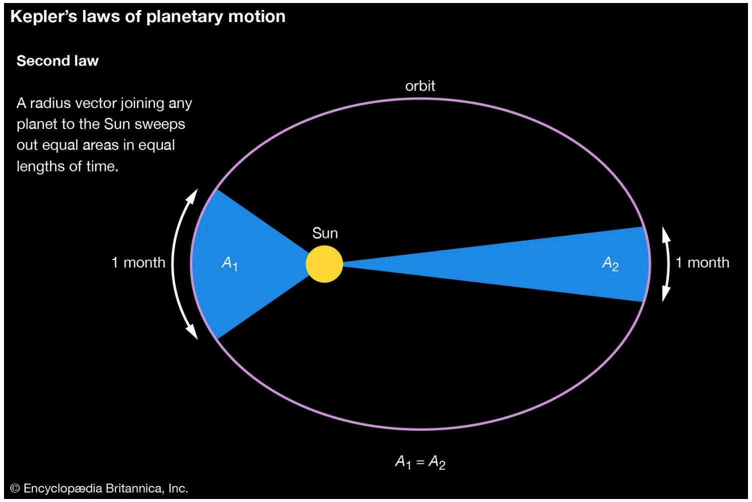
$$\boxed{\frac{d^2u}{d\varphi^2} + u = \frac{1}{L^2u^2} \frac{\partial \phi}{\partial r}\left(\frac{1}{u}\right)} \Rightarrow \boxed{\frac{d^2u}{d\varphi^2} + u = \frac{GM}{L^2}}$$

Harmonic equation,
with frequency 1

Kepler laws (1609-1619) :

EXERCICE

- The orbit of a planet is an ellipse with the Sun at one of the two foci.
- A line segment joining a planet and the Sun sweeps out equal areas during equal intervals of time.
- The square of a planet’s orbital period is proportional to the cube of the length of the semi-major axis of its orbit.



Kepler’s laws of planetary motion

Second law

A radius vector joining any planet to the Sun sweeps out equal areas in equal lengths of time.

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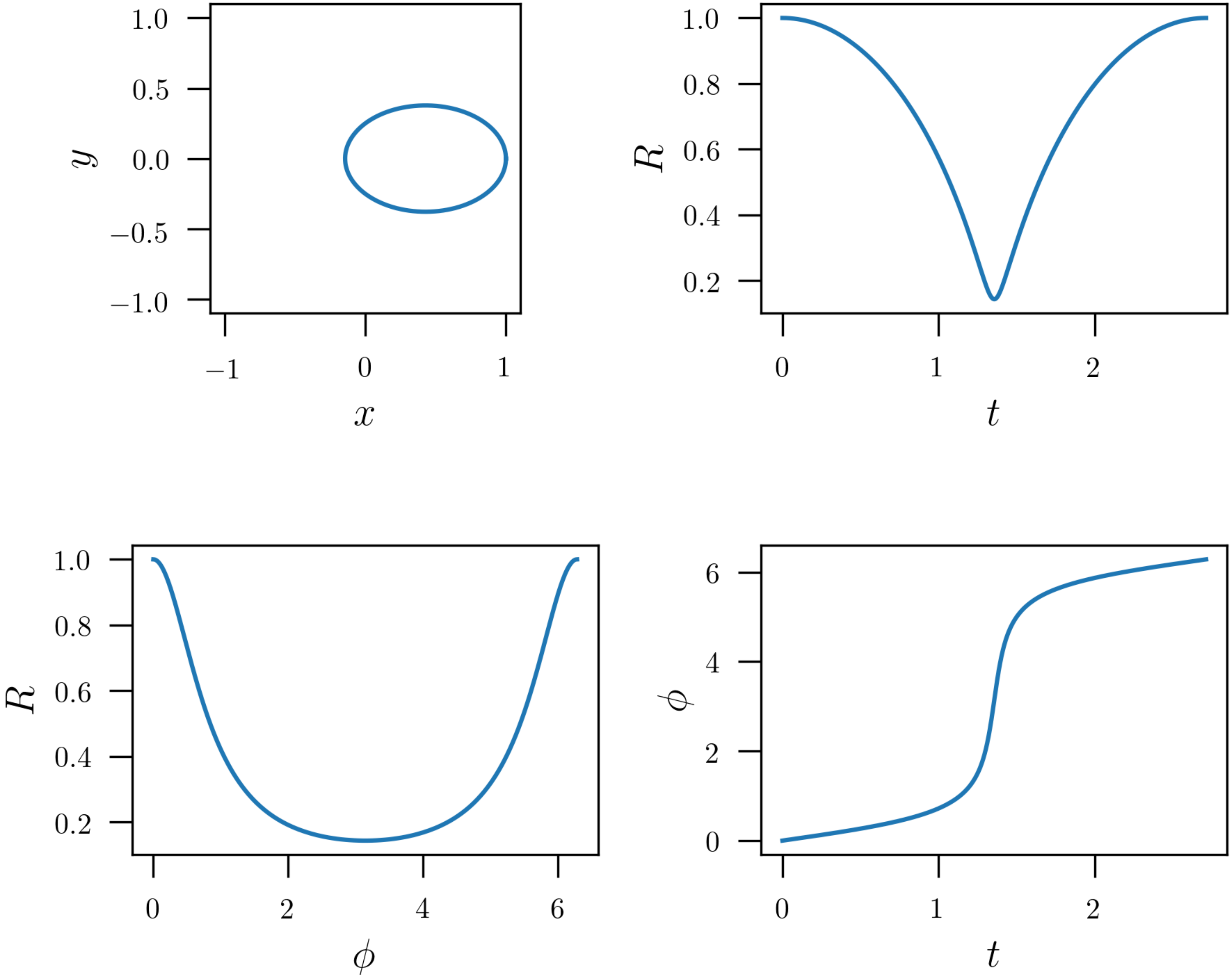
Radial and azimuthal periods:

$$T_r = T_\phi$$

Keplerian orbits (point mass)



② Homogeneous sphere ρ_0, R_0 (Harmonic oscillations)

$$\phi(r) = \underbrace{-2\pi G \rho_0 R_0^2}_{\text{cte} \rightarrow 0} + \frac{2}{3}\pi G \rho_0 r^2$$

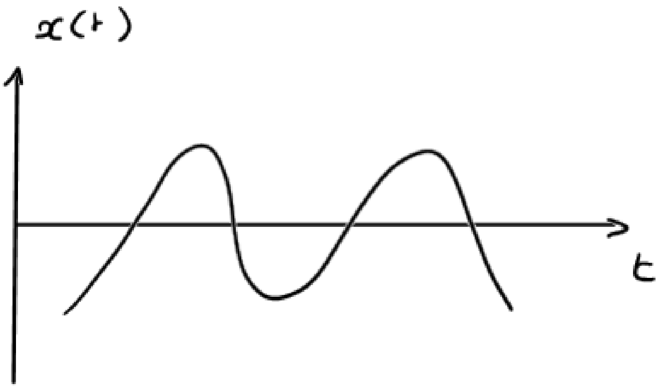
$$\phi(r) = \frac{1}{2}\Omega^2 r^2 \qquad \text{with } \Omega = \sqrt{\frac{4}{3}\pi G \rho_0}$$

Equations of motion (in carthesian coordinates)

$$L(x, y, \dot{x}, \dot{y}) = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\dot{y}^2 - \frac{1}{2}\Omega^2(x^2 + y^2)$$

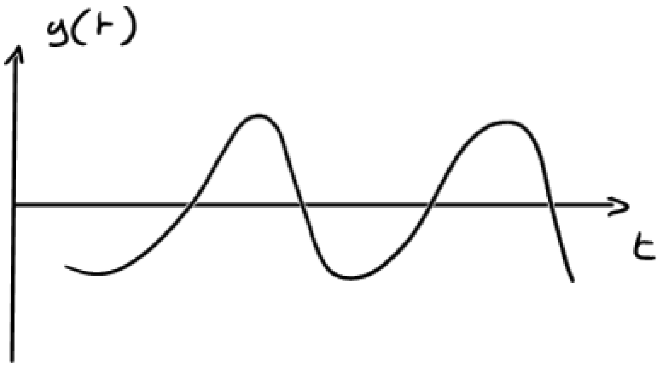
$$\begin{cases} \ddot{x} = -\Omega^2 x \\ \ddot{y} = -\Omega^2 y \end{cases} \qquad \begin{cases} x(t) = X\cos(\Omega t + \varepsilon_x) \\ y(t) = Y\cos(\Omega t + \varepsilon_y) \end{cases}$$

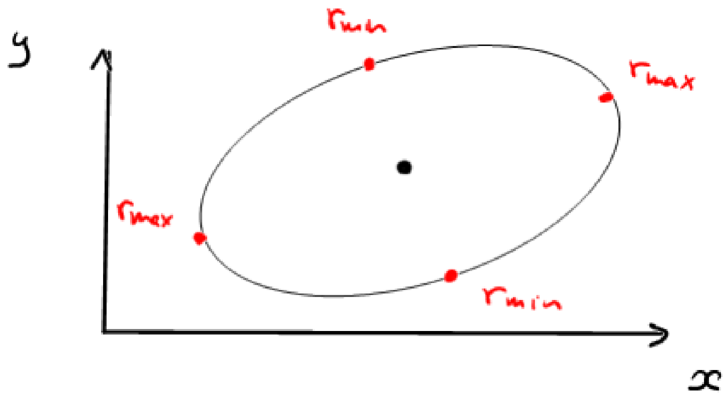
$X, Y, \varepsilon_x, \varepsilon_y$ constants fixed by the initial conditions



same period

$\Rightarrow$ closed orbits (ellipse)

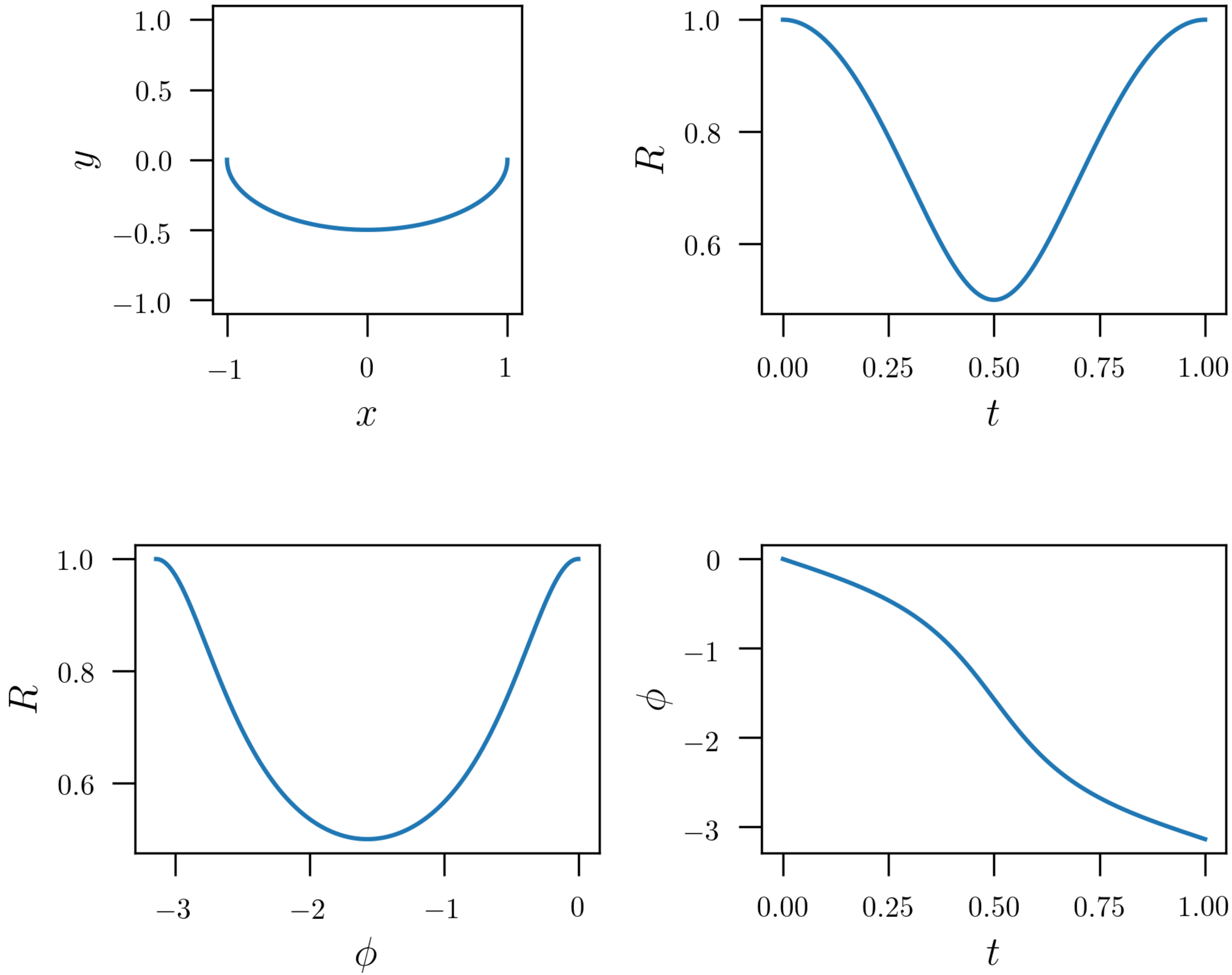




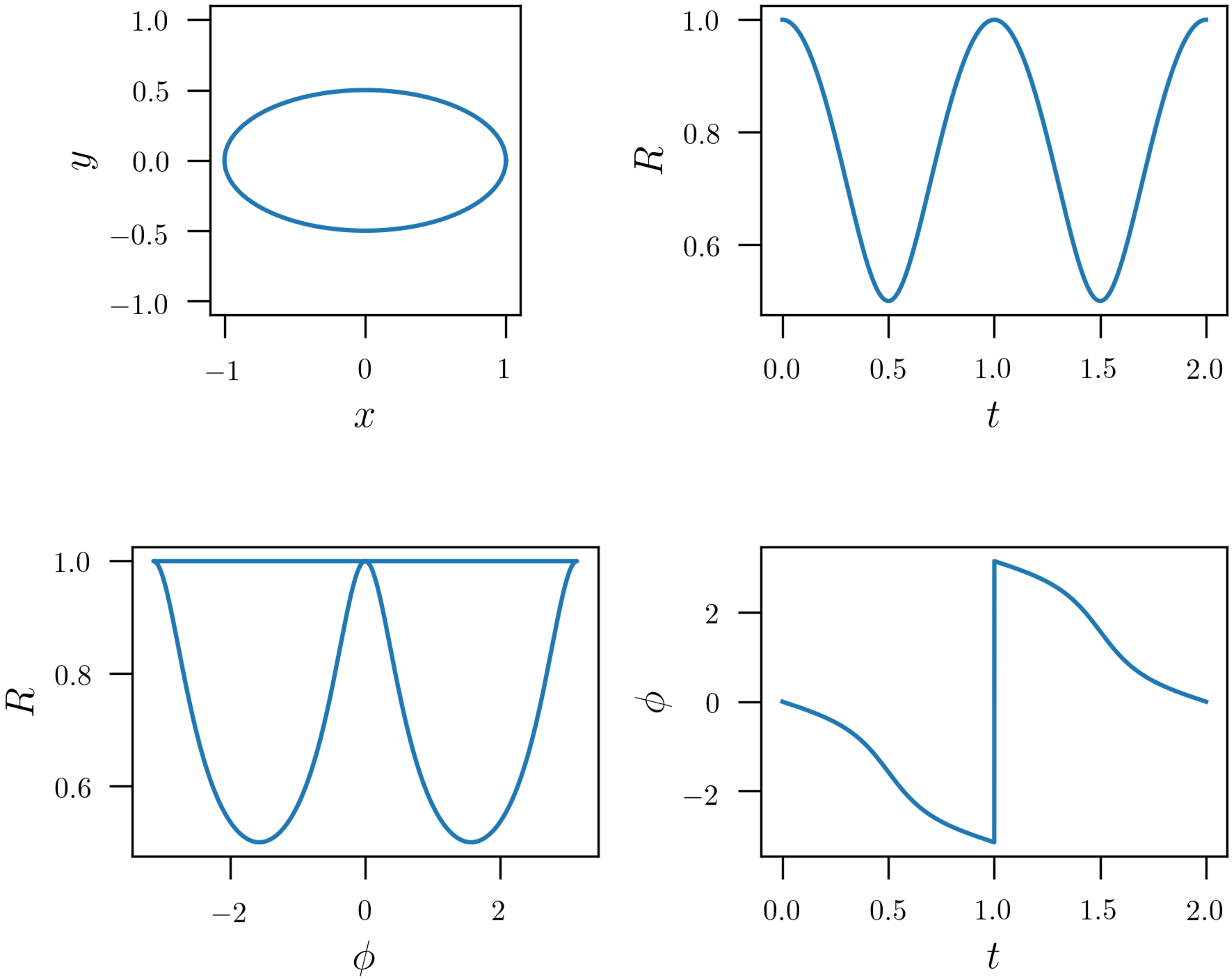
Periods

$$T_\varphi = \frac{2\pi}{\Omega} \qquad T_r = \frac{1}{2} T_\varphi = \frac{\pi}{\Omega}$$

