

Astrophysics IV: Stellar and galactic dynamics

Solutions

Problem 1:

We set up our coordinates such that the slab lies on the $z = 0$ plane. As the mass distribution is discontinuous, we cannot easily rely on the Poisson equation to derive the corresponding potential. We instead use Gauss's law:

$$\int_S \vec{\nabla}\Phi \cdot d\vec{S} = 4\pi G M_S, \quad (1)$$

where S is any surface and M_S is the mass enclosed by the surface S . Let us define S to be the surface of a cylinder perpendicular to the plane $z = 0$. By symmetry (the surface density of the plane is constant) :

$$\vec{\nabla}\Phi = \frac{\partial}{\partial z}\Phi(z) \cdot \vec{e}_z \quad \text{and} \quad \frac{\partial}{\partial z}\Phi(z) = -\frac{\partial}{\partial z}\Phi(-z). \quad (2)$$

Thus, in the integral (1) the surface perpendicular to the plane $z = 0$ does not contribute and we get :

$$\int_S \vec{\nabla}\Phi \cdot d\vec{S} = 2 \frac{\partial}{\partial z}\Phi(z) \Delta s. \quad (3)$$

where Δs is the surface of the cylinder parallel to the plane $z = 0$. The mass enclosed in the cylinder is :

$$M_S = \Delta s \Sigma_0 \quad (4)$$

and (3) with (4) and (1) give :

$$2 \frac{\partial}{\partial z}\Phi(z) \Delta s = 4\pi G \Delta s \Sigma_0. \quad (5)$$

This leads to :

$$\frac{\partial}{\partial z}\Phi(z) = 2\pi G \Sigma_0, \quad (6)$$

and after integration :

$$\Phi(z) = 2\pi G \Sigma_0 z + \text{const.} \quad (7)$$

Problem 2:

We consider a wire aligned with the x axis. As the mass distribution is discontinuous, we cannot rely on the Poisson equation to derive the corresponding potential. We instead rely on the Gauss Theorem :

$$\int_S \vec{\nabla}\Phi \cdot d\vec{S} = 4\pi G M_S, \quad (8)$$

where S is any surface and M_S is the mass enclosed by the surface S . Let us define S to be the surface of a cylinder of length Δx and radius R , with its symmetry axis being the axis x , i.e., the wire. The surface Δs parallel to the x axis is:

$$\Delta s = 2\pi R \Delta x, \quad (9)$$

and the enclosed mass is :

$$M_S = \lambda_0 \Delta x. \quad (10)$$

By symmetry (the linear density of the wire is constant) :

$$\vec{\nabla}\Phi = \frac{\partial}{\partial R}\Phi(R) \vec{e}_R, \quad (11)$$

where \vec{e}_R is perpendicular to the axis x . With (9), (10) and (11), the Gauss theorem becomes :

$$\int_S \vec{\nabla}\Phi \cdot d\vec{S} = 2\pi R \Delta x \frac{\partial}{\partial R}\Phi(R) = 4\pi G \lambda_0 \Delta x, \quad (12)$$

which leads to :

$$\frac{\partial}{\partial R}\Phi(R) = 2G \frac{\lambda_0}{R}, \quad (13)$$

and after integrating over the radius R :

$$\Phi(R) = 2G \lambda_0 \ln(R) + \text{const}, \quad (14)$$

Problem 3:

In spherical coordinates, the Kepler potential is written as:

$$\Phi(r) = -\frac{GM}{r}, \quad (15)$$

where M is the central mass.

As we want to describe an orbit, i.e., a function $r(\phi)$, it is convenient to use the differential equation derived in the lecture, that relates $u(\phi)$

$$\frac{d^2}{d^2\varphi}u + u = \frac{1}{L^2 u^2} \frac{\partial}{\partial r}\Phi(1/u), \quad (16)$$

where $u = \frac{1}{r}$ and L is the norm of the angular momentum. The derivative of the Keplerian potential is:

$$\frac{\partial}{\partial r}\Phi(r) = \frac{GM}{r^2} = GMu^2. \quad (17)$$

Inserting equation (17) into (16), we get:

$$\frac{d^2}{d^2\varphi}u + u = \frac{GM}{L^2}. \quad (18)$$

This equation is a second order linear differential equation with a constant forcing term. Notice that this equation has the same form than the harmonic oscillator with a constant forcing term. Therefore, the general solution is:

$$u(\phi) = C \cos(\varphi - \varphi_0) + \frac{GM}{L^2}. \quad (19)$$

We can express the solution in terms of r to yield:

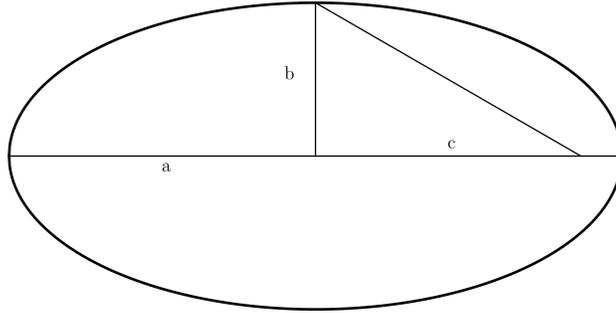
$$r(\varphi) = \frac{1}{C \cos(\varphi - \varphi_0) + \frac{GM}{L^2}}. \quad (20)$$

