Exercise Set 4: Solution
Quantum Computation

Exercise 1 *IBM Q practice: Implementation and tests with the Toffoli gate*

Please refer to the Jupyter Notebook on Moodle.

Exercise 2 *Square-root of the NOT gate*

(a) The solutions to the eigenvalue-eigenvector equation are:

$$\lambda_0 = 1, \quad w^{(0)} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \quad \text{and} \quad \lambda_1 = -1, \quad w^{(1)} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$$

so $\Lambda = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$ and $W = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}$.

(b) We deduce from the above that a possible V is

$$V = W\sqrt{\Lambda}\,W^\dagger = \frac{1}{2}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}\begin{pmatrix}1 & 0\\0 & i\end{pmatrix}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1+i & 1-i\\1-i & 1+i\end{pmatrix}$$

and indeed, one can check directly that $V^2 = U$.

(c) Yes, V is also unitary, as $V^\dagger = \frac{1}{2}\begin{pmatrix}1-i & 1+i\\1+i & 1-i\end{pmatrix}$ and $VV^\dagger = I$.
Alternatively, one can see that V is unitary by using the fact that the composition of unitaries is unitary (prove it!). In our case, we look at the expression

$$V = W\sqrt{\Lambda}W^\dagger \ . \tag{1}$$

$\sqrt{\Lambda}$ is a unitary matrix as a diagonal matrix with entries of size 1 on its diagonal. W is unitary as a symmetric real matrix.

Remark: Please note that we have already encountered $W = H$, $\Lambda = Z$ and $\sqrt{\Lambda} = S$!

Exercise 3 *Deutsch-Josza's algorithm with noisy Hadamard gates*

(a) First observe that $H_\varepsilon^\dagger = H_\varepsilon$, so

$$H_\varepsilon H_\varepsilon^\dagger = H_\varepsilon^2 = \frac{1}{2}\begin{pmatrix}\sqrt{1+\varepsilon} & \sqrt{1-\varepsilon}\\\sqrt{1-\varepsilon} & -\sqrt{1+\varepsilon}\end{pmatrix}^2 = \frac{1}{2}\begin{pmatrix}1+\varepsilon+1-\varepsilon & 0\\0 & 1-\varepsilon+1+\varepsilon\end{pmatrix} = I$$

(b) The state of the system after the first passage of the Hadamard gates is given by

$$\begin{aligned}
|\psi_1\rangle &= H_\varepsilon |0\rangle \otimes H_\varepsilon |0\rangle \otimes H |1\rangle \\
&= \frac{1}{2}\left(\sqrt{1+\varepsilon}\,|0\rangle + \sqrt{1-\varepsilon}\,|1\rangle\right) \otimes \left(\sqrt{1+\varepsilon}\,|0\rangle + \sqrt{1-\varepsilon}\,|1\rangle\right) \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}} \\
&= \frac{1}{2}\left((1+\varepsilon)\,|00\rangle + \sqrt{1-\varepsilon^2}\,(|01\rangle + |10\rangle) + (1-\varepsilon)\,|11\rangle\right) \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}}
\end{aligned}$$

Let us write this state as

$$|\psi_1\rangle = \sum_{x\in\{0,1\}^2} \beta_x\,|x\rangle \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

where $\beta_{00} = \frac{1+\varepsilon}{2}$, $\beta_{01} = \beta_{10} = \frac{\sqrt{1-\varepsilon^2}}{2}$ and $\beta_{11} = \frac{1-\varepsilon}{2}$. Then the output of the circuit (before the measurement) is given by

$$|\psi_4\rangle = \frac{1}{2}\sum_{y\in\{0,1\}^2}\left(\sum_{x\in\{0,1\}^2} \beta_x\,(-1)^{f(x)+x\cdot y}\right)|y\rangle \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

So the probability that the output state is $|00\rangle$ when f is constant is given by

$$|\alpha_{00}|^2 = \left(\frac{(1+\varepsilon) + 2\sqrt{1-\varepsilon^2} + (1-\varepsilon)}{4}\right)^2 = \left(\frac{1+\sqrt{1-\varepsilon^2}}{2}\right)^2$$

(c) From the above expression, using successively the approximations $\sqrt{1-x} \simeq 1 - \frac{x}{2}$ and $(1-x)^2 \simeq 1 - 2x$, both valid for x small, we obtain

$$|\alpha_{00}|^2 \simeq \left(1 - \frac{\varepsilon^2}{4}\right)^2 \simeq 1 - \frac{\varepsilon^2}{2}$$

So the error probability $\delta \simeq \frac{\varepsilon^2}{2}$. In order to ensure $\delta \leq 0.1$, ε should be taken less than 0.33; for $\delta \leq 0.01$, $\varepsilon \leq 0.14$ is needed.