Homegeneous sphere (harmonic)



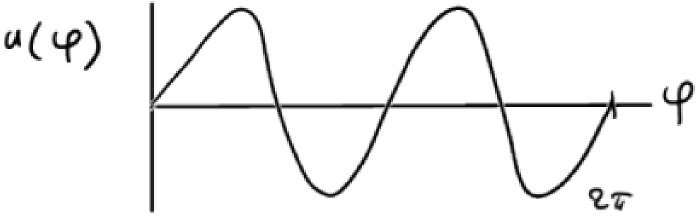
Homegeneous sphere (harmonic)

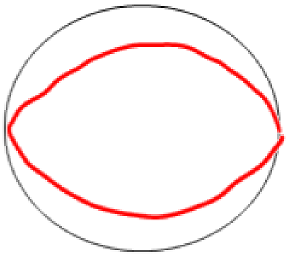


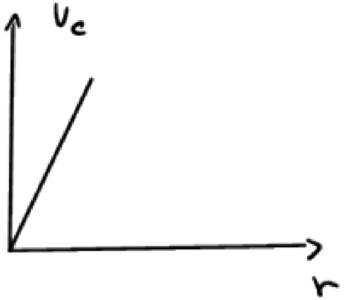
Important Remarks

Homogeneous sphere

$$\boxed{T_r = \frac{1}{2} T_\varphi}$$



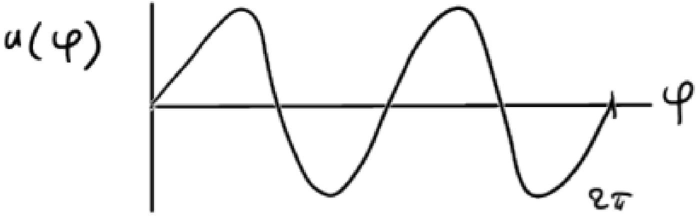


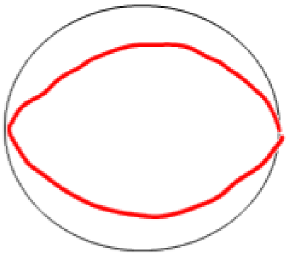


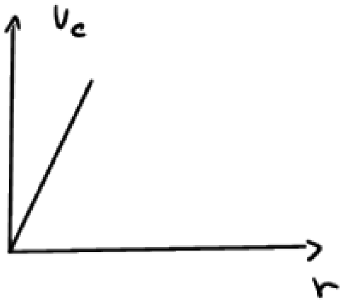
Important Remarks

Homogeneous sphere

$$\boxed{T_r = \frac{1}{2} T_\varphi}$$

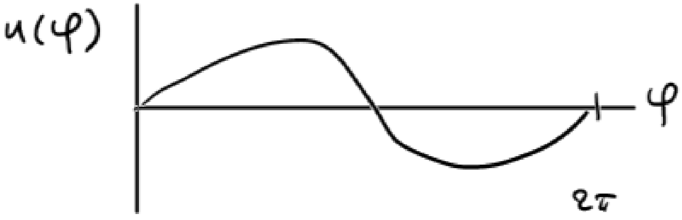


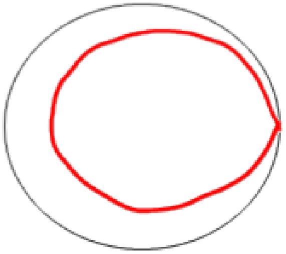


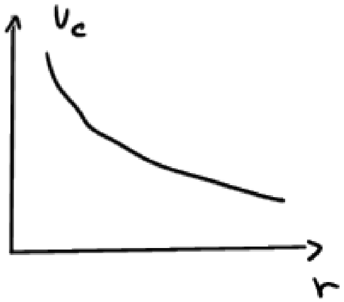


Keplerian potential

$$\boxed{T_r = T_\varphi}$$



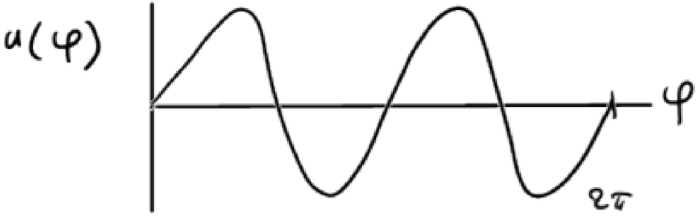


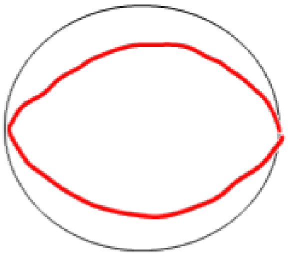


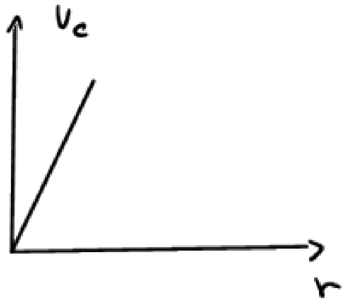
Important Remarks

Homogeneous sphere

$$\boxed{T_r = \frac{1}{2} T_\varphi}$$

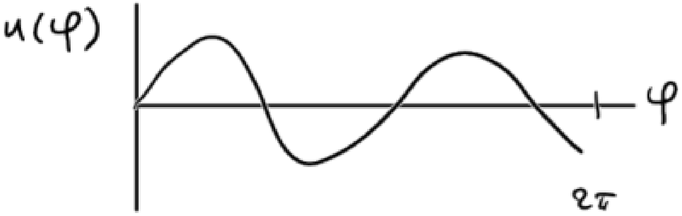


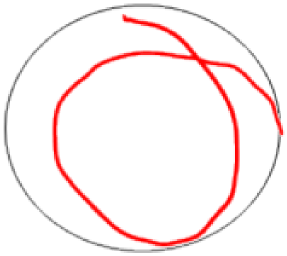


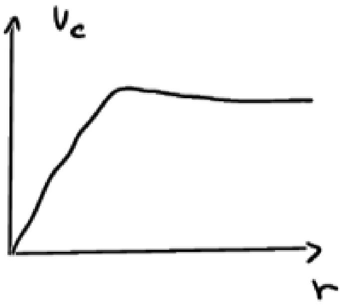


Galaxy

$$\boxed{\frac{1}{2} T_\varphi < T_r < T_\varphi}$$

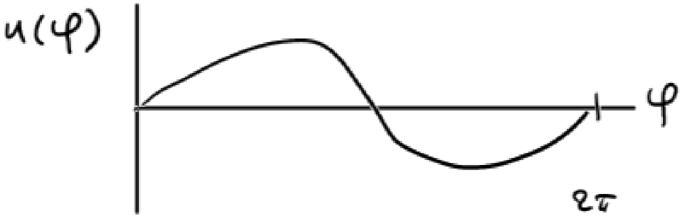


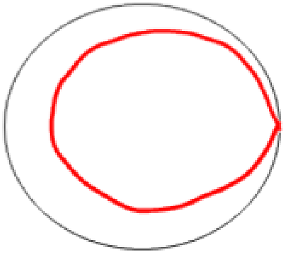


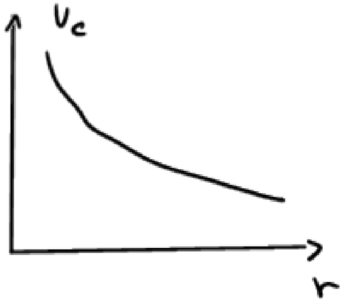


Keplerian potential

$$\boxed{T_r = T_\varphi}$$







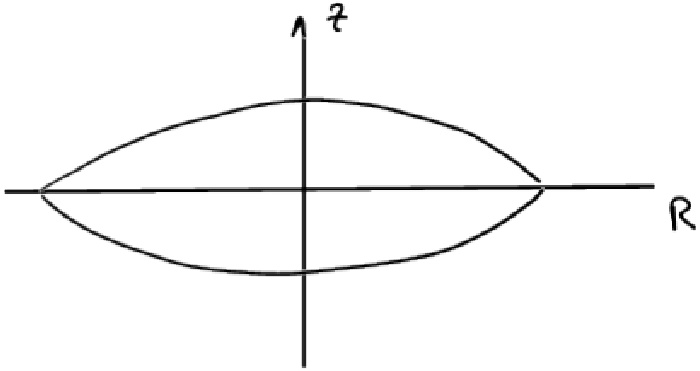
Stellar orbits

Axisymmetric Systems

Orbits in axisymmetric potentials

Axisymmetric potential

$\phi(\bar{x}) = \phi(R, |z|)$

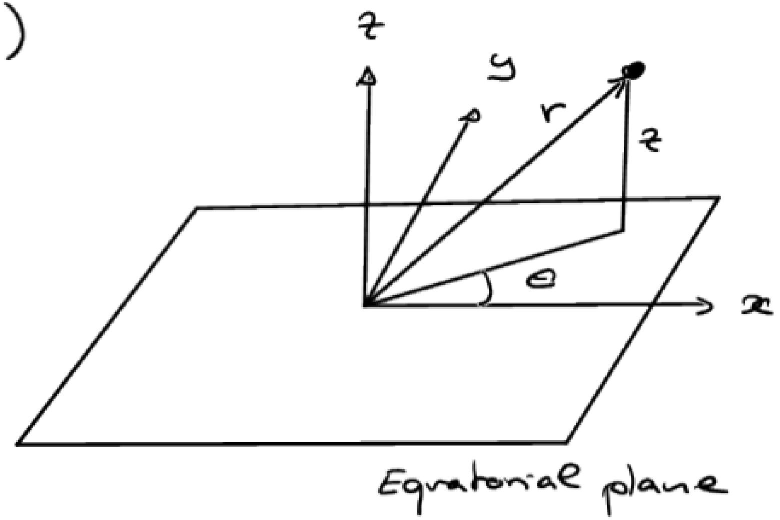


- symmetry of revolution around z
- reflection symmetry with respect to the $z = 0$ plane

Description of the dynamics

Cylindrical coordinates

(R, θ, z)



Orbits in the equatorial plane

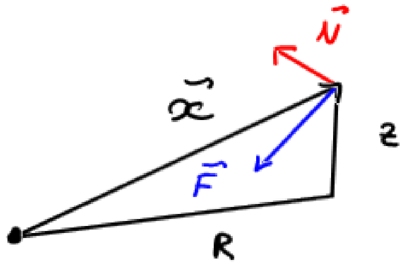
$\forall t,\ \underline{z = 0}$

$$\boxed{\phi(R, |z| = 0) \equiv \phi(R)}$$

The potential seen by the stars is similar to a spherical potential

- description of the orbits in polar coordinates r, φ
- recycle all results developped for spherical potentials

Angular momentum derivative



$$\frac{d\vec{L}}{dt} = \vec{x} \times \vec{g}(\vec{x}) = \vec{N}$$

$$\vec{x} = R\,\vec{e}_R + z\,\vec{e}_z$$

$$\vec{g}(\vec{x}) = -\vec{\nabla}\phi(x)$$

$$= -\frac{\partial\phi}{\partial R}\vec{e}_R - \frac{1}{R}\underbrace{\cancel{\frac{\partial\phi}{\partial\theta}}}_{=0}\vec{e}_\theta - \frac{\partial\phi}{\partial z}\vec{e}_z$$

$$\boxed{\frac{d\vec{L}}{dt} = \left(z\frac{\partial\phi}{\partial R} - R\frac{\partial\phi}{\partial z}\right)\vec{e}_\theta} \quad ①$$

But

$$\begin{cases} \dot{\vec{e}}_R = \dot{\theta}\,\vec{e}_\theta \\ \dot{\vec{e}}_\theta = -\dot{\theta}\,\vec{e}_R \\ \dot{\vec{e}}_z = 0 \end{cases}$$

$$\vec{L} = L_R\,\vec{e}_R + L_\theta\,\vec{e}_\theta + L_z\,\vec{e}_z$$

$$\frac{d\vec{L}}{dt} = \dot{L}_R\,\vec{e}_R + L_R\,\dot{\theta}\,\vec{e}_\theta + \dot{L}_\theta\,\vec{e}_\theta - L_\theta\,\dot{\theta}\,\vec{e}_R + \dot{L}_z\,\vec{e}_z$$

$$= \left(\dot{L}_R - L_\theta\,\dot{\theta}\right)\vec{e}_R + \left(\dot{L}_\theta - L_R\,\dot{\theta}\right)\vec{e}_\theta + \dot{L}_z\,\vec{e}_z$$

comparing with ①

$$\begin{cases} \dot{L}_z = 0 & \Rightarrow \boxed{L_z = cte} \\ \dot{L}_R - L_\theta\,\dot{\theta} = 0 & \Rightarrow \boxed{L_z = cte} \end{cases}$$

