Stellar orbits

1^{st} part

Outlines

Idealized but useful models

- infinite slab with oscillatory surface density, tightly wound spiral

Orbits

- some generalities

Lagrangian and Hamiltonian mechanics

- Euler-Lagrange equations
- Hamilton’s equations

Orbits in spherical potentials

- angular momentum conservation
- equations of motion
- radial orbits
- non radial orbits

Examples of orbits in spherical potentials

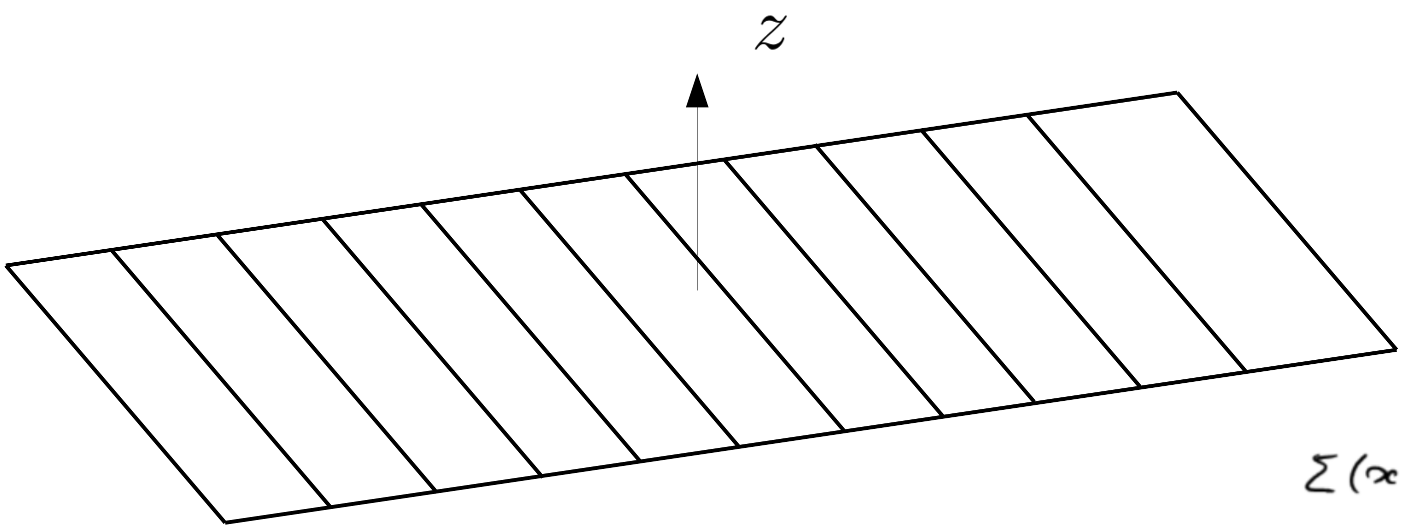
- Keplerian orbits
- orbits in an homogeneous sphere
- important remarks

Potential Theory

Idealized but useful models

NGC 1232

Potential of an infinite slab with an oscillatory surface density



$$k = |\vec{k}| = \frac{2\pi}{\lambda}$$

$$\Sigma(x,y) = \Sigma_1 \,\mathrm{Re}\left(e^{i\left(\vec{k}\cdot\vec{x}\right)}\right)$$

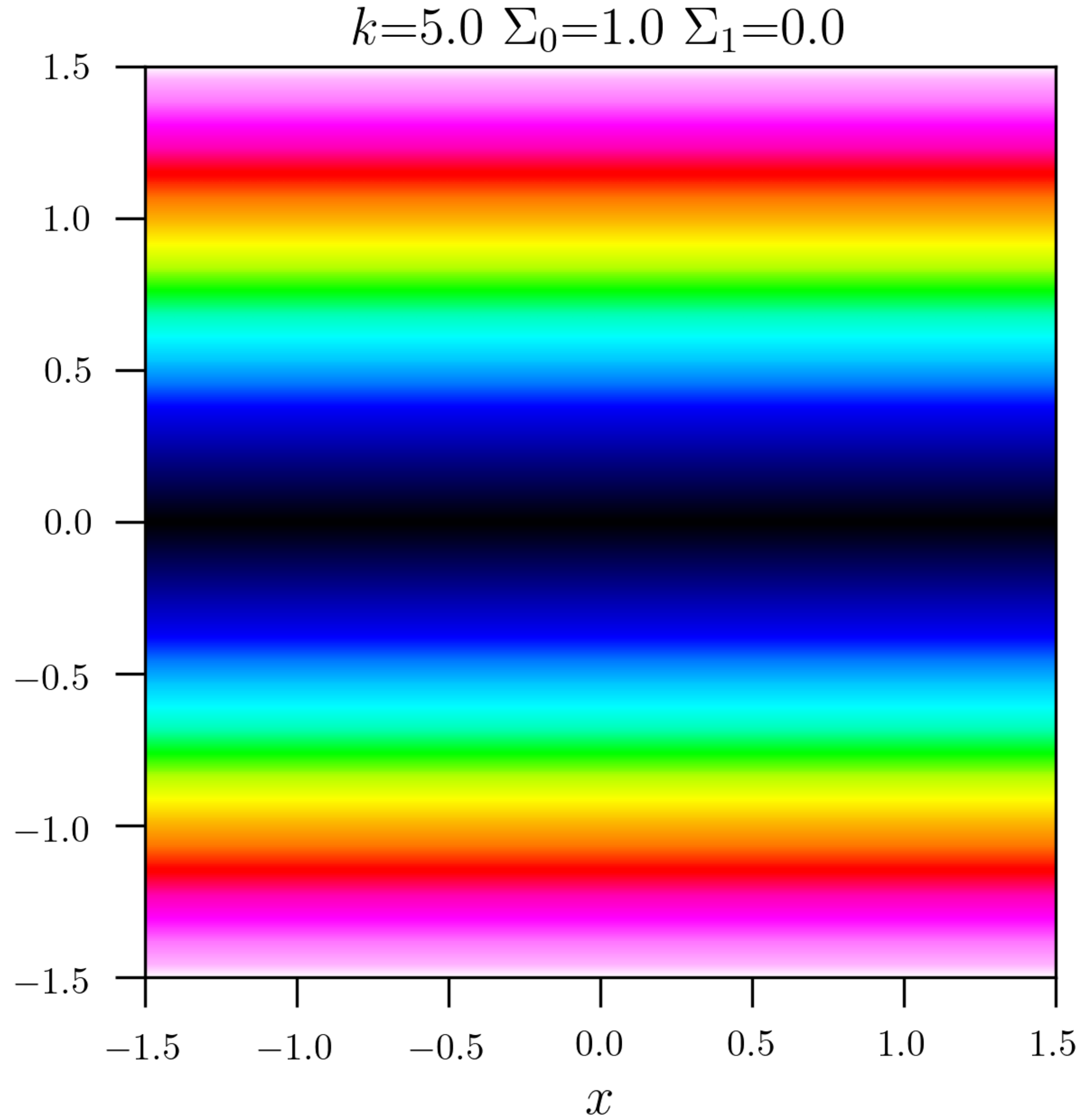
! will be negative !