We can rewrite the latter equation with the following parametrisation:

$$\begin{cases} e = \frac{CL^2}{GM} & \text{eccentricity} \\ a = \frac{L^2}{GM(1 - e^2)} & \text{semi-major axis.} \end{cases} \quad (21)$$

This leads to:

$$r(\varphi) = \frac{a(1 - e^2)}{1 + e \cos(\varphi)}. \quad (22)$$



Now, our goal is to demonstrate that the latter equation defines an ellipse. Recall that the equation of an ellipse is given by:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad (23)$$

where the foci are at

$$c = \pm\sqrt{a^2 - b^2},$$

and the eccentricity is defined as

$$e = \frac{c}{a}.$$

Using these relations, we can write:

$$\begin{aligned} e^2 &= \frac{c^2}{a^2} = \frac{a^2 - b^2}{a^2} = 1 - \frac{b^2}{a^2} \\ y^2 &= b^2 - \frac{b^2}{a^2}x^2 = \frac{b^2}{a^2}(a^2 - x^2) = (1 - e^2)(a^2 - x^2). \end{aligned}$$

Now, we apply a coordinate transformation. Let $x = x' + ae$ ($= x' + c$). This gives

$$y^2 = (1 - e^2) (a^2 - (x' + ae)^2) . \quad (24)$$

Finally, we show that the equation of Keplerian orbits (22) can be written in the same form as (24).

Using the following coordinate definition, $x' = r \cos(\varphi)$, $y = r \sin(\varphi)$, we get:

$$\begin{aligned} r(1 + e \cos(\varphi)) &= r + er \cos(\varphi) = r + ex' \\ &= a(1 - e^2) \\ r^2 &= a^2(1 - e^2)^2 + e^2 x'^2 - 2a(1 - e^2)ex' \\ &= x'^2 + y^2 \\ y^2 &= a^2(1 - e^2) + x'^2(e^2 - 1) - 2a(1 - e^2)ex' \\ &= (1 - e^2)[a^2(1 - e^2) - x'^2 - 2aex'] \\ &= (1 - e^2)[a^2 - a^2e^2 - (x' + ae)^2 + a^2e^2] \\ &= (1 - e^2)[a^2 - (x' + ae)^2] \end{aligned}$$

which is exactly equation (24) again.

Problem 4:

First law : The orbit of a planet is an ellipse with the Sun at one of the two foci. This was shown in question 1.

Second law : A line segment joining a planet and the Sun sweeps out equal areas during equal intervals of time. Consider the Sun to be at the centre of the coordinate system and a planet at the position $\vec{x}(t)$ with a velocity $\vec{v}(t)$. Consider first the areas sweeps out during an infinitesimal time dt . This area will be:

$$\delta A = \frac{1}{2} |\vec{x}(t) \times d\vec{x}(t)| , \quad (25)$$

where $d\vec{x} = \vec{v}dt$. So,

$$\delta A = \frac{1}{2} dt |\vec{x}(t) \times \vec{v}(t)| = \frac{1}{2} dt |\vec{L}| , \quad (26)$$

with \vec{L} , the angular momentum (consider a body of unit mass). As the latter is conserved in a spherical potential, δA is independent of the time and of the position along the orbit. We can thus write for any interval time ΔT such that $\Delta T = t_2 - t_1$:

$$A = \int_{t_1}^{t_2} \delta A = \frac{1}{2} |\vec{L}| \int_{t_1}^{t_2} dt = \frac{1}{2} |\vec{L}| \Delta T , \quad (27)$$

which demonstrates the law.

Third law : The square of a planet's orbital period is proportional to the cube of the length of the semi-major axis of its orbit. From the previous law, we got a result of the form

$$A = \frac{1}{2}L\Delta T,$$

with L the magnitude of the angular momentum of a test particle of unit mass. For a full orbit, $\Delta T \equiv T$ is the period, and A is the area of the ellipse:

$$A = \pi ab = \pi a^2 \sqrt{1 - e^2}.$$

Let us now turn our attention to L . There are different ways of calculating it, but we will use the Vis-Viva equation:

$$v^2(r) = GM \left(\frac{2}{r} - \frac{1}{a} \right).$$

Let's take , e.g., $r = r_{\min}$:

$$v^2(r_{\min}) = GM \left(\frac{2}{r_{\min}} - \frac{1}{a} \right) = GM \left(\frac{2a - r_{\min}}{r_{\min}a} \right)$$

but $2a - r_{\min}$ is r_{\max} , and we also have $r_{\min}r_{\max} = b^2$. Together we get:

$$v^2(r_{\min}) = \frac{GM}{a} \left(\frac{b}{r_{\min}} \right)^2$$

So we have

$$L = L(r_{\min}) = \sqrt{\frac{GM}{a}} b = \sqrt{\frac{GM}{a}} a \sqrt{1 - e^2}$$

Thus the period is

$$T = \frac{2A}{L} = 2 \frac{\pi a^2 \sqrt{1 - e^2}}{\sqrt{\frac{GM}{a}} a \sqrt{1 - e^2}} = 2 \frac{\pi a^{3/2}}{\sqrt{GM}},$$

or

$$T^2 = \frac{4\pi^2}{GM} a^3.$$

Throughout this exercise, we took a test particle of unit mass to make dealing with the units easier. (Usually, $L = mrv$ and not only $L = rv$ which we used here.)