EXERCICE

The z-component of the angular momentum is conserved

Orbits that moves outside the equatorial plane

Cylindrical coordinates

$$\begin{cases} x = R\cos\theta \\ y = R\sin\theta \\ z = z \end{cases} \qquad \begin{aligned} \dot{x} &= \dot{R}\cos\theta - R\sin\theta\,\dot{\theta} \\ \dot{y} &= \dot{R}\sin\theta + R\cos\theta\,\dot{\theta} \\ \dot{z} &= \dot{z} \end{aligned}$$

$$\underline{\dot{x}^2 + \dot{y}^2 = \dot{R}^2 + R^2\dot{\theta}^2}$$

Lagrangian (specific) in cylindrical coordinates

$$\mathcal{L} = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2\right) + \phi\left(\sqrt{x^2+y^2}\right) = \frac{1}{2}\left(\dot{R}^2 + R^2\dot{\theta}^2 + \dot{z}^2\right) - \phi(R, z)$$

Lagrange equations

$$\boxed{\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\vec{q}}}\right) - \frac{\partial \mathcal{L}}{\partial \vec{q}} = 0}$$

$$\vec{q} = \begin{cases} R \\ \theta \\ z \end{cases} \qquad \dot{\vec{q}} = \begin{cases} \dot{R} \\ \dot{\theta} \\ \dot{z} \end{cases}$$

Lagrange equations

$$\begin{cases} \ddot{R} = R\dot{\theta}^2 - \dfrac{\partial \phi}{\partial R} & \text{①} \\ \dfrac{d}{dt}\left(R^2\dot{\theta}\right) = \left(-\cancel{\dfrac{\partial \phi}{\partial \theta}}\right) = 0 & \text{②} \\ \ddot{z} = -\dfrac{\partial \phi}{\partial z} & \text{③} \end{cases}$$

② $R^2\dot{\theta} = \text{cte} = L_z$

The z-component of the angurar momentum is conserved

Solution

$$\boxed{\theta(t) = L_z \int_{t_0}^{t_1} \frac{1}{R^2(t)}\, dt}$$

① + ③ two coupled through $\phi(\underset{\uparrow}{R}, \underset{\uparrow}{z})$ equations for R and z

Hamiltonian/Energy

$$\boxed{H(\vec{q},\vec{p},t) := \vec{p}\cdot\dot{\vec{q}} - \mathcal{L}(\vec{q},\dot{\vec{q}},t)}$$

$$\vec{q} = \begin{cases} R \\ \theta \\ z \end{cases} \qquad \dot{\vec{q}} = \begin{cases} \dot{R} \\ \dot{\theta} \\ \dot{z} \end{cases} \qquad \vec{p} = \begin{cases} \dfrac{\partial\mathcal{L}}{\partial\dot{R}} = \dot{R} \\ \dfrac{\partial\mathcal{L}}{\partial\dot{\theta}} = R^2\dot{\theta} \\ \dfrac{\partial\mathcal{L}}{\partial\dot{z}} = \dot{z} \end{cases}$$

$$\boxed{p_\theta = R^2\dot{\theta} = L_z}$$

$$\boxed{H(R,\dot{R},\theta,\dot{\theta},z,\dot{z}) = \frac{1}{2}\left(\dot{R}^2 + R^2\dot{\theta}^2 + \dot{z}^2\right) + \phi(R,z) = E}$$

E (Energy) is conserved

as $\mathcal{L}$ is time independant

ϕ

Effective potential

with $L_z = R^2\dot{\theta}$

$L_z^2 = R^4\dot{\theta}^2$

Definition

$$\phi_{\mathrm{eff}}(R,z) = \phi(R,z) + \frac{L_z^2}{2R^2}$$

$$\begin{cases} \dfrac{\partial \phi_{\mathrm{eff}}}{\partial R} = \dfrac{\partial \phi}{\partial R} - \dfrac{L_z^2}{R^3} \equiv \dfrac{\partial \phi}{\partial R} - R\dot{\theta}^2 \\[2ex] \dfrac{\partial \phi_{\mathrm{eff}}}{\partial z} = \dfrac{\partial \phi}{\partial z} \end{cases}$$

The equations of motion ① + ③ becomes

$$\boxed{\begin{cases} \ddot{R} = -\dfrac{\partial \phi_{\mathrm{eff}}}{\partial R}(R,z) \\[2ex] \ddot{z} = -\dfrac{\partial \phi_{\mathrm{eff}}}{\partial z}(R,z) \end{cases}}$$

The 3D motion of a star in an axisymmetric potential is reduced to a 2D motion in the meridian plane (R,z)

phase space 6D → 4D

Hamiltonian in the meridian plane

Those equations of motion may be derived from the lagrangian

$$\mathcal{L}(R, \dot{R}, z, \dot{z}) = \frac{1}{2}\dot{R}^2 + \frac{1}{2}\dot{z}^2 - \phi_{\text{eff}}(R, z)$$

The corresponding Hamiltonian writes $(p_R = \dot{R}, \; p_z = \dot{z})$

$$\boxed{\begin{aligned} H(R, \dot{R}, z, \dot{z}) &= \frac{1}{2}\left(\dot{R}^2 + \dot{z}^2\right) + \phi_{\text{eff}}(R, z) \\ &\equiv \frac{1}{2}\left(\dot{R}^2 + \dot{z}^2\right) + \phi(R, z) + \frac{L_z^2}{2R^2} \\ &\equiv \underbrace{\frac{1}{2}\left(\dot{R}^2 + \dot{z}^2\right)}_{\text{kinetic energy in the orbital plane}} + \phi(R, z) + \frac{1}{2}R^2\dot{\theta}^2 = \underbrace{E}_{\text{orbit's total energy}} \end{aligned}}$$

E is conserved as ϕ_{eff} is time independent

Illustration in the $z = 0$ plane

for $R \to \infty$

$$\Phi_{\rm eff} = \Phi + \underbrace{\frac{L_z^2}{2}\frac{1}{R^2}}_{\to 0} \sim \Phi$$

for $R \to 0$

$$\Phi_{\rm eff} = \underbrace{\Phi}_{\text{bounded}} + \underbrace{\frac{L_z^2}{2}\frac{1}{R^2}}_{\text{diverges}} \sim \frac{1}{R^2}$$

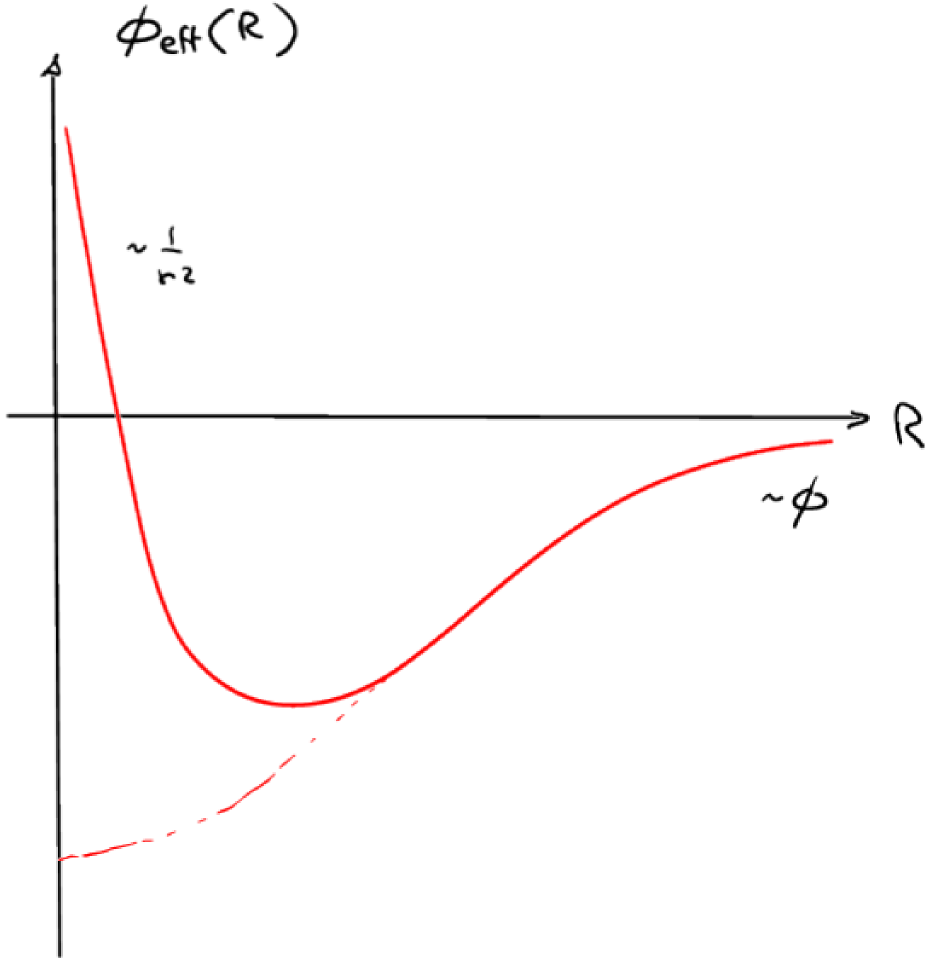


Illustration in the $z = 0$ plane

$$\boxed{E = \frac{1}{2}\dot{R}^2 + \phi_{\rm eff}(R)}$$

4 cases

① $E > \phi_{\rm eff}(\infty)$ except at $E = \phi_{\rm eff}$

$\dot{R} \neq 0$ unbound orbits

② $E < \min(\phi_{\rm eff}(R))$ $\dot{R}^2 < 0$

impossible

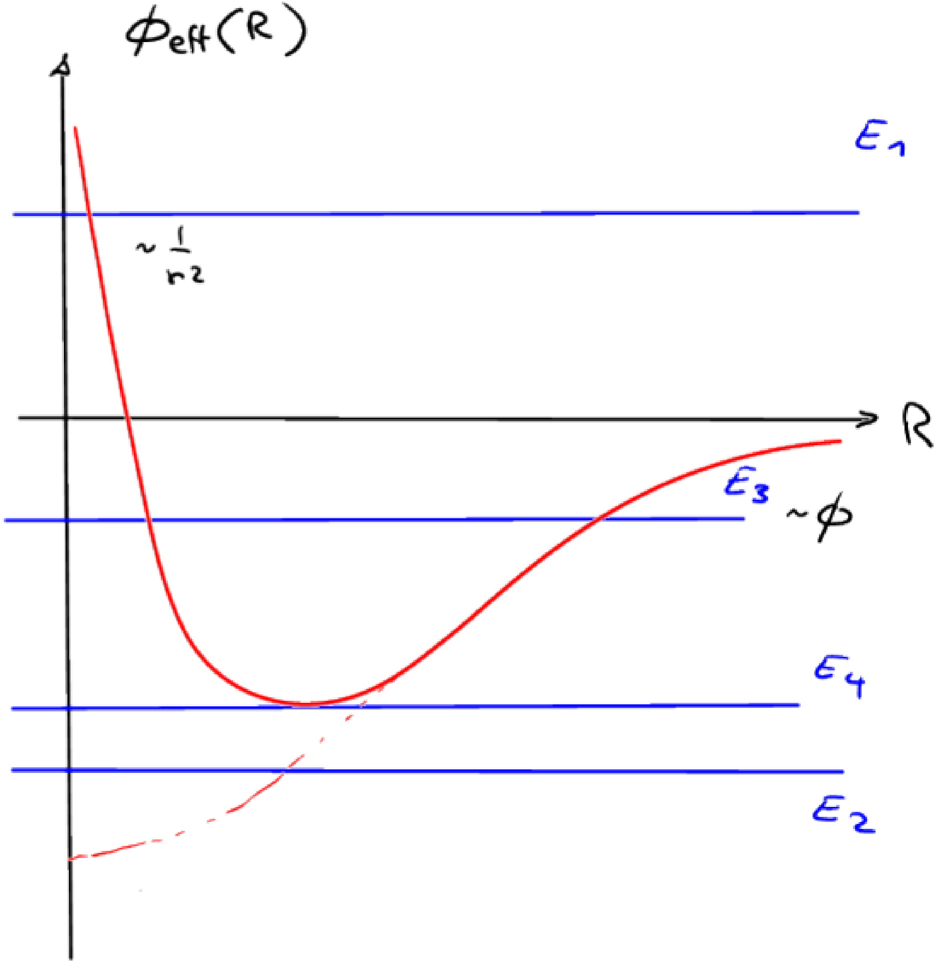
③ $\min(\phi_{\rm eff}(R)) < E < \phi_{\rm eff}(\infty)$

orbit bounded between

R_1 and R_2 (where $\dot{R} = 0$)

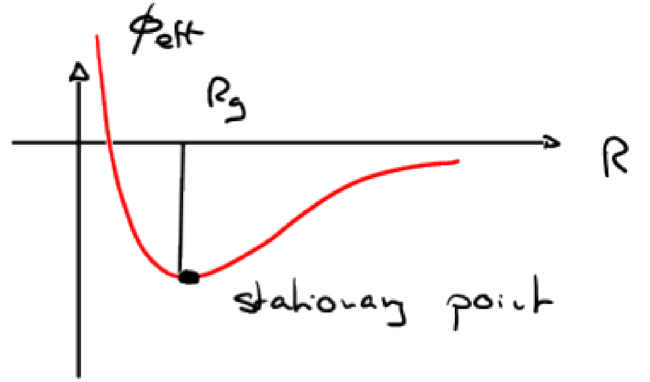
④ $E = \min(\phi_{\rm eff}(R))$ (stationary point)

$R_1 = R_2$ (circular orbit)



Stationary point

$\dot{R} = \ddot{R} = 0 \qquad \dot{z} = \ddot{z} = 0$



from

$$\begin{cases} \ddot{R} = -\dfrac{\partial \Phi_{\rm eff}}{\partial R}(R,z) \\ \ddot{z} = -\dfrac{\partial \Phi_{\rm eff}}{\partial z}(R,z) \end{cases}$$

$$\begin{cases} \dfrac{\partial \Phi_{\rm eff}}{\partial R} = 0 = \dfrac{\partial \Phi}{\partial R} - \dfrac{L_z^2}{R^3} = 0 \\ \dfrac{\partial \Phi_{\rm eff}}{\partial z} = 0 = \dfrac{\partial \Phi}{\partial z} = 0 \end{cases} \quad \longrightarrow \quad \text{by symmetry where } z = 0$$

$$R_g \text{ such that } \left.\frac{\partial \Phi}{\partial R}\right)_{R_g,0} = \frac{L_z^2}{R_g^3} = R_g \dot{\theta}^2 \overset{V_\theta = R\dot{\theta}}{=} \frac{V_\theta^2(R_g)}{R_g} \underset{V_c^2 = R \left.\frac{\partial \Phi}{\partial R}\right)_{R,0}}{=} \frac{V_c^2(R_g)}{R_g}$$

R_g : guiding center

The stationary point in R_g in the meridional plane corresponds to a circular orbit

Circular orbits

angular speed $\dot{\theta} = \frac{L_z}{R_g^2}$

angular momentum L_z

energy $\phi_{eff} + \frac{L_z^2}{2R_g}$

Note

For a given angular momentum L_z, the circular orbit is the one that minimize the energy.