$$\boxed{\Phi(x,y,z) = -\frac{2\pi G\,\Sigma_1}{|\vec{k}|}\mathrm{Re}\left(e^{i\left(\vec{k}\cdot\vec{x}\right)}\right)e^{-|\vec{k}|\,z}}$$

Potential of an Infinite slab

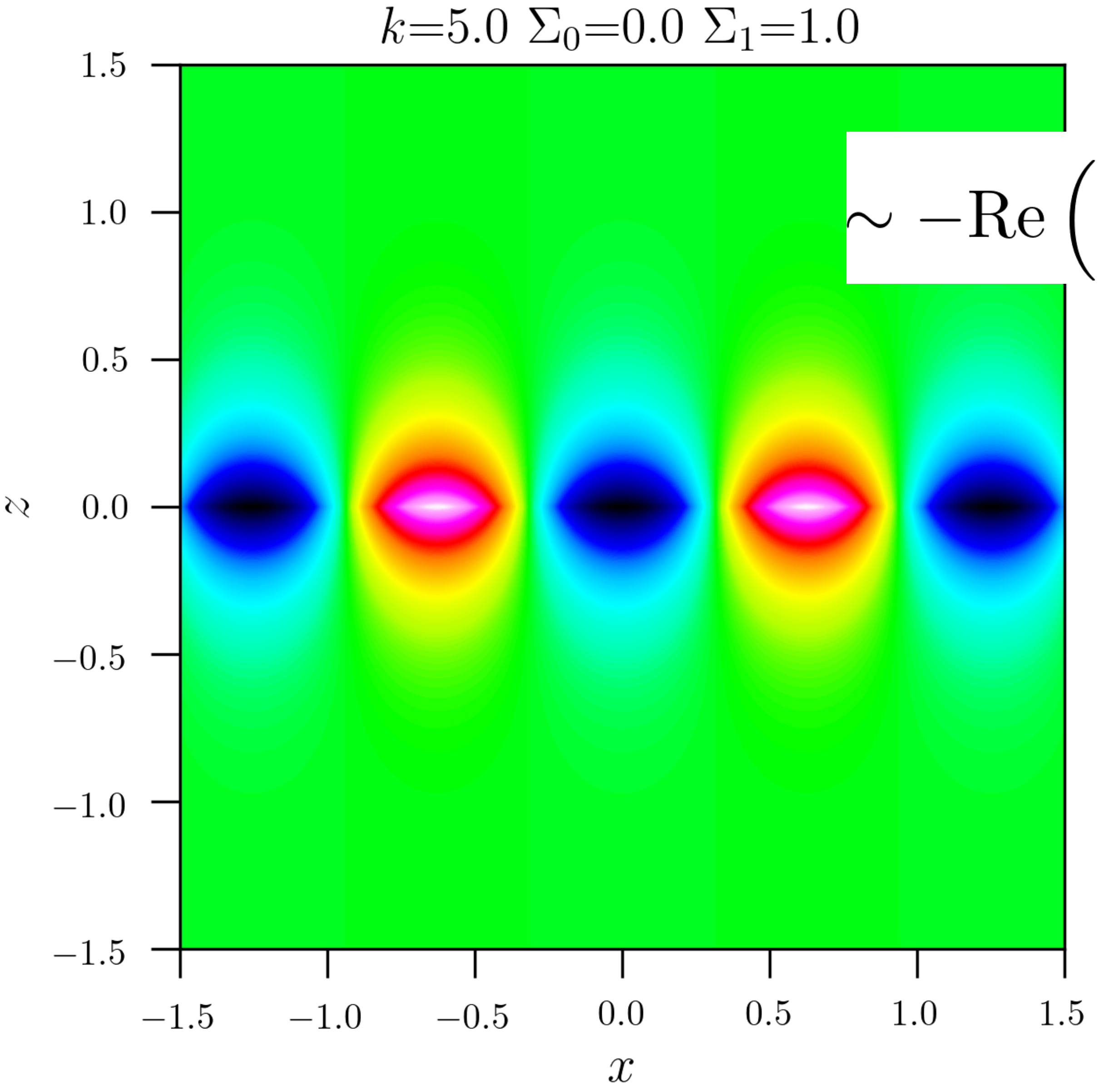
$$\Sigma(x) = \Sigma_0 + \Sigma_1 Re\left(e^{ikx}\right)$$