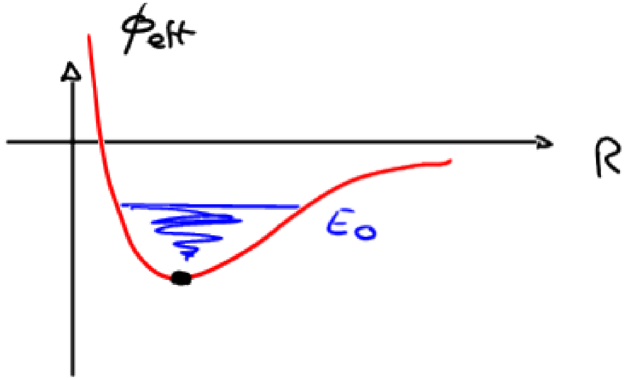
EXERCICE

① $E_0 = \frac{1}{2}\dot{R}^2 + \frac{1}{2}\dot{z}^2 + \phi + \frac{L_z^2}{2R}$

② Dissipate energy $\rightsquigarrow 0$ $L_z = cte$

$\dot{z} \searrow$ $\dot{R} \searrow$

③ circular orbit



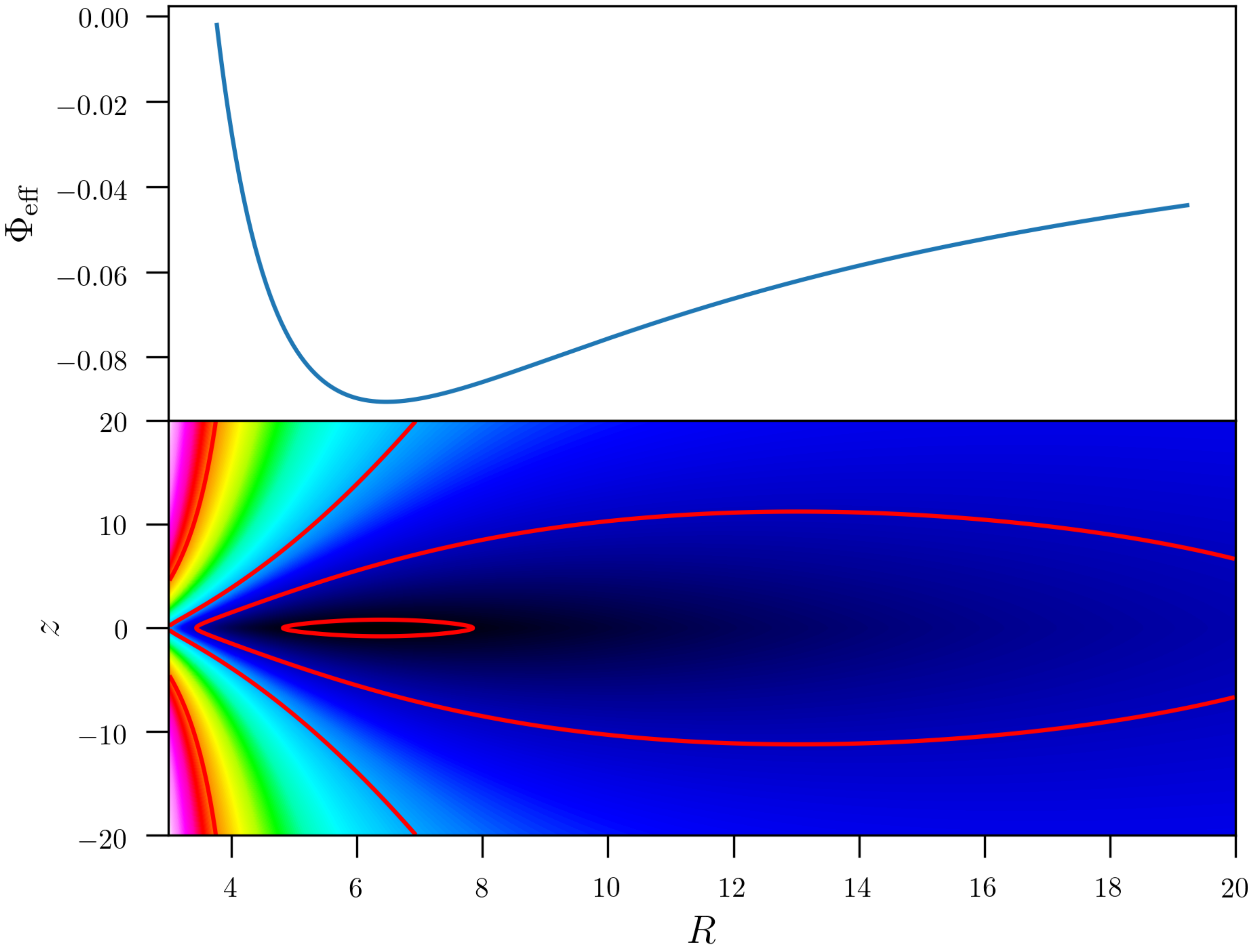
Examples

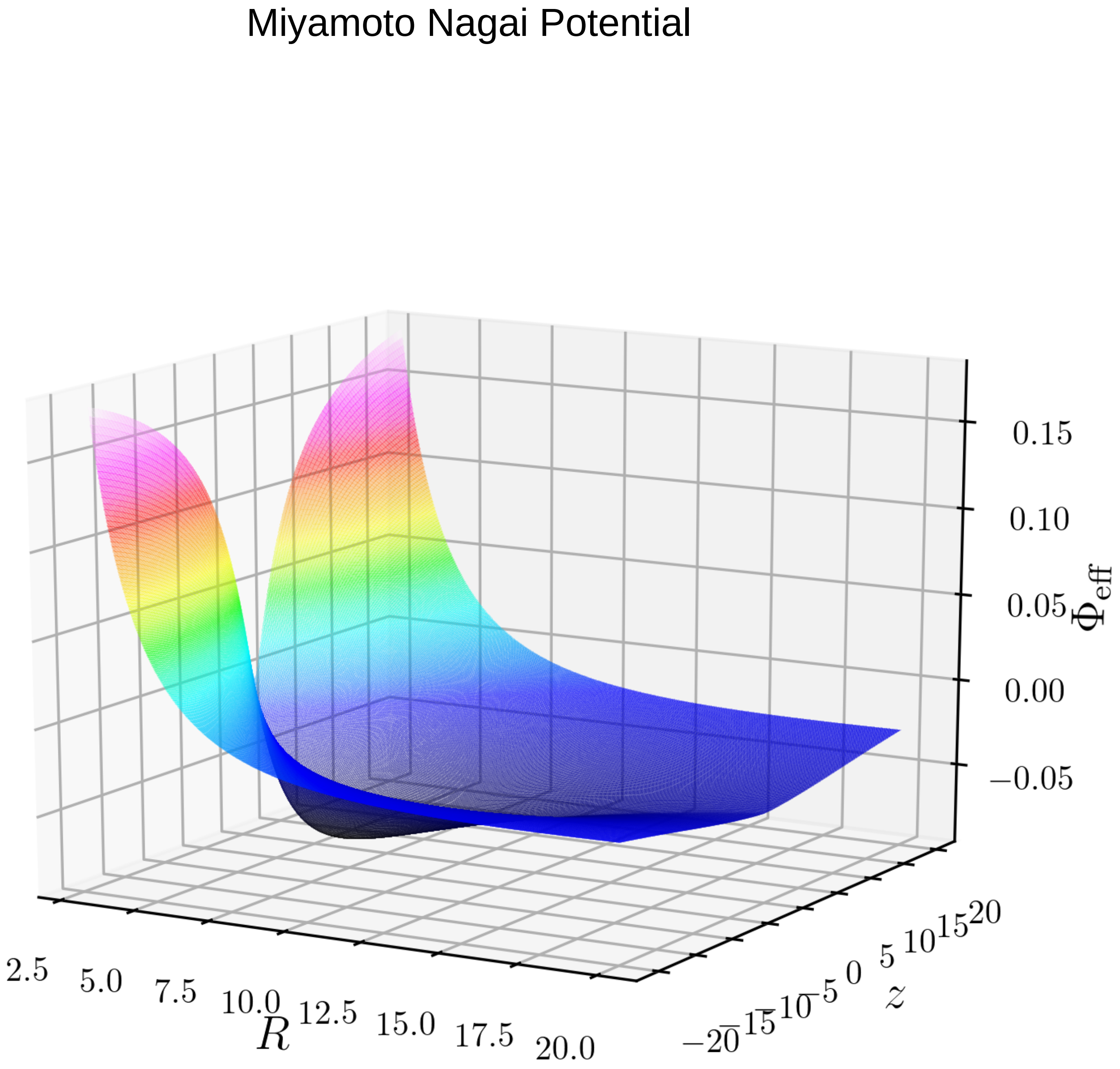
① Miyamoto - Nagai potential

$$\phi(R, z) = - \frac{GM}{\sqrt{R^2 + \left(a + \sqrt{z^2+b^2}\right)^2}}$$

$$\phi_{eff}(R, z=0) = - \frac{GM}{\sqrt{R^2 + (a+b)^2}} + \frac{L_z^2}{2R^2}$$

Miyamoto Nagai Potential




Miyamoto Nagai Potential
0.15
0.10
0.05
0.00
−0.05
Φ_{eff}
2.5
5.0
7.5
10.0
12.5
15.0
17.5
20.0
R
−20
−15
−10
−5
0
5
10
15
20
z

Examples

① Miyamoto - Nagai potential

$$\phi(R, z) = -\frac{GM}{\sqrt{R^2 + \left(a + \sqrt{z^2+b^2}\right)^2}}$$

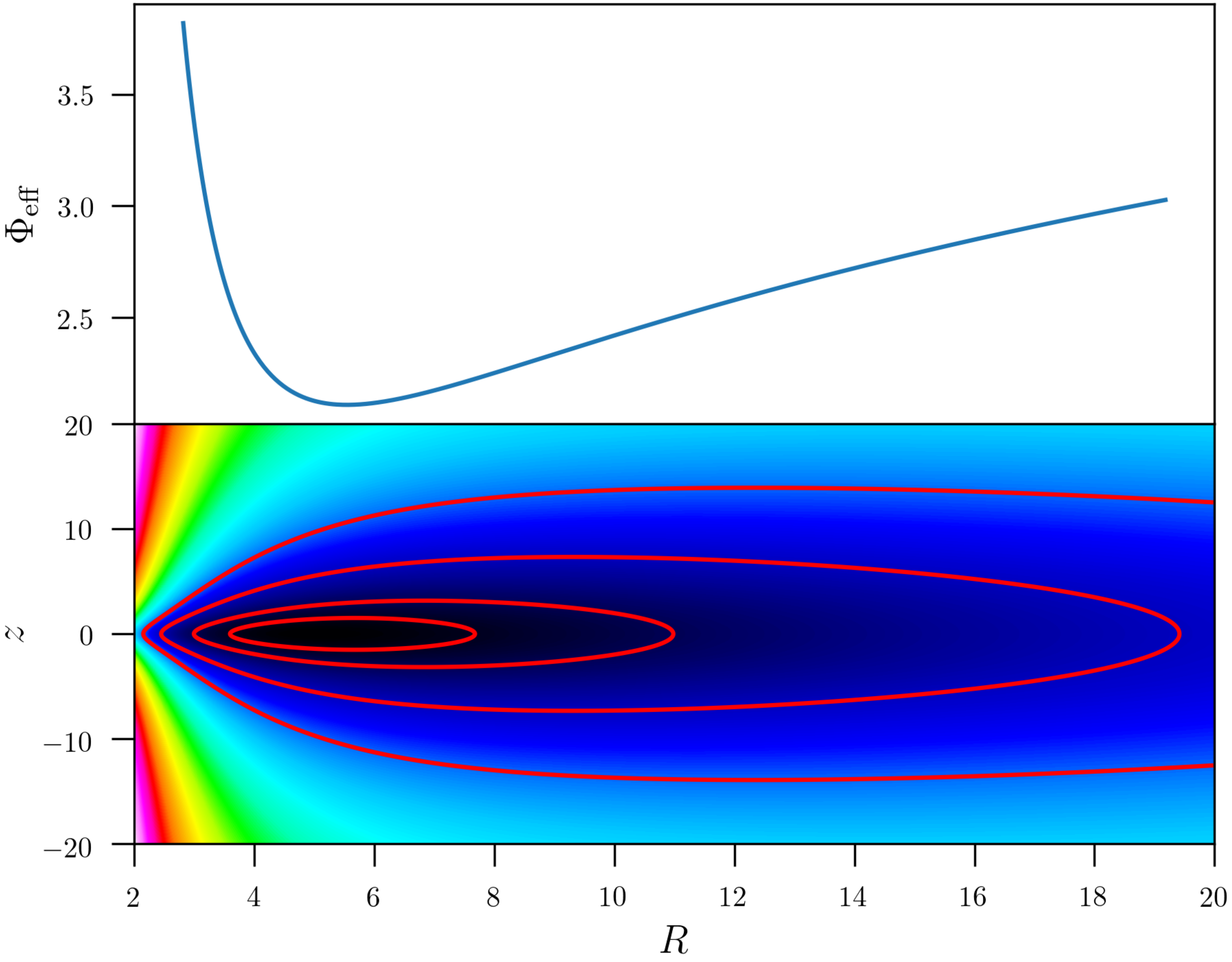
$$\phi_{eff}(R, z=0) = -\frac{GM}{\sqrt{R^2 + (a+b)^2}} + \frac{L_z^2}{2R^2}$$