$\sim \Sigma_0 \, |z|$



Potential of an Infinite slab

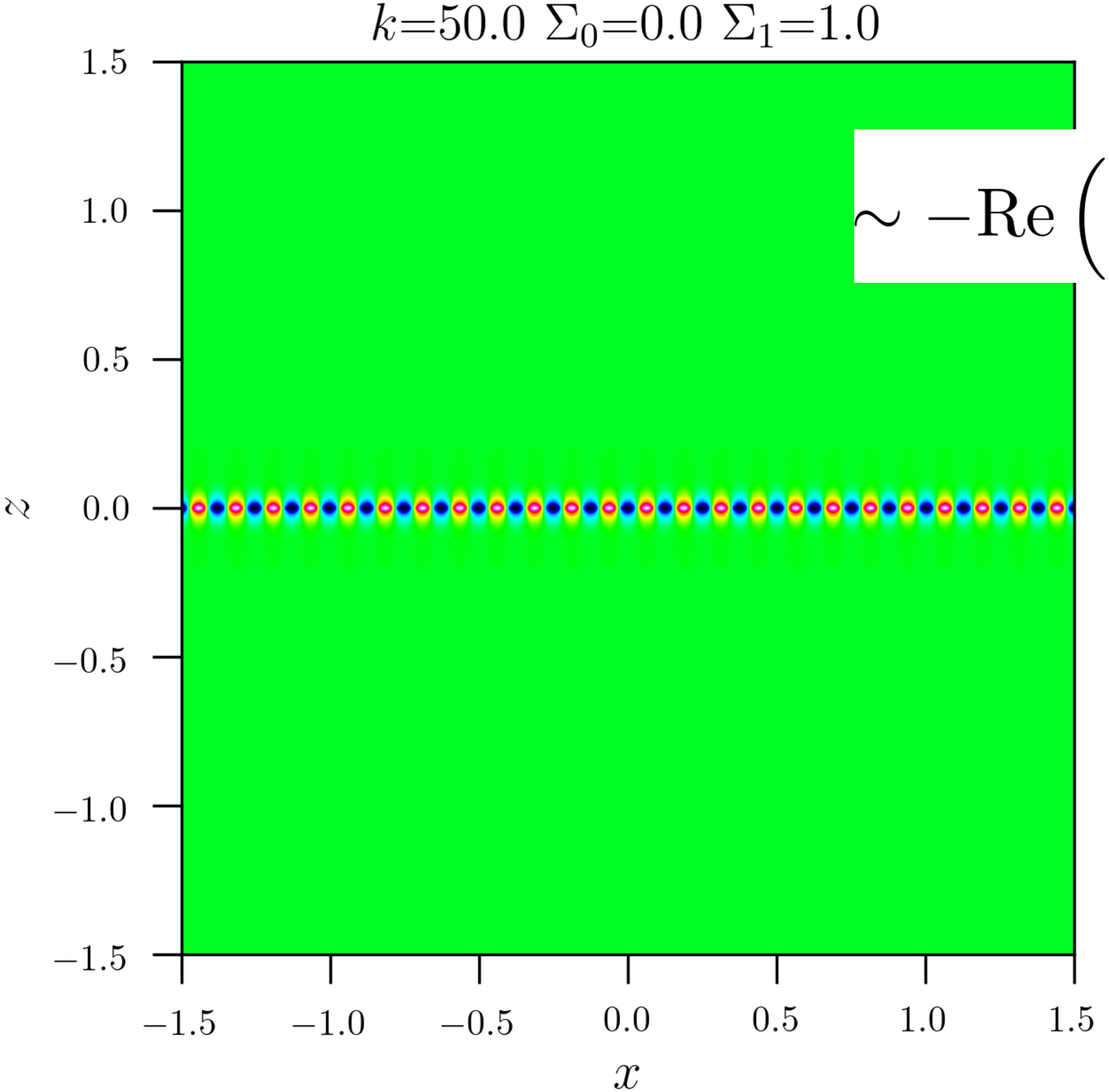
$$\Sigma(x) = \Sigma_0 + \Sigma_1 Re\left(e^{ikx}\right)$$



$$\sim -\mathrm{Re}\left(e^{i\left(\vec{k}\cdot\vec{x}\right)}\right)\frac{e^{-k\,|z|}}{k}$$

Potential of an Infinite slab

$$\Sigma(x) = \Sigma_0 + \Sigma_1 Re\left(e^{ikx}\right)$$

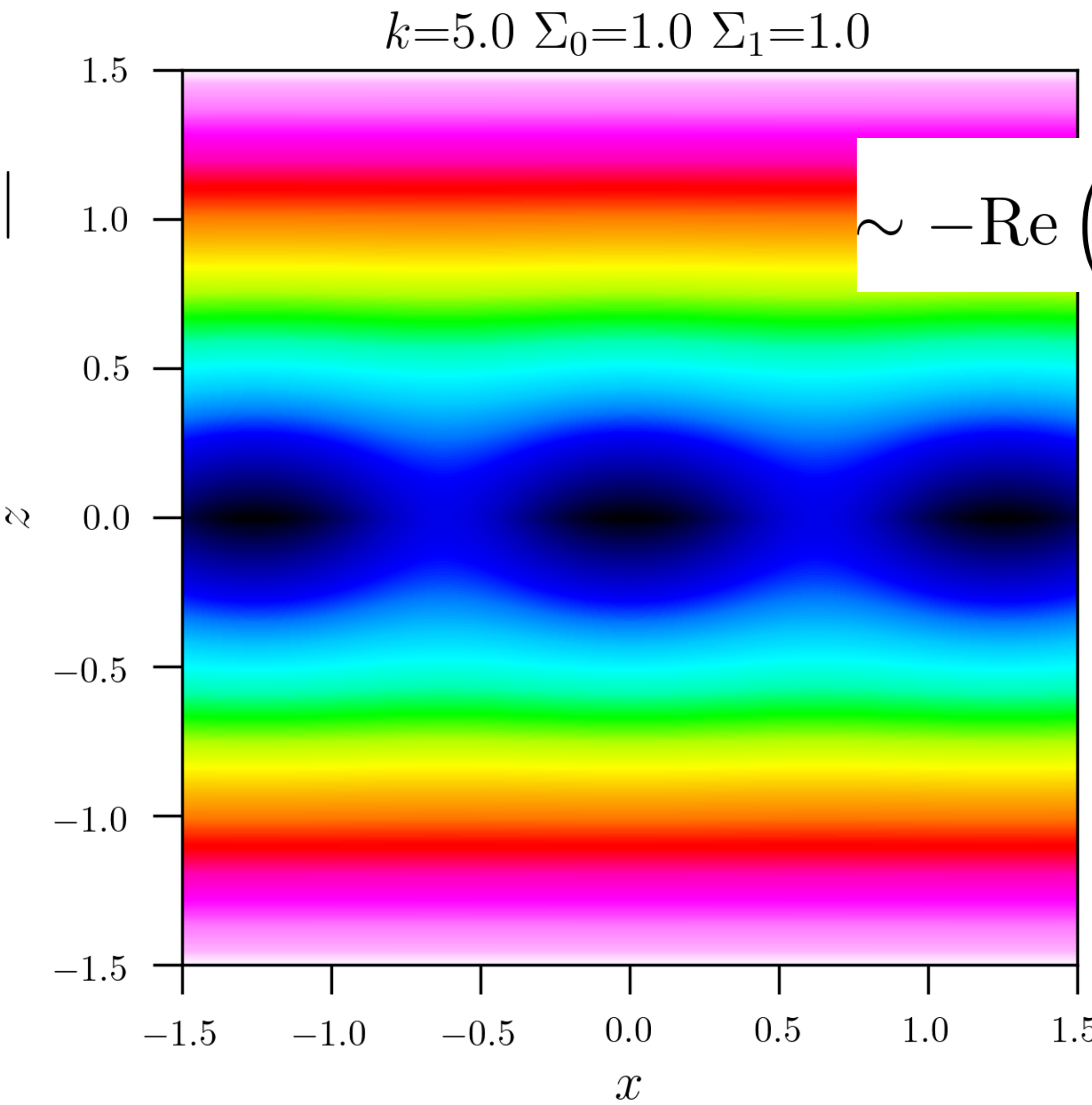


$$\sim -\mathrm{Re}\left(e^{i\left(\vec{k}\cdot\vec{x}\right)}\right)\frac{e^{-k\,|z|}}{k}$$

Potential of an Infinite slab

$$\Sigma(x) = \Sigma_0 + \Sigma_1 Re\left(e^{ikx}\right)$$

$$\sim \Sigma_0\,|z|$$

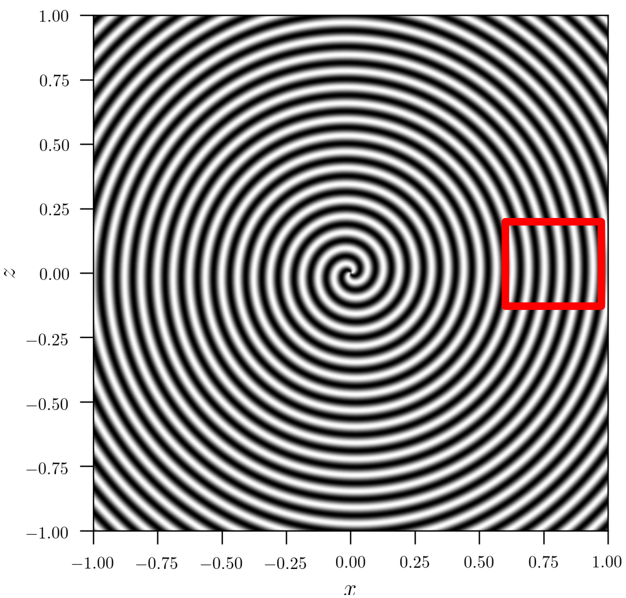


$$\sim -\mathrm{Re}\left(e^{i\left(\vec{k}\cdot\vec{x}\right)}\right)\frac{e^{-k\,|z|}}{k}$$

Potential of an infinite slab with a tightly wound spiral pattern

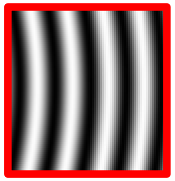
m=2

$$\Sigma(R,\phi) = H(R)\,\mathrm{Re}\left(e^{i[m\,\phi + f(R)]}\right)$$



if $$\left|\frac{\partial f}{\partial R}\cdot R\right| \ll 1$$

WKB approximation
(Wentzel,Kramers,Brillouin)

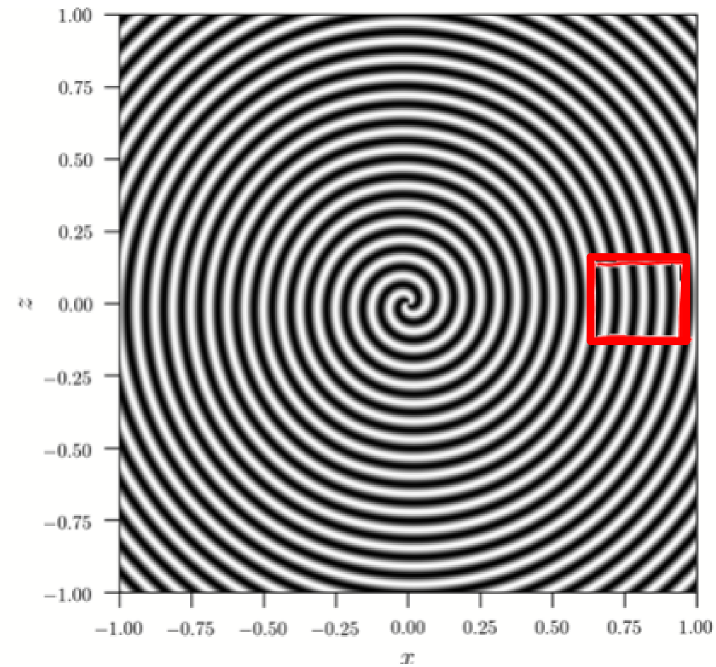


$$\boxed{\Phi(R,\phi) = -\frac{2\pi G\,\Sigma_0}{\left|\frac{\partial f}{\partial R}\right|}\,H(R)\,\mathrm{Re}\left(e^{if(R)}\right)\,e^{-\left|\frac{\partial f}{\partial R}\cdot z\right|}}$$

COMPLEMENT

Potential of an infinite slab with a tightly wound spiral pattern

$m=2$



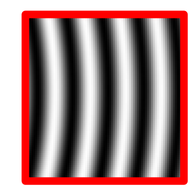
$$\boxed{\Sigma(R,\phi) = R_e\Big(\underbrace{H(R)}_{\text{slow variation}}\ \underbrace{e^{i(m\theta+g(R))}}_{\text{rapid variation}}\Big)}$$

<ins>Note</ins> $\boxed{m\theta + g(R) = \text{cte}}$

describe a spiral $\quad g(R)$ = shape function

<ins>Idea</ins>: WKB approximation

far from the center, Σ is nearly $\sim e^{i(kx)}$



<ins>Indeed</ins> Developping $g(R)$ around R_0 gives

$$g(R) \cong g(R_0) + \left.\frac{\partial g}{\partial R}\right|_{R_0}(R-R_0)$$

COMPLEMENT

For $\theta = 0$

$$\Sigma(R,\theta) = \underbrace{\mu(R_0)\, e^{i f(R_0)}}_{\text{no radial dependency}} \; \underbrace{e^{i \left.\frac{\partial f}{\partial R}\right|_{R_0} (R-R_0)}}_{e^{ikx}}$$

$$\left\{ \begin{array}{l} k = \left.\dfrac{\partial f}{\partial R}\right|_{R_0} \\ x = R - R_0 \end{array} \right.$$

<u>We directely have the solution from the infinite slab</u>

$$\phi(R,\theta) = - \frac{2\pi G}{\left|\frac{\partial f}{\partial R}\right|}\, \mu(R_0)\, e^{i f(R_0)}\, e^{i \left.\frac{\partial f}{\partial R}\right|_{R_0} (R-R_0)}\, e^{-\left|\frac{\partial f}{\partial R} z\right|}$$

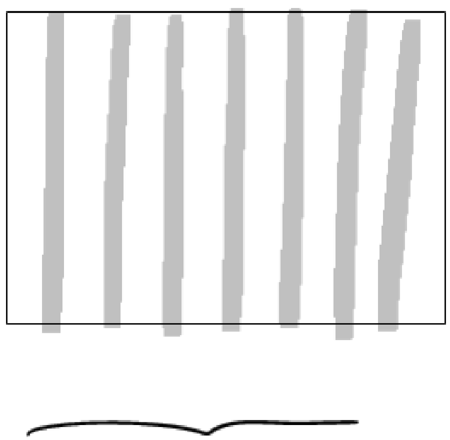
<u>Choosing</u> $R_0 = R$

$$\boxed{\phi(R,\theta) = - \frac{2\pi G}{\left|\frac{\partial f}{\partial R}\right|}\, \mu(R)\, e^{i f(R)}\, e^{-\left|\frac{\partial f}{\partial R} z\right|}}$$

COMPLEMENT

Validity of the approximation

- we want a large number of "oscillations" over a small radius compared to R



$\sim R$

$$\boxed{\left|\frac{\partial \delta}{\partial R}\right| \cdot R \gg 1}$$

Generalities

Stellar orbits

Why studying stellar orbits ?