② Logarithmic potential

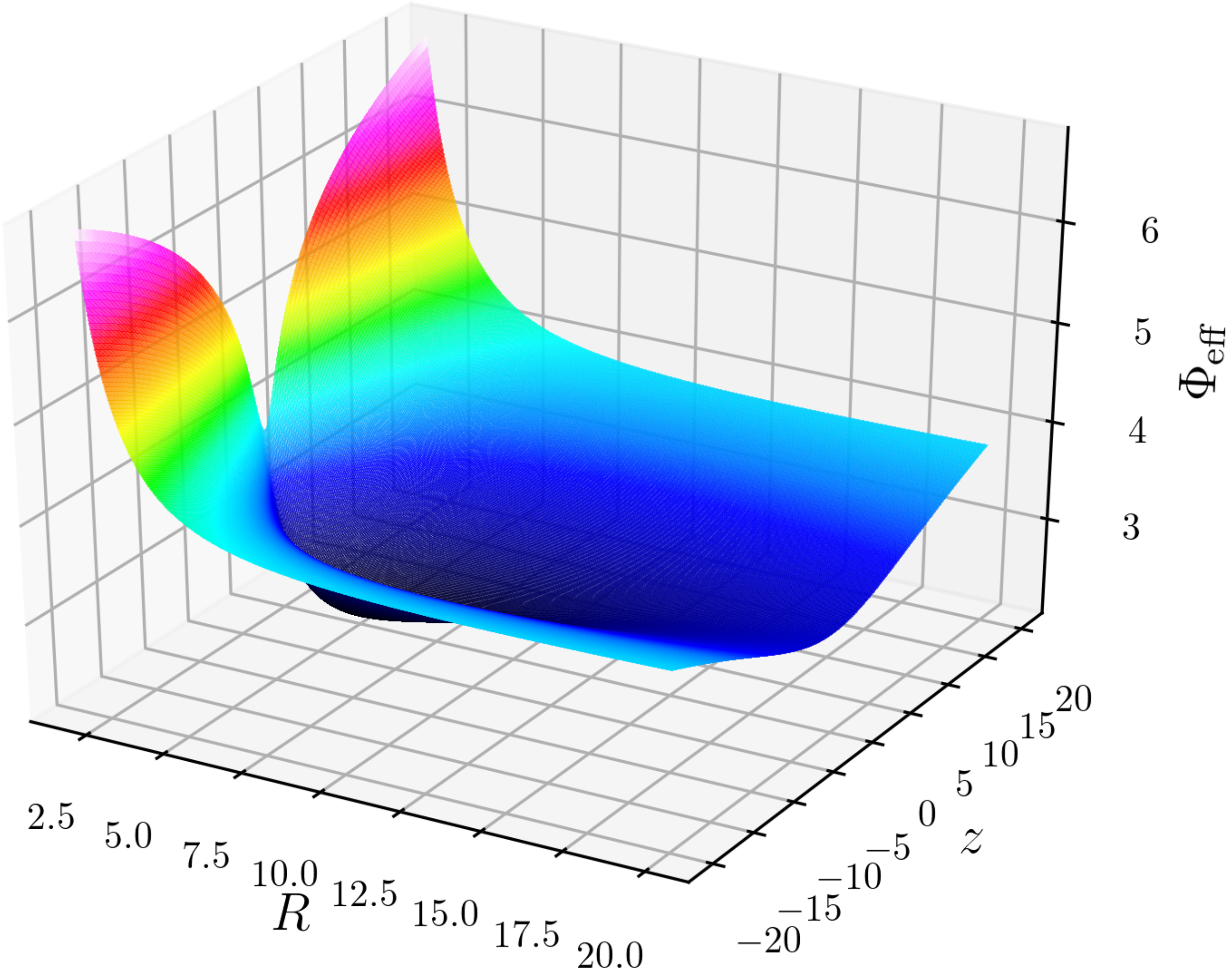
$$\phi(R, z) = \frac{1}{2} V_0^2 \ln\left(R^2 + \frac{z^2}{q^2}\right)$$

$$\phi_{eff}(R, z=0) = \frac{1}{2} V_0^2 \ln\left(R^2\right) + \frac{L_z^2}{2R^2}$$

Logarithmic Potential



Logarithmic Potential



General solutions for the equations of motion

$$\boxed{\begin{cases} \ddot{R} = -\dfrac{\partial \Phi_{\text{eff}}}{\partial R}(R,z) \\[2ex] \ddot{z} = -\dfrac{\partial \Phi_{\text{eff}}}{\partial z}(R,z) \end{cases}}$$

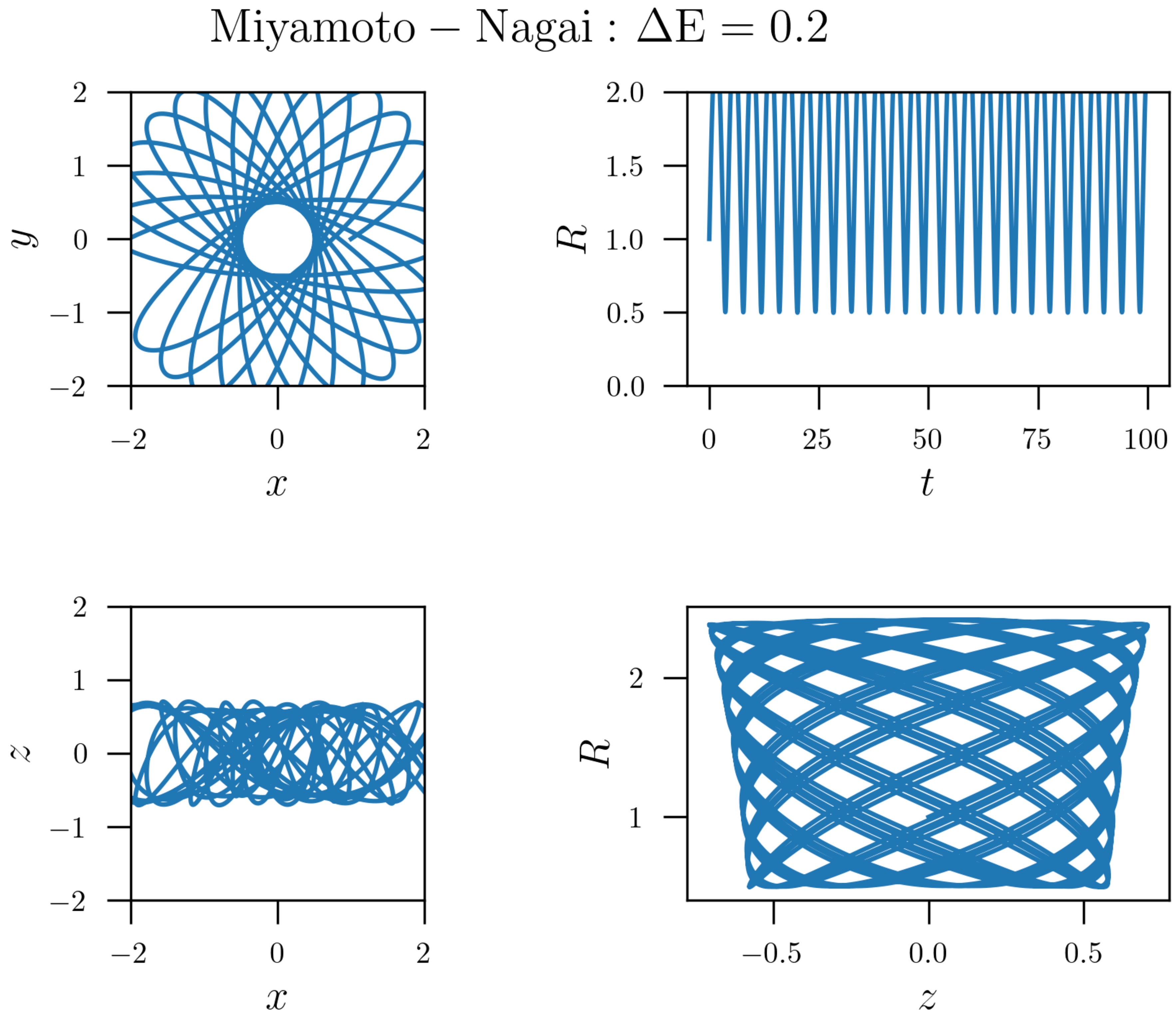
no simple solutions ☹

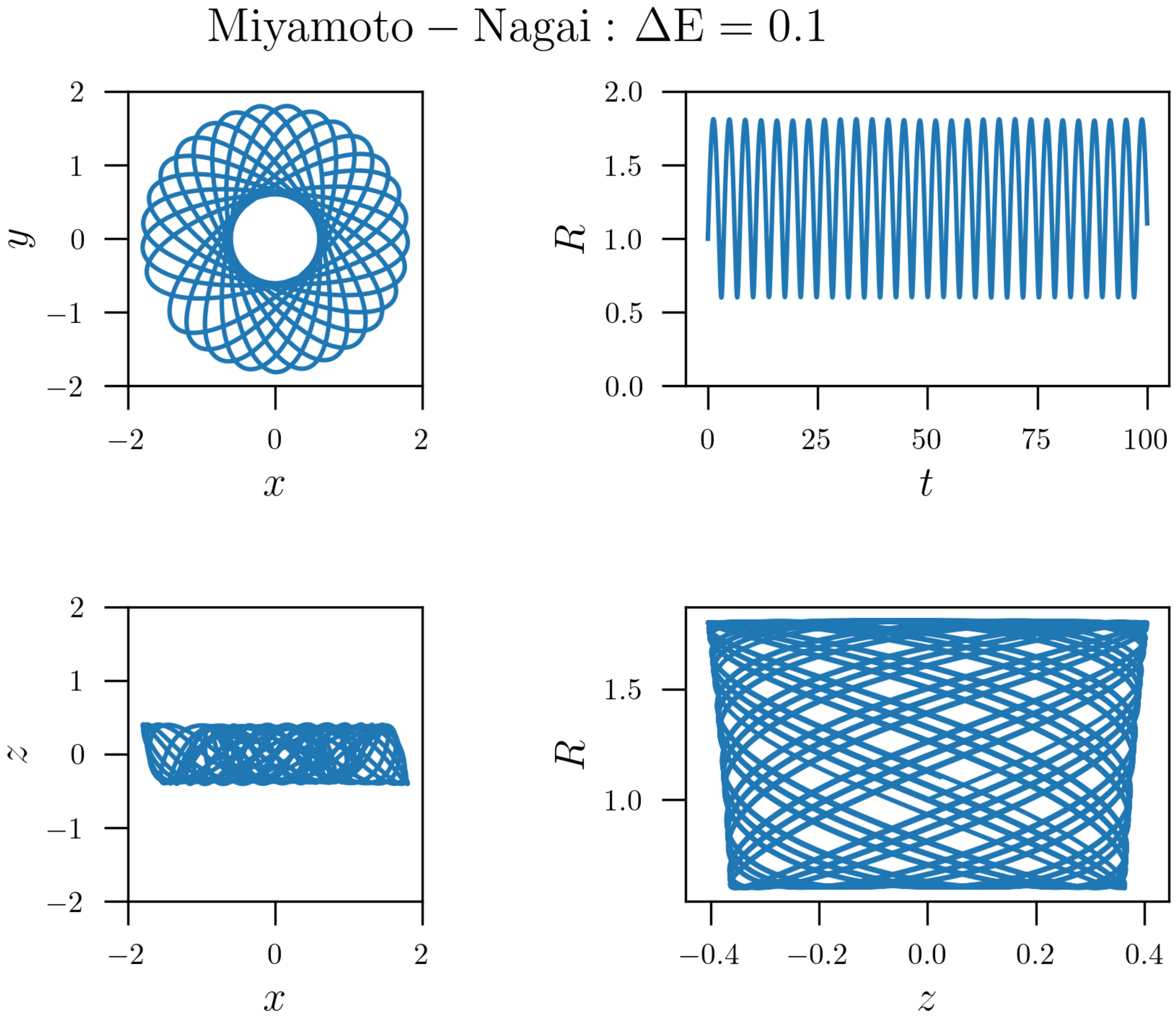
need numerical integration

Hamilton's Equations

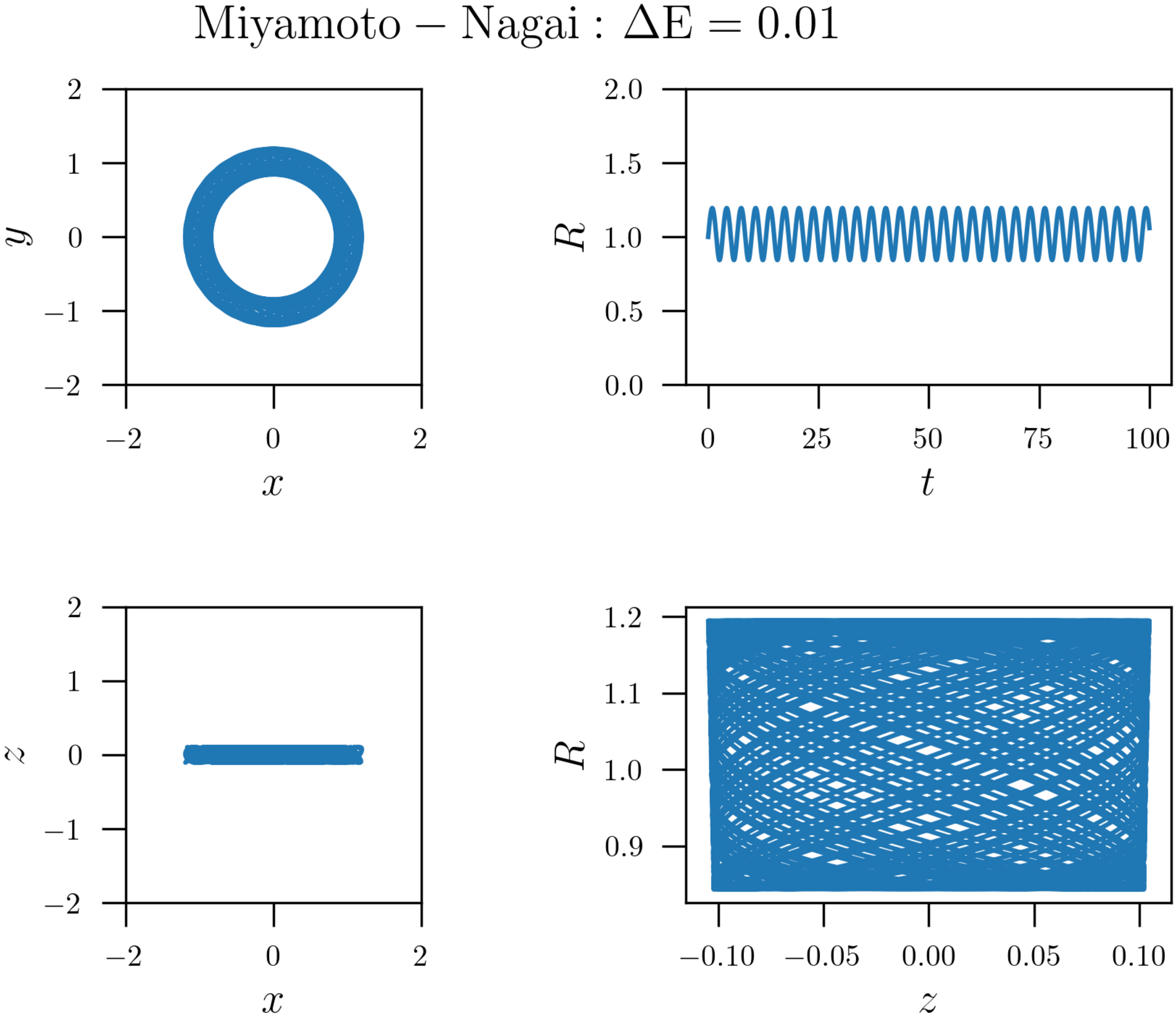
$$\vec{q} = \begin{cases} R \\ z \end{cases} \qquad \dot{\vec{q}} = \begin{cases} \dot{R} \\ \dot{z} \end{cases} \qquad \vec{p} = \begin{cases} \dot{R} \\ \dot{z} \end{cases}$$