- understand the motion of stars in stellar systems and galaxies

  → understand the observed kinematics
  → constraints the mass model

- orbits are the backbone of galaxies !

We will assume :

- a smoothed gravitational field

Stellar orbits

Definitions

- trajectory solution of the equation of motion

$$\ddot{\vec{x}} = -\vec{\nabla}\Phi(\vec{x})$$

defined on a finite interval:

$$\vec{x}(t), \vec{x}(t_0) = \vec{x}_0, \dot{\vec{x}}(t_0) = \dot{\vec{x}}_0, t \in [t_0, t_1]$$

- orbit a trajectory defined on an infinite time interval

$$\vec{x}(t), \vec{x}(t_0) = \vec{x}_0, \dot{\vec{x}}(t_0) = \dot{\vec{x}}_0, t \in [-\infty, \infty[$$

- periodic orbit a closed orbit

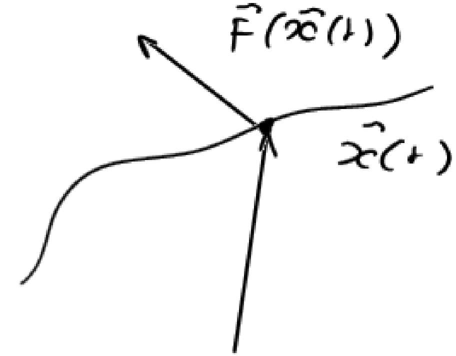
$$\forall\, t, \exists\, T, \vec{x}(t+T) = \vec{x}(t),\ \ \dot{\vec{x}}(t+T) = \dot{\vec{x}}(t)$$

- stationary point a point such that:

$$\ddot{\vec{x}} = \dot{\vec{x}} = 0$$

Lagrangian and Hamiltonian mechanics

Lagrangian Mechanics



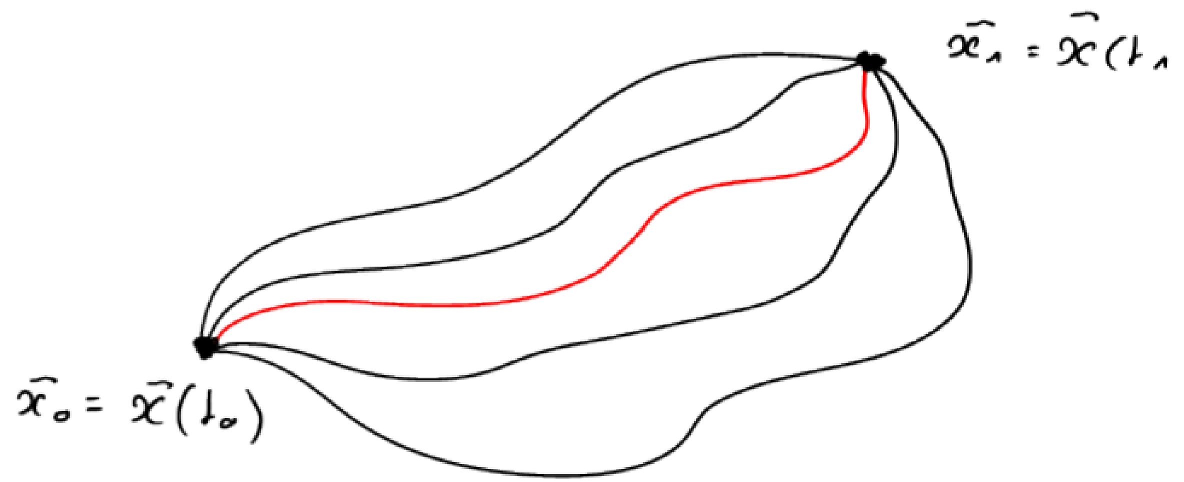
Assume a mass point moving in a force field $\vec{F}(\vec{x})$

Definition Lagrangian, a scalar function of $\vec{x}$, $\dot{\vec{x}}$, t

$$\boxed{\mathcal{L}(\vec{x}, \dot{\vec{x}}, t) = K - V = \frac{1}{2} m \dot{\vec{x}}^2 - V(\vec{x}, t)}$$

Principle of least action or Hamiltonian principle

The motion of the particle from $\vec{x}_0$ to $\vec{x}_1$ is along a curve $\vec{x}(t)$ such that $\vec{x}(t_0) = \vec{x}_0$, $\vec{x}(t_1) = \vec{x}_1$ that is an extremal of the action I.



$$\boxed{I = \int_{t_0}^{t_1} \mathcal{L}(\vec{x}, \dot{\vec{x}}, t)\, dt = \int_{t_0}^{t_1} K(t) - V(t)\, dt}$$

Euler - Lagrange equation

The trajectory is an extremal of I if and only if

$$\boxed{\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\vec{x}}}\right) - \frac{\partial \mathcal{L}}{\partial \vec{x}} = 0}$$

With cathesian coordinates, we get :

$$\boxed{m\ddot{\vec{x}} = -\vec{\nabla} V(\vec{x})}$$

which is nothing else than the second Newton law.

However : $\mathcal{L}$ can be a fuction of arbitrary coordinates $(\vec{q}, \dot{\vec{q}})$ "generalized" coordinates $\mathcal{L}(\vec{q}, \dot{\vec{q}})$.

$$\boxed{\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\vec{q}}}\right) - \frac{\partial \mathcal{L}}{\partial \vec{q}} = 0}$$

Lagrange's equations

We can easily write equations of motions in any coord. system.

Hamiltonian mechanics

Note: Lagrangian mechanics generate 2^{nd} order differential equations

$$m\ddot{\vec{x}} = -\vec{\nabla}V(\vec{x})$$

It is always possible to split a 2^{nd} order differential equation into two first order differential equations.

This is what is done in Hamiltonian mechanics

Definition

(1) For $\vec{q}, \dot{\vec{q}}$, a set of generalized coordinates, the generalized momentum are:

$$\boxed{\vec{p} := \frac{\partial \mathcal{L}}{\partial \dot{\vec{q}}}}$$

Note: inverting $\vec{p} = \vec{p}(\vec{q}, \dot{\vec{q}})$, it is possible to write $\dot{\vec{q}} = \dot{\vec{q}}(\vec{p}, \vec{q})$

(2) Hamiltonian The scalar function

$$\boxed{H(\vec{q}, \vec{p}, t) := \vec{p} \cdot \dot{\vec{q}} - \mathcal{L}(\vec{q}, \dot{\vec{q}}, t)}$$

Note: $\dot{\vec{q}}$ is replaced by $\vec{q}, \vec{p}$ through the definition of $\vec{p}$

Hamilton Equations

$$H(\vec{q},\vec{p},t) = \vec{p}\cdot\dot{\vec{q}} - \mathcal{L}(\vec{q},\dot{\vec{q}},t)$$

with $\dot{\vec{q}} = \dot{\vec{q}}(\vec{q},\vec{p})$

(1)
$$\frac{\partial H}{\partial \vec{p}} = \dot{\vec{q}} + \vec{p}\cancel{\frac{\partial \dot{\vec{q}}}{\partial \vec{p}}} - \cancel{\underbrace{\frac{\partial \mathcal{L}}{\partial \dot{\vec{q}}}}_{:=\vec{p}}\frac{\partial \dot{\vec{q}}}{\partial \vec{p}}}$$

$$\boxed{\frac{\partial H}{\partial \vec{p}} = \dot{\vec{q}}}$$

(2)
$$\frac{\partial H}{\partial \vec{q}} = -\frac{\partial \mathcal{L}}{\partial \vec{q}}$$

with EL : $\frac{d}{dt}\Big(\underbrace{\frac{\partial \mathcal{L}}{\partial \dot{\vec{q}}}}_{\vec{p}}\Big) - \frac{\partial \mathcal{L}}{\partial \vec{q}} = 0$

$$\boxed{\frac{\partial H}{\partial \vec{q}} = -\dot{\vec{p}}}$$

(3)
$$\boxed{\frac{\partial H}{\partial t} = \frac{\partial \mathcal{L}}{\partial t}}$$

In conclusion, we have transformed a set of 2^{nd} order differential equations into 2x more 1^{st} order differential equations:

$$\boxed{\begin{aligned} \dot{\vec{q}} &= \frac{\partial H}{\partial \vec{p}} \\ \dot{\vec{p}} &= -\frac{\partial H}{\partial \vec{q}} \end{aligned}}$$

$$\frac{\partial \mathcal{L}}{\partial t} = -\frac{\partial H}{\partial t}$$

Hamilton's equations

Hamiltonian conservation

Lets compute the time derivative of $H(\vec{q},\vec{p},t)$

$$\frac{d}{dt} H(\vec{q},\vec{p},t) = \frac{\partial H}{\partial \vec{q}} \frac{d\vec{q}}{dt} + \frac{\partial H}{\partial \vec{p}} \frac{d\vec{p}}{dt} + \frac{\partial H}{\partial t}$$

$$- \dot{\vec{p}} \cdot \dot{\vec{q}} + \dot{\vec{q}}\, \dot{\vec{p}} = 0$$

If $\mathcal{L}$ is time independant, i.e. $\mathcal{L} = \mathcal{L}(\vec{q}, \dot{\vec{q}})$

$(\equiv V(\vec{q})$ is time independant$)$

$\Rightarrow$

$$\boxed{\text{By construction, } H(\vec{q},\vec{p}) \text{ is conserved along a trajectory}}$$

Maupertuis principle

first variation

The Hamilton principle $$\delta \int_{t_0}^{t_1} \mathcal{L}(\vec{q}, \dot{\vec{q}}, t)\, dt = 0$$

For a constant energy H $\qquad$ cte

$$\delta \int_{t_0}^{t_1} \mathcal{L}(\vec{q}, \dot{\vec{q}}, t)\, dt = \delta \int_{t_0}^{t_1} \mathcal{L}(\vec{q}, \dot{\vec{q}}, t) + \underbrace{H\left(\vec{q}, \frac{\partial \mathcal{L}}{\partial \dot{\vec{q}}}\right)}\, dt$$

$$= \delta \int_{t_0}^{t_1} \mathcal{L}(\vec{q}, \dot{\vec{q}}, t) + \vec{p}\cdot\dot{\vec{q}} - \mathcal{L}(\vec{q}, \dot{\vec{q}}, t)\, dt$$

$$= \delta \int_{t_0}^{t_1} \vec{p}\cdot\dot{\vec{q}}\, dt = \delta \int_{\vec{q}_0}^{\vec{q}_1} \vec{p}\cdot d\vec{q}$$

change of variable $d\vec{q} = \dot{\vec{q}}\cdot dt$

So $$\boxed{\delta \int_{\vec{q}_0}^{\vec{q}_1} \vec{p}\cdot d\vec{q} = 0}$$

Maupertuis principle

COMPLEMENT

Definitions

for a system with n-dimensions

Configuration space $\quad (q_1 \ldots q_n) \quad$ n-dimensions

Momentum space $\quad (p_1 \ldots p_n) \quad$ n-dimensions

Phase space $\quad (q_1 \ldots q_n, p_1 \ldots p_n) \quad$ $2n$-dimensions

$$\equiv (w_1 \ldots w_{2n})$$

Note As Hamilton's equations are 1st order differential equations, a trajectory is uniquely defined by a point in the phase space

COMPLEMENT

Poisson brackets

two operators A, B

$$[A, B] := \frac{\partial A}{\partial \vec{q}} \frac{\partial B}{\partial \vec{p}} - \frac{\partial A}{\partial \vec{p}} \frac{\partial B}{\partial \vec{q}} = \sum_i^n \frac{\partial A}{\partial q_i} \frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i} \frac{\partial B}{\partial q_i}$$

Hamilton's equations

$$\boxed{\dot{w}_\alpha = [w_\alpha, H]} = \frac{\partial w_\alpha}{\partial \vec{q}} \frac{\partial H}{\partial \vec{p}} - \frac{\partial w_\alpha}{\partial \vec{p}} \frac{\partial H}{\partial \vec{q}}$$

$$= \begin{cases} \dot{q}_\alpha = \dfrac{\partial H}{\partial p_\alpha} \\[2ex] \dot{p}_\alpha = -\dfrac{\partial H}{\partial q_\alpha} \end{cases}$$

Gradient of the phase space velocity

$$\boxed{\vec{\nabla}_w \dot{\vec{w}} = 0}$$

$$\dot{\vec{w}} = \begin{pmatrix} \dot{\vec{q}} \\ \dot{\vec{p}} \end{pmatrix} \qquad \vec{\nabla}_w = \begin{pmatrix} \frac{\partial}{\partial \vec{q}} \\ \frac{\partial}{\partial \vec{p}} \end{pmatrix}$$