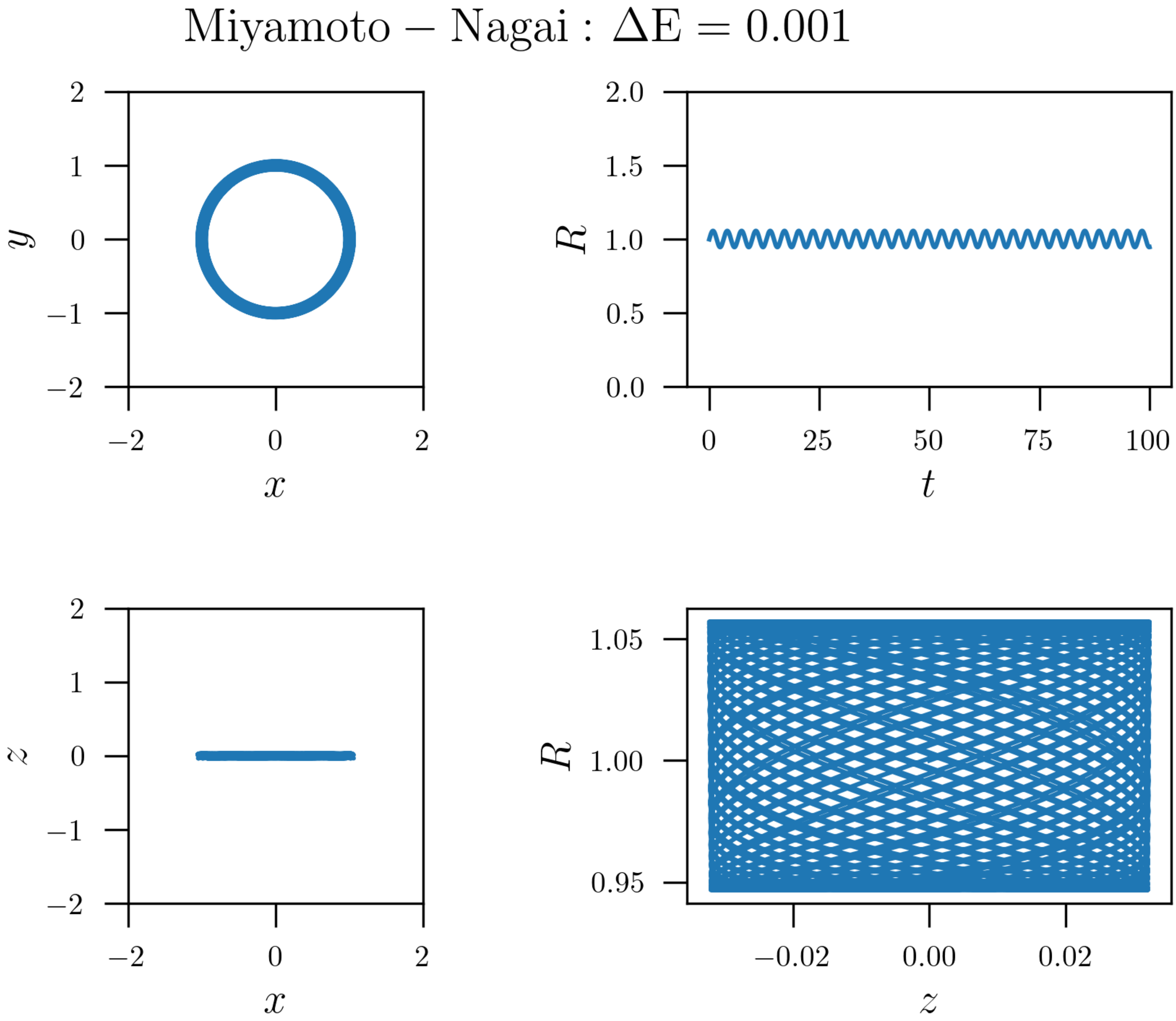
$$\begin{cases} \dot{q}_R = p_R & \equiv \dot{R} \\[1ex] \dot{q}_z = p_z & \equiv \dot{z} \\[1ex] \dot{p}_R = -\dfrac{\partial \Phi_{\text{eff}}}{\partial q_R}(q_R, q_z) & \equiv -\dfrac{\partial \Phi_{\text{eff}}}{\partial R}(R,z) \\[2ex] \dot{p}_z = -\dfrac{\partial \Phi_{\text{eff}}}{\partial q_z}(q_R, q_z) & \equiv -\dfrac{\partial \Phi_{\text{eff}}}{\partial z}(R,z) \end{cases}$$

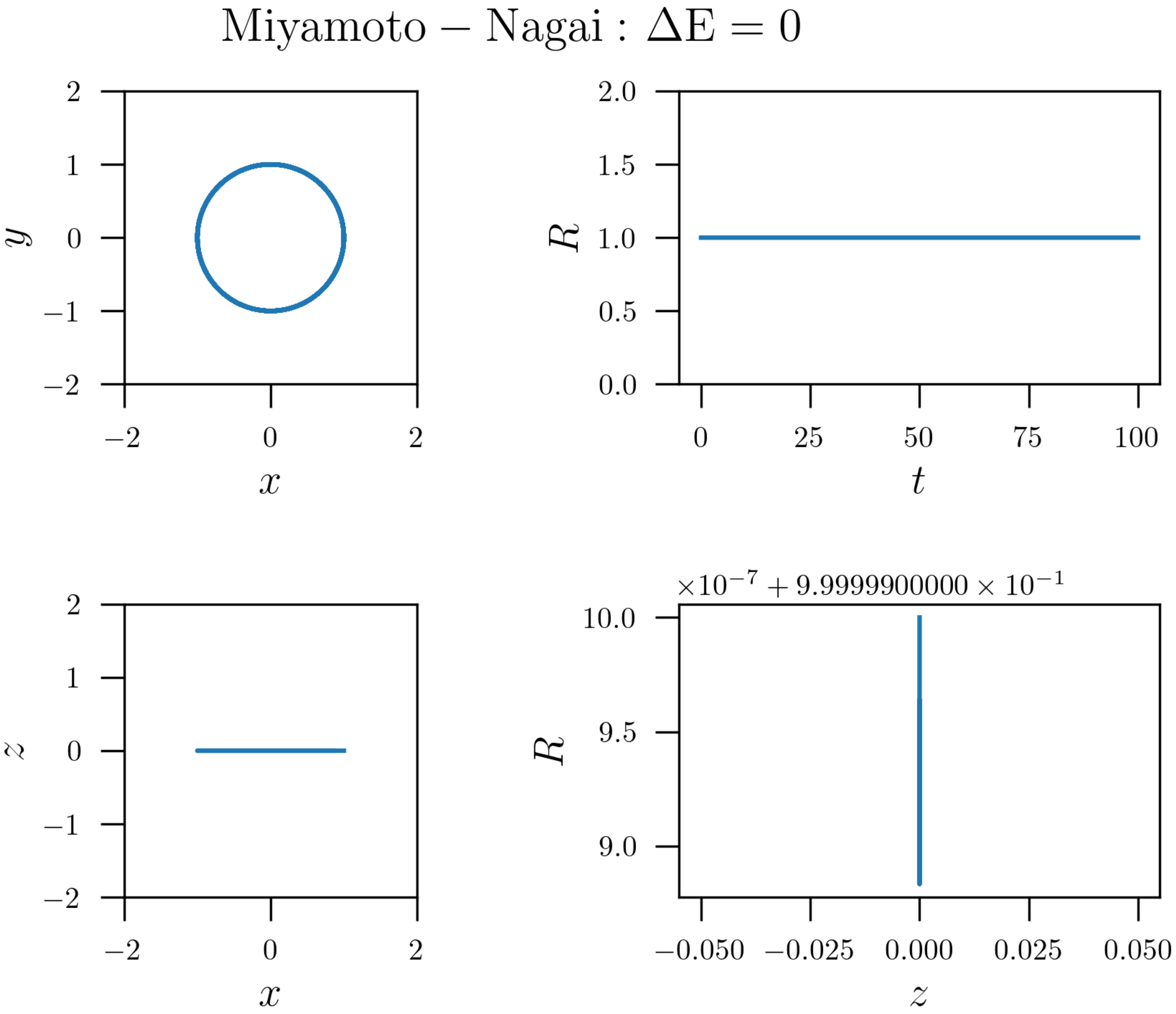

Miyamoto − Nagai : ΔE = 0.2
y
x
2
1
0
−1
−2
−2
0
2
R
t
2.0
1.5
1.0
0.5
0.0
0
25
50
75
100
z
x
2
1
0
−1
−2
−2
0
2
R
z
2
1
−0.5
0.0
0.5



Miyamoto – Nagai : $\Delta \mathrm{E} = 0.1$


Miyamoto – Nagai : ΔE = 0.01
y
x
2
1
0
−1
−2
−2
0
2
R
t
2.0
1.5
1.0
0.5
0.0
0
25
50
75
100
z
x
2
1
0
−1
−2
−2
0
2
R
z
1.2
1.1
1.0
0.9
−0.10
−0.05
0.00
0.05
0.10


$\mathrm{Miyamoto-Nagai:\Delta E=0.001}$
y
x
2
1
0
−1
−2
−2
0
2
R
t
2.0
1.5
1.0
0.5
0.0
0
25
50
75
100
z
x
2
1
0
−1
−2
−2
0
2
R
z
1.05
1.00
0.95
−0.02
0.00
0.02

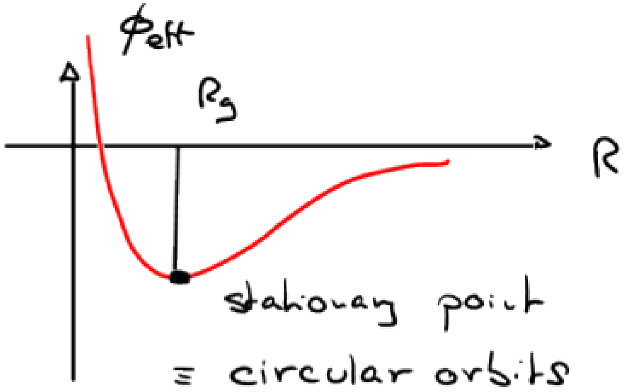

Miyamoto – Nagai : ΔE = 0
y
x
R
t
z
x
R
z
×10^{−7} + 9.9999900000 × 10^{−1}
2
1
0
−1
−2
2.0
1.5
1.0
0.5
0.0
0
25
50
75
100
10.0
9.5
9.0
−0.050
−0.025
0.000
0.025
0.050

Stellar orbits

Nearly circular orbits

Nearly circular orbits

From the previous study of orbits in axisymmetric potentials



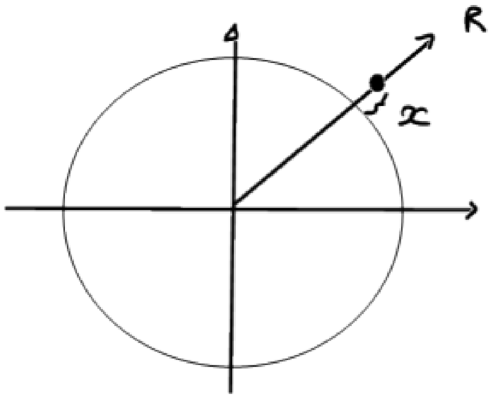
Goal Study orbits in the neighbourhood of circular orbits

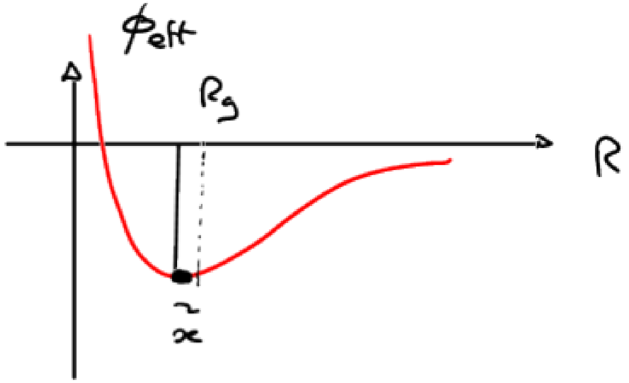
Justifications In a disk galaxy, many stars are found in nearly circular orbits

Recall R_g : the guiding center

$$\boxed{R_g \text{ such that } \left.\frac{\partial \phi}{\partial R}\right)_{R_g,0} = \frac{L_z^2}{R_g^3} = R_g \dot{\theta}^2}$$

We define $x := R - R_g$ the distance to the guiding center R_g





Taylor expansion of ϕ_{eff} around $R = R_g$, $z = 0$

$$\phi_{eff}(R,z) \cong \phi_{eff}(R_g,0) + \underbrace{\frac{\partial \phi_{eff}(R_g,0)}{\partial R}}_{=0 \quad \text{min}} (R-R_g) + \underbrace{\frac{\partial \phi_{eff}(R_g,0)}{\partial z}}_{=0 \quad \text{sym.}} z$$

$$+ \frac{1}{2} \frac{\partial^2 \phi_{eff}(R_g,0)}{\partial R^2} (R-R_g)^2 + \frac{1}{2} \frac{\partial^2 \phi_{eff}(R_g,0)}{\partial z^2} z^2$$

$$+ \frac{1}{2} \underbrace{\frac{\partial^2 \phi_{eff}}{\partial z \, \partial R}}_{=0} (R_g,0) (R-R_g) z + O(((R-R_g)z)^3)$$

$= 0$ $\phi_{eff}(R,z)$ must be sym. with respect to $z = 0$

$$\boxed{\phi_{\text{eff}}(R, z) \simeq \phi_{\text{eff}}(R_g, 0) + \frac{1}{2}\frac{\partial^2 \phi_{\text{eff}}}{\partial R^2}(R_g, 0)\, x^2 + \frac{1}{2}\frac{\partial^2 \phi_{\text{eff}}}{\partial z^2}(R_g, 0)\, z^2}$$

Definition

$$\begin{cases} \kappa^2(R_g) = \left(\dfrac{\partial^2 \phi_{\text{eff}}}{\partial R^2}\right)_{(R_g, 0)} \\[2ex] \nu^2(R_g) = \left(\dfrac{\partial^2 \phi_{\text{eff}}}{\partial z^2}\right)_{(R_g, 0)} \end{cases}$$

$$[\phi] = \left(\frac{m}{s}\right)^2$$

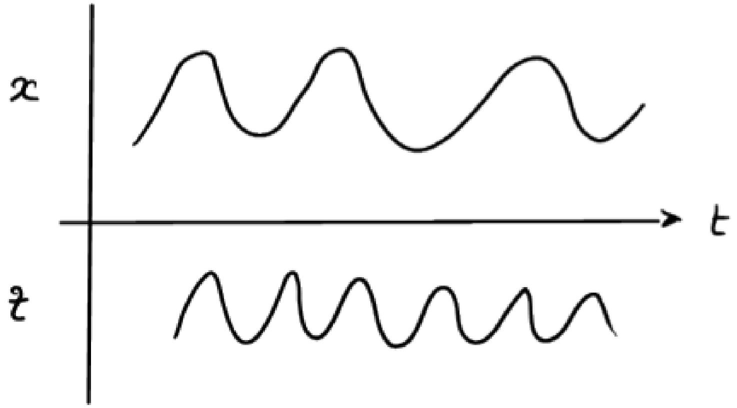
$$\left[\left(\frac{\partial^2 \phi}{\partial R^2}\right)^{\frac{1}{2}}\right] \equiv \left[\left(\frac{\partial^2 \phi}{\partial z^2}\right)^{\frac{1}{2}}\right] = \frac{1}{s}$$

frequency

Equations of motion near R_g

$$\begin{cases} \ddot{R} = -\dfrac{\partial \phi_{\text{eff}}}{\partial R}(R, z) \\[2ex] \ddot{z} = -\dfrac{\partial \phi_{\text{eff}}}{\partial z}(R, z) \end{cases} \Rightarrow \boxed{\begin{cases} \ddot{x} = -\kappa^2(R_g)\, x \\[2ex] \ddot{z} = -\nu^2(R_g)\, z \end{cases}}$$

$$\begin{cases} \ddot{x} = -\varkappa^2(R_g)\, x \\ \ddot{z} = -\nu^2(R_g)\, z \end{cases}$$