Demonstration

$$\vec{\nabla}_w \dot{\vec{w}} = \frac{\partial}{\partial \vec{q}} \dot{\vec{q}} + \frac{\partial}{\partial \vec{p}} \dot{\vec{p}}$$

Hamilton equations

$$= \frac{\partial}{\partial \vec{q}} \frac{\partial}{\partial \vec{p}} H - \frac{\partial}{\partial \vec{p}} \frac{\partial}{\partial \vec{q}} H = 0$$

#

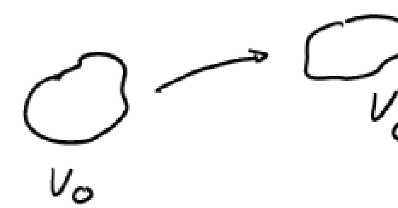
Time evolution operator

It is the possible to define a time evolution operator $\mathcal{H}_t$ that will bring $(\tilde{q}_0, \tilde{p}_0)$ to $(\tilde{q}(t), \tilde{p}(t))$

$$(\tilde{q}(t), \tilde{p}(t)) = \mathcal{H}_t(\tilde{q}_0, \tilde{p}_0) \equiv \tilde{w}(t) = \mathcal{H}_t(\tilde{w}_0)$$

$\mathcal{H}_t$ will map:

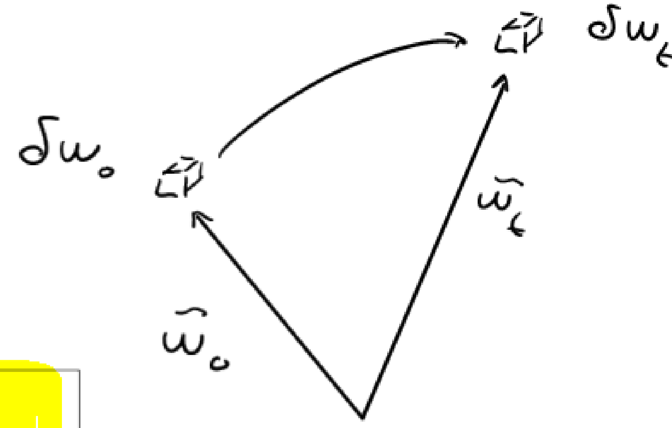
- any 2D surface S_0 in the phase space to an other 2D surface S_t in the phase space.



- any 2N-D volume V_0 in the phase space to an other 2N-D volume V_t in the phase space.

Phase space volume conservation

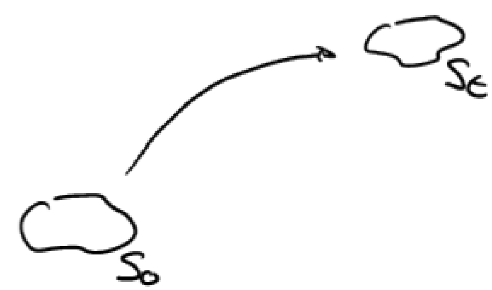
$$\boxed{\delta \omega_0 = \delta \omega_t}$$



The volume on any arbitrary region in phase space is conserved by a Hamiltonian flow.

Poincaré invariant theorem

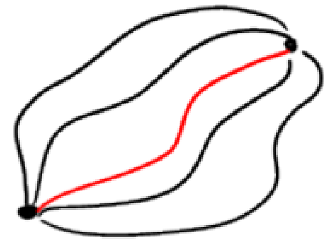
$$\boxed{\iint_{S_0} d\hat{q} \cdot d\hat{p} = \iint_{S_t} d\hat{q} \, d\hat{p}}$$



Three visions of mechanics

① Newtonian $\qquad m\ddot{\vec{x}} = \vec{F}(\vec{x})$

② Langrangian



③ Hamiltonian

$$\boxed{\begin{aligned} \dot{\vec{q}} &= \frac{\partial H}{\partial \vec{p}} \\ \dot{\vec{p}} &= -\frac{\partial H}{\partial \vec{q}} \end{aligned}}$$

Laplace paradigm

specifying one point in the phase space at one moment in time $(\vec{x}(t_0), \vec{v}(t_0))$ is enough to determine the evolution of the system.

But $\vec{v}$ is not local in time !

$\vec{q}$ and $\vec{p}$ are !

Orbits in Spherical Systems

Orbits in spherical potentials

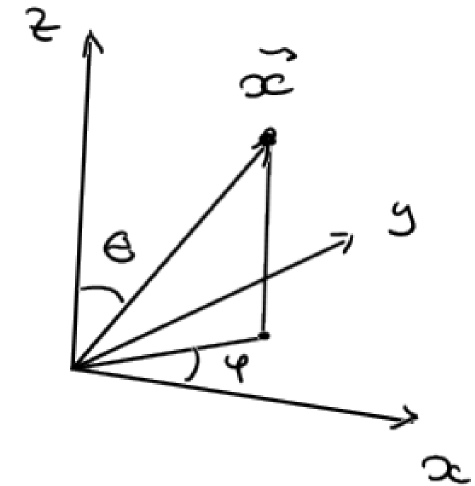
$\phi(\vec{x}) = \phi(r)$

Spherical coordinates

$$\begin{cases} x = r\cos\varphi\sin\theta \\ y = r\sin\varphi\sin\theta \\ z = r\cos\theta \end{cases}$$

$\vec{x} = r\,\vec{e}_r = \vec{r}$

$r = \sqrt{x^2 + y^2 + z^2}$



Equation of motion (Newton law)

$$\boxed{\frac{d^2}{dt^2}(\vec{x}) = \vec{g}(\vec{x}) \equiv g(r)\,\vec{e}_r}$$

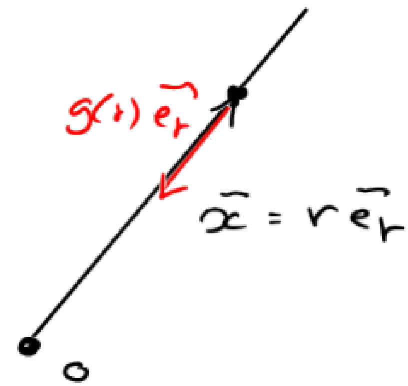
$$g(\vec{x}) = -\vec{\nabla}\phi(\vec{x}) = -\frac{\partial}{\partial r}\phi(r)\,\vec{e}_r - \frac{1}{r}\cancel{\frac{\partial}{\partial\theta}\phi(r)}\,\vec{e}_\theta - \frac{1}{r\sin\theta}\cancel{\frac{\partial}{\partial\varphi}\phi(r)}\,\vec{e}_\varphi$$

$$= g(r)\,\vec{e}_r \quad \text{with} \quad g(r) = -\frac{\partial}{\partial r}\phi(r)$$