Two decoupled harmonic oscillators with frequencies $\varkappa$ and ν

$\varkappa$: epicycle (radial) frequency

ν : vertical frequency

Expressions of κ and ν from the total potential

$$\kappa^2(R_g) = \left.\frac{\partial^2\phi_{\rm eff}}{\partial R^2}\right)_{(R_g,0)} = \left.\frac{\partial^2\phi}{\partial R^2}\right)_{(R_g,0)} + 3\,\frac{L_z^2}{R_g^4}$$

$$L_z^2 = V_c^2 R_g^2 = R_g^3 \left.\frac{\partial\phi}{\partial R}\right)_{R_g}$$

$$= \left.\frac{\partial^2\phi}{\partial R^2}\right)_{(R_g,0)} + \frac{3}{R_g}\left.\frac{\partial\phi}{\partial R}\right)_{(R_g,0)}$$

circ. frequency $\Omega^2 = \frac{1}{R_g}\left.\frac{\partial\phi}{\partial R}\right)_{R_g}$

$$= \left.\frac{\partial^2\phi}{\partial R^2}\right)_{(R_g,0)} + 3\,\Omega^2$$

$\Vert$

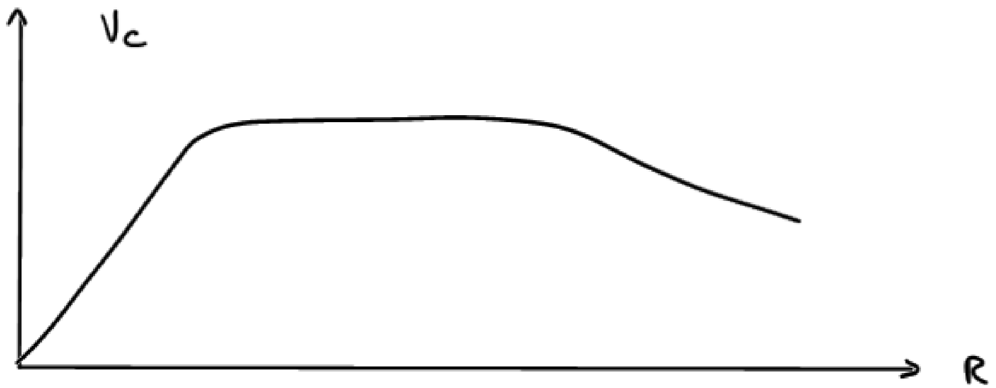
$\Omega^2 = \frac{V_c^2}{R^2}$

$$= \left(R\,\frac{\partial(\Omega^2)}{\partial R} + 4\,\Omega^2\right)_{(R_g,0)}$$

$$= \left(\frac{1}{R}\,\frac{\partial(V_c^2)}{\partial R} + 2\,\Omega^2\right)_{(R_g,0)} = \left(\frac{1}{R}\,\frac{\partial(V_c^2)}{\partial R} + 2\,\frac{V_c^2}{R^2}\right)_{(R_g,0)}$$

$$\nu^2(R_g) = \left.\frac{\partial^2\phi_{\rm eff}}{\partial z^2}\right)_{(R_g,0)} = \left.\frac{\partial^2\phi}{\partial z^2}\right)_{(R_g,0)}$$

Note : $\varkappa$ depends only on V_c



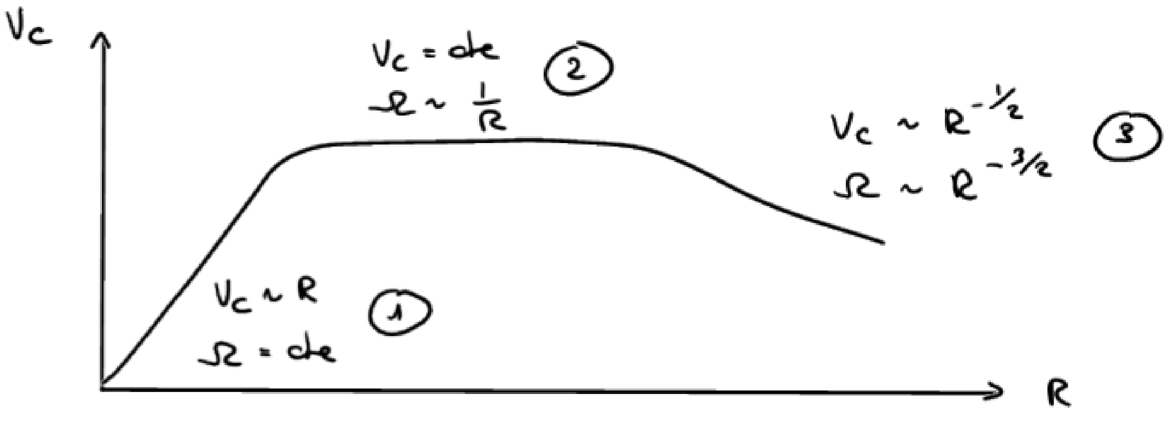
$\varkappa$ obtained by derivating V_c^2

Periods :

$$\begin{cases} \text{radial} & T_R := \dfrac{2\pi}{\varkappa} \\ \text{vertical} & T_z := \dfrac{2\pi}{\nu} \\ \text{azimuthal} & T_\theta := \dfrac{2\pi}{\Omega} \end{cases}$$

Radial dependency of $\varkappa$, γ for a typical galaxy

$$\boxed{\Omega = \frac{V_c}{R}}$$



- ① near the center

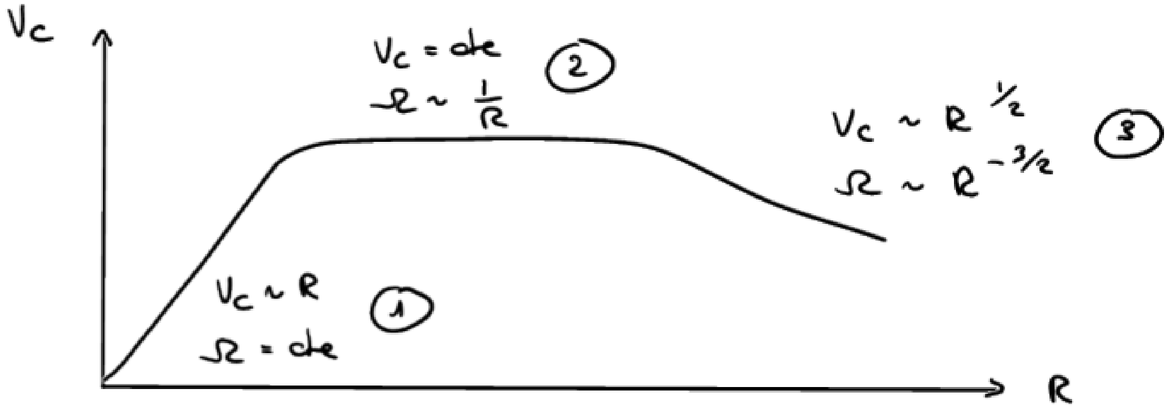
$$V_c \sim R \quad (\text{rigid rotation}) \quad \Rightarrow \quad \Omega = cte$$

$$\varkappa^2 = R \frac{d}{dR}(\Omega^2) + 4\Omega^2 \quad \Rightarrow \quad \varkappa^2 = 4\Omega^2$$

$$\boxed{\varkappa \sim 2\Omega}$$

Radial dependency of κ, ν for a typical galaxy

$$\boxed{\Omega = \frac{V_c}{R}}$$



• ② flat rotation part

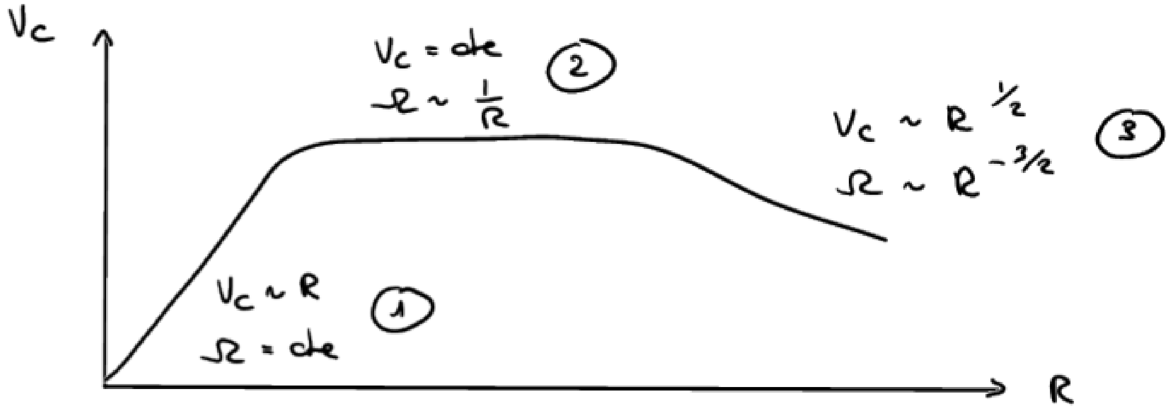
$$V_c = \text{cte} \quad \rightarrow \quad \Omega \sim \frac{1}{R}$$

$$\kappa^2 = \frac{1}{R}\frac{\partial}{\partial R}(V_c^2) + 2\Omega^2 \quad \rightarrow \quad \kappa^2 = 2\Omega^2$$

$$\boxed{\kappa \sim \sqrt{2}\,\Omega}$$

Radial dependency of $\varkappa$, ν for a typical galaxy

$$\boxed{\Omega = \frac{V_c}{R}}$$



- ③ further out

$V_c \sim R^{-1/2}$ (Keplerian decrease)

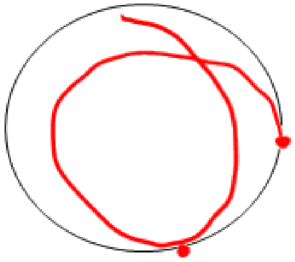
$$\Omega = \frac{V_c}{R} \sim R^{-3/2} \qquad \Omega^2 \sim R^{-3} \qquad \frac{\partial}{\partial R}(\Omega^2) = -3\,\frac{\Omega^2}{R}$$

$$\varkappa^2 = \underbrace{R\,\frac{\partial}{\partial R}(\Omega^2)}_{-3\Omega^2} + 4\Omega^2$$

$$\boxed{\varkappa = \Omega}$$

Thus, in general

$$\boxed{\Omega \leqslant \varkappa \leqslant 2\,\Omega}$$



Orbital motions

$$\begin{cases} \ddot{x} = -\kappa^2(R_g)\, x \\ \ddot{z} = -\nu^2(R_g)\, z \end{cases}$$

$+ \quad R^2\dot{\theta} = L_z$

Solutions

① motion in z

$$z(t) = Z\cos(\nu t + \xi)$$

② motion in x

$$x(t) = X\cos(\kappa t + \alpha)$$

Note valid only for small oscillations

as long as $\nu^2 = \frac{\partial^2\phi}{\partial z^2} \simeq$ cte

ie $\rho_{disk} \simeq$ cte $\quad \left(\nu^2 = \frac{\partial^2\phi}{\partial z^2} = 4\pi G\rho\right)$

$\to$ $z <$ disk scale lenght

~ 300 pc

③ motion in Θ

$$\boxed{L_z = R^2\dot{\Theta}}$$

$$\Theta(t) = L_z \int_{t_0}^{t} dt' \, \frac{1}{R^2(t')} = L_z \int_{t_0}^{t} dt' \, \frac{1}{\left(R_g + x(t')\right)^2}$$

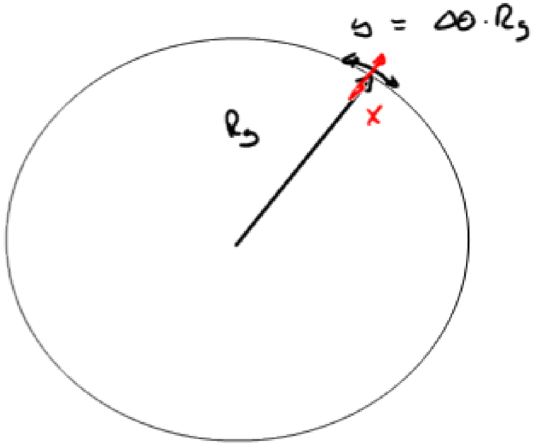
$$= \frac{L_z}{R_g^2} \int_{t_0}^{t} dt' \, \frac{1}{\left(\frac{x}{R_g} + 1\right)^2} \overset{\text{Taylor}}{\simeq} \Omega_g \int_{t_0}^{t} dt' \left(1 - \frac{2x(t)}{R_g}\right)$$

$$\Omega_g = \frac{L_z}{R_g^2}$$

introducing $x(t) = X\cos(\kappa t + \alpha)$

$$\boxed{\Theta(t) = \underbrace{\Omega_g \cdot t}_{\substack{\text{motion of the}\\ \text{guiding center}\\ \text{along the circular}\\ \text{orbit}}} - \underbrace{\frac{2\Omega_g}{\kappa}\frac{X}{R_g}\sin(\kappa t + \alpha)}_{\text{oscillations}} + \Theta_0}$$

New carthesian system x, y, z with an origin that follows the guiding center



$$\begin{cases} R(t) = R_g \\ \theta(t) = \Omega_g t + \theta_0 \end{cases}$$

Then, from

$$\theta(t) = \Omega_g \cdot t \underbrace{- \frac{2\Omega_g}{\varkappa} \frac{X}{R_g} \sin(\varkappa t + \alpha)}_{\Delta\theta = \frac{y}{R_g}} + \theta_0$$

$$y = -\frac{2\Omega_g}{\varkappa} X \sin(\varkappa t + \alpha)$$

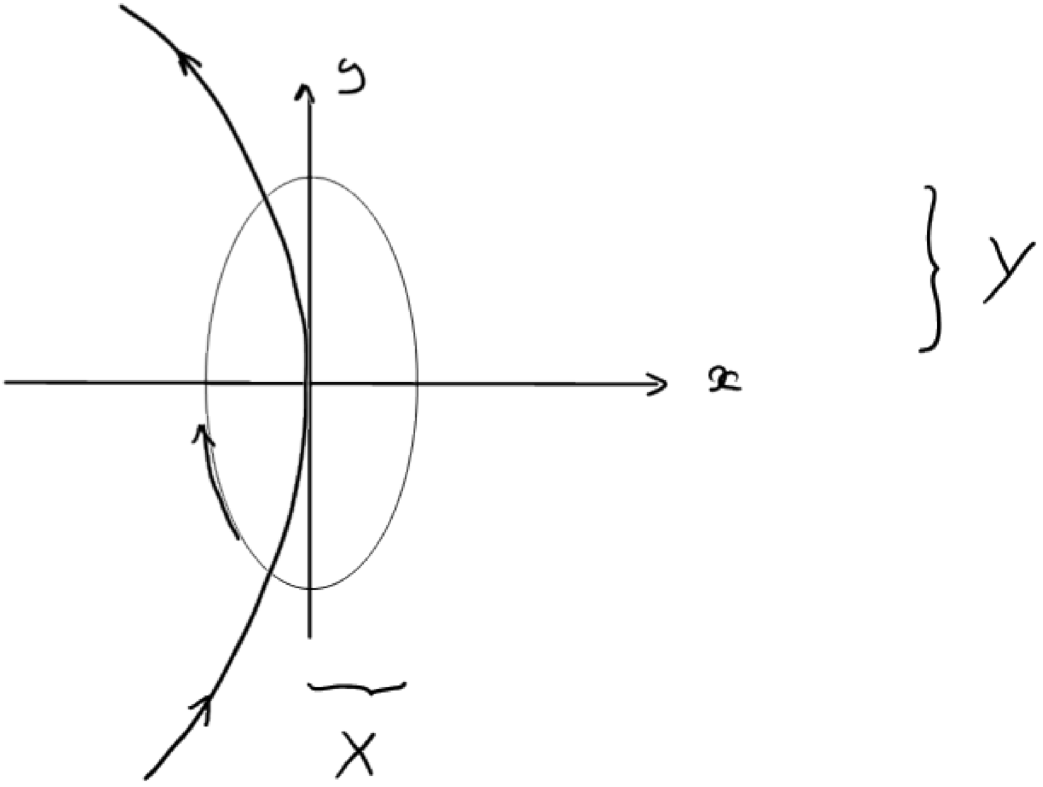
$$\boxed{y(t) = -Y \sin(\varkappa t + \alpha)} \qquad Y := \frac{2\Omega_g}{\varkappa} X$$

Complete solution

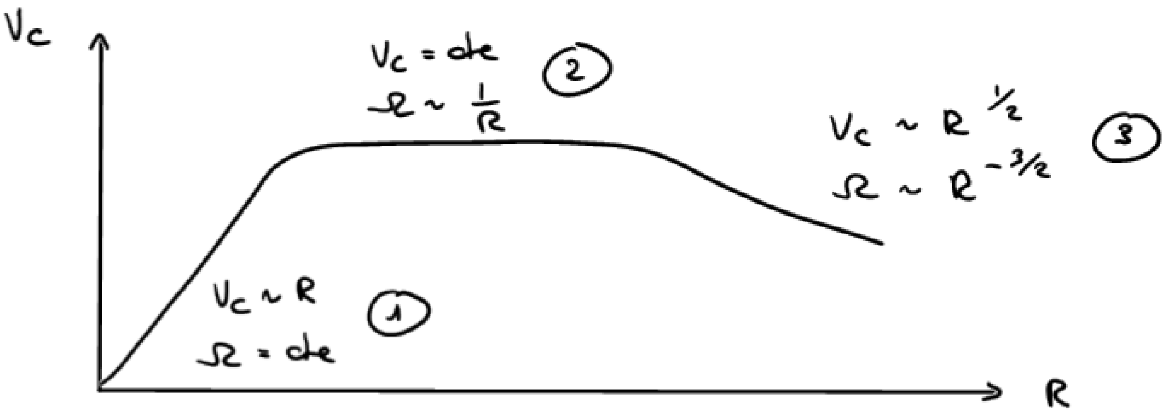
$$\boxed{\begin{cases} x(t) = X\cos(\varkappa t + \alpha) \\ y(t) = -Y\sin(\varkappa t + \alpha) \\ z(t) = Z\cos(\nu t + \xi) \end{cases}}$$

ellipse

$$Y = \frac{2\Omega_g}{\varkappa} X$$



Radial dependency for a typical galaxy

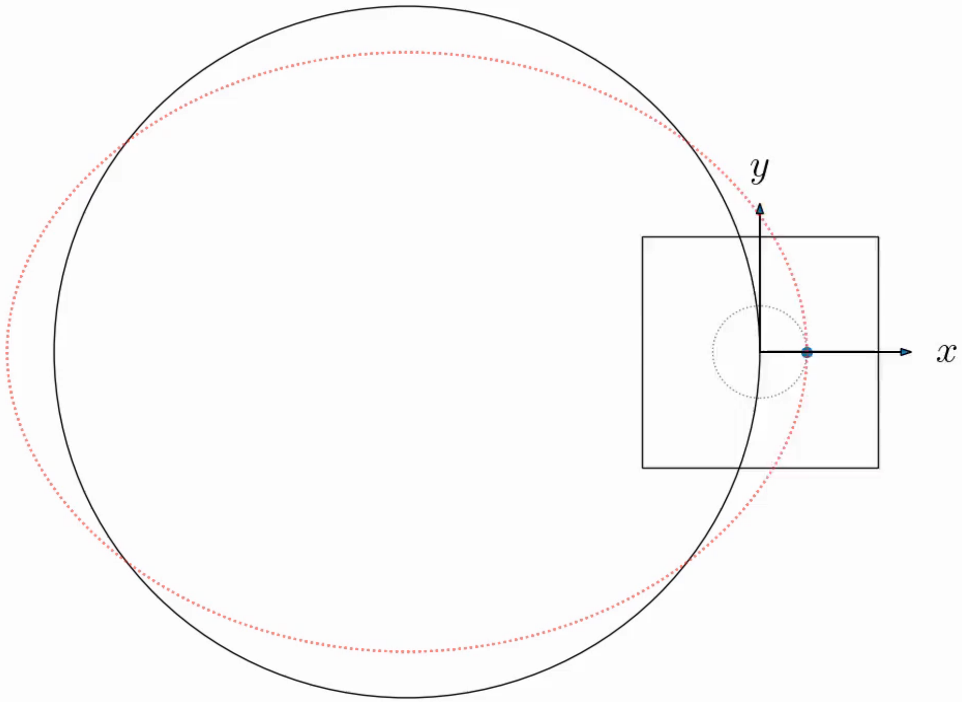


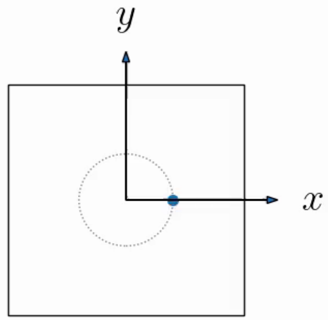
① near the center $\quad \kappa = 2\Omega \quad \frac{X}{Y} = 1 \quad$ circle

② flat rotation part $\quad \kappa = \sqrt{2}\,\Omega \quad \frac{X}{Y} = \frac{\sqrt{2}\,\Omega}{2\Omega} \quad X < Y$

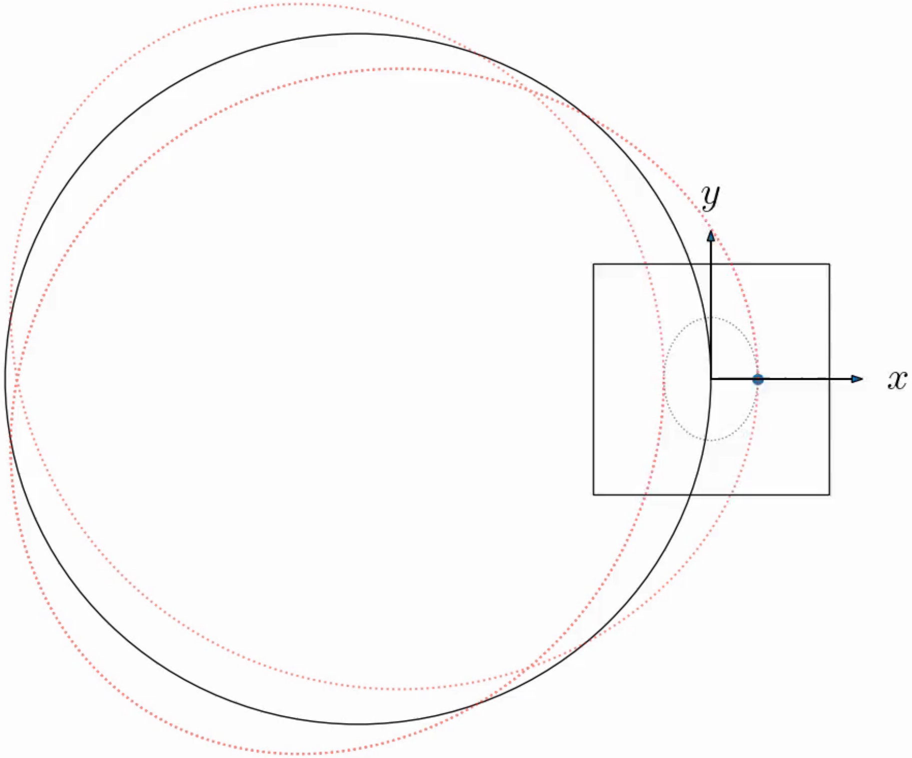
③ further out $\quad \kappa = \Omega \quad \frac{X}{Y} = \frac{\Omega}{2\Omega} \quad X < Y$

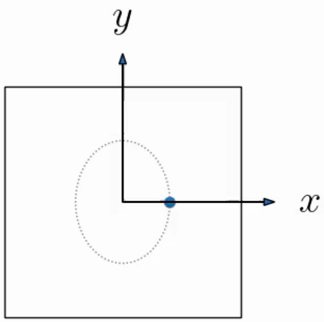
$$\kappa/\Omega = 2.0$$



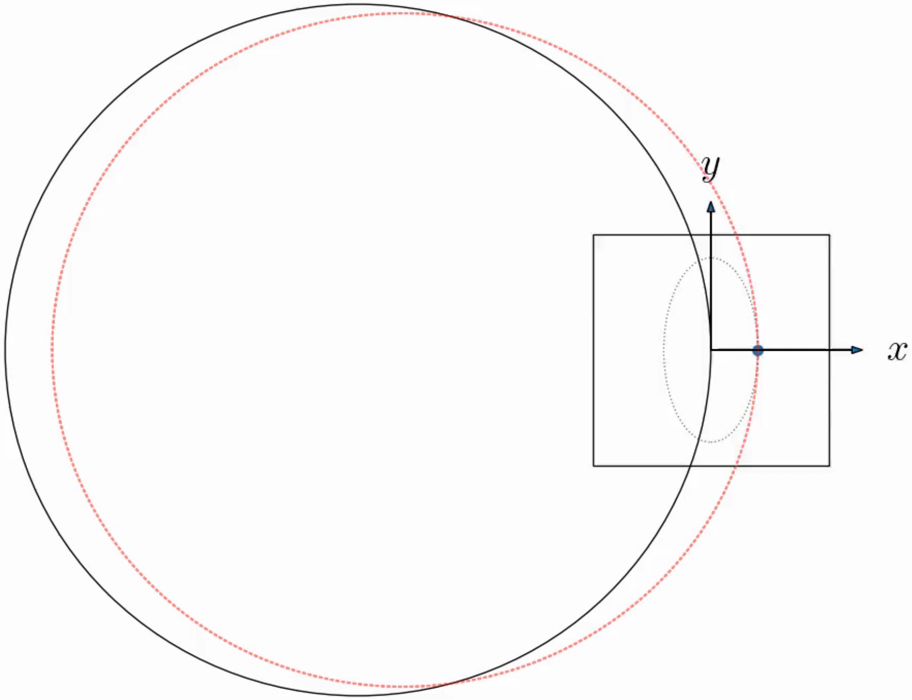


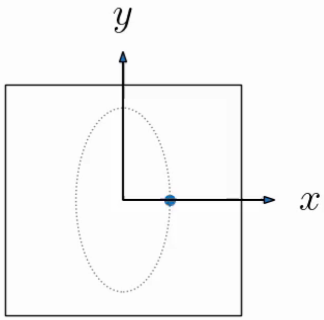
$$\kappa/\Omega = 1.5$$

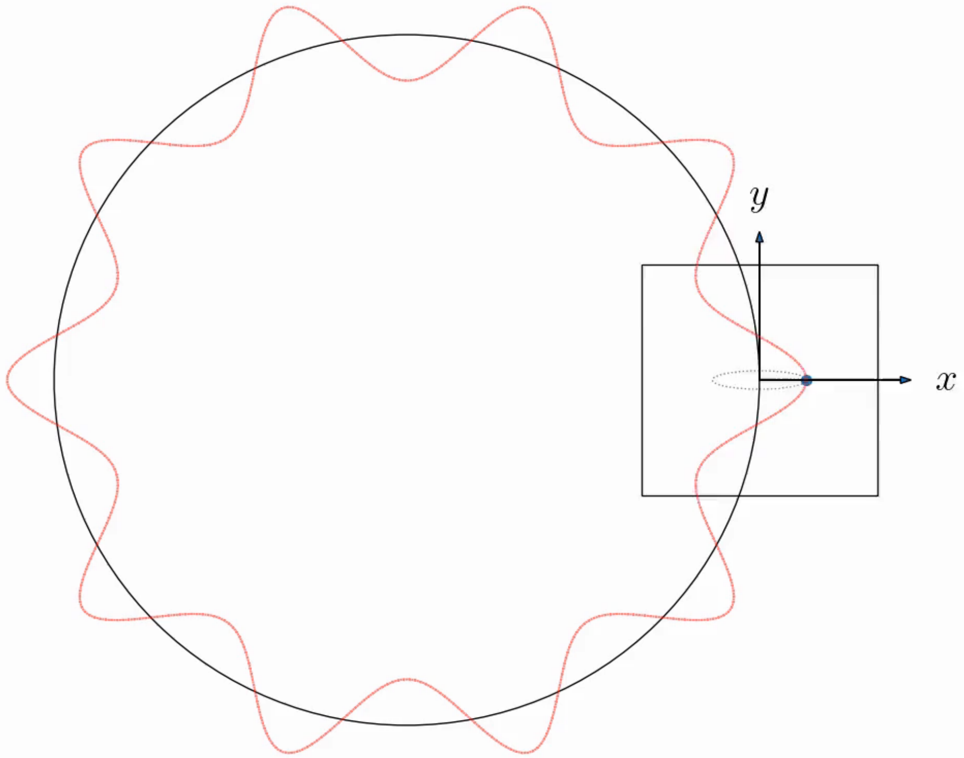




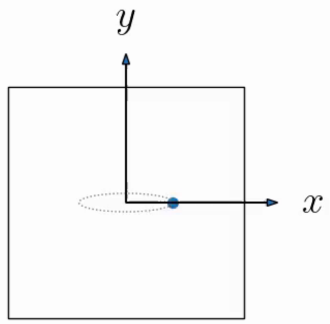
$\kappa/\Omega = 1.0$







$$\kappa/\Omega = 10.0$$



The End