The force generated by a spherical potential is central

Angular momentum conservation

Tork of the force



$$\vec{N} := \vec{x} \times \vec{F} = \vec{r} \times \vec{g}(r)\,\vec{e_r}$$

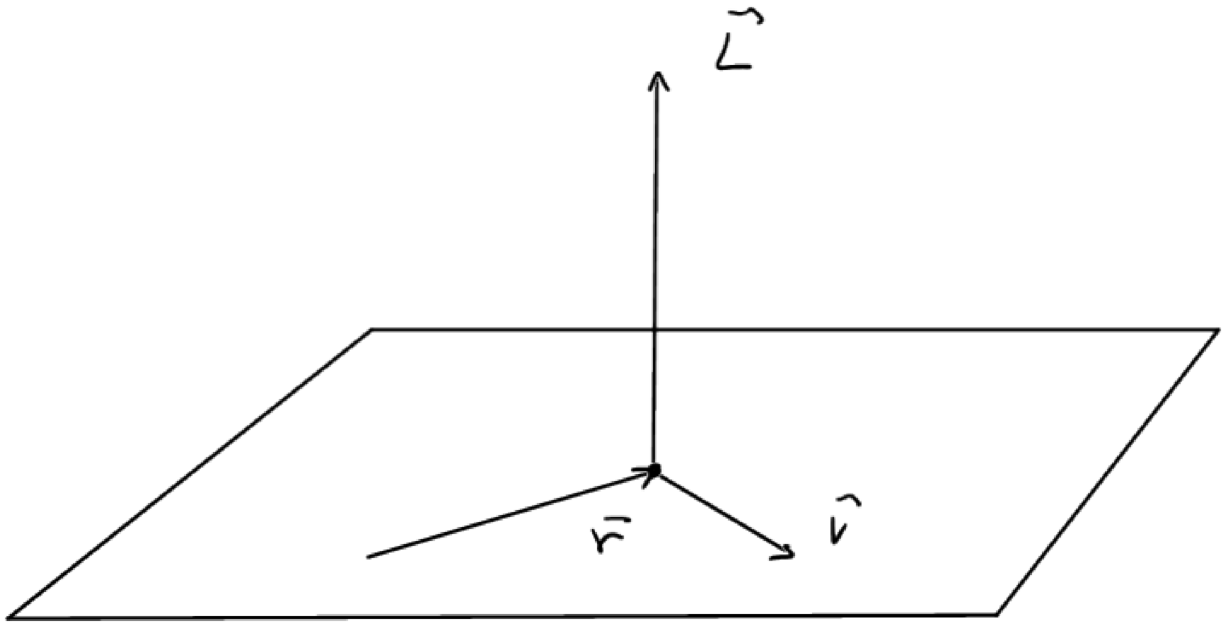
$$= r\vec{e_r} \times g(r)\,\vec{e_r} = 0$$

A spherical potential induces no tork

So, as $\frac{d}{dt}(\vec{L}) = \vec{N}$ $\quad\quad$ $\underline{\underline{\vec{L} = cte}}$

In a spherical system, the angular momentum of a particle is conserved ! $\quad \vec{L} = cte$

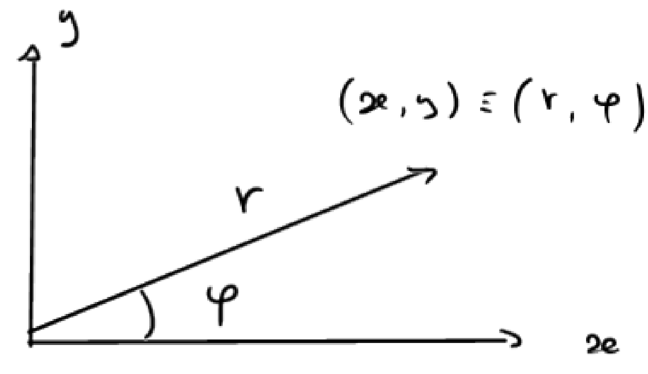
Corollary As $\vec{L}$ is conserved the orbit of a particle is limited to a plane (the orbital plane)



2D problem

Equations of motion in the orbital plane

Polar coordinates (in the orbital plane)



$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases} \qquad \begin{cases} \dot{x} = \dot{r}\cos\varphi - r\sin\varphi\,\dot{\varphi} \\ \dot{y} = \dot{r}\sin\varphi + r\cos\varphi\,\dot{\varphi} \end{cases}$$

Lagrangian (specific) in polar coordinates

$$\mathcal{L} = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2\right) + \phi\left(\sqrt{x^2+y^2}\right) = \frac{1}{2}\left(\dot{r}^2 + (r\dot{\varphi})^2\right) - \phi(r)$$

Lagrange equations

$$\boxed{\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\vec{q}}}\right) - \frac{\partial \mathcal{L}}{\partial \vec{q}} = 0}$$

$$\begin{cases} \ddot{r} - r\dot{\varphi}^2 + \dfrac{\partial \phi}{\partial r} = 0 \\ \dfrac{d}{dt}\left(r^2\dot{\varphi}\right) = 0 \end{cases}$$

Angular momentum

$$\boxed{r^2 \dot{\varphi} = |\vec{L}| = L}$$

Indeed

in spherical coordinates

$$\vec{x} = r\,\vec{e_r}$$

$$\vec{v} = \dot{r}\,\vec{e_r} + r\dot{\varphi}\,\vec{e_\varphi}$$

$$\vec{L} = \vec{x} \times \vec{v} = r\,\vec{e_r} \times \left(\dot{r}\,\vec{e_r} + r\dot{\varphi}\,\vec{e_\varphi} \right)$$

$$= r^2 \dot{\varphi}\,\vec{e_z}$$

Hamiltonian/Energy

$$\boxed{H(\vec{q},\vec{p},t) := \vec{p}\cdot\dot{\vec{q}} - \mathcal{L}(\vec{q},\dot{\vec{q}},t)}$$

$$\vec{q} = \begin{cases} r \\ \varphi \end{cases} \qquad \dot{\vec{q}} = \begin{cases} \dot{r} \\ \dot{\varphi} \end{cases} \qquad \vec{p} = \begin{cases} \frac{\partial \mathcal{L}}{\partial \dot{r}} = \dot{r} = p_r \\ \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = r^2\dot{\varphi} = p_\varphi \end{cases}$$

$$H(r,\varphi,\dot{r},r^2\dot{\varphi}) = \dot{r}^2 + r^2\dot{\varphi}^2 - \frac{1}{2}\left(\dot{r}^2 + (r\dot{\varphi})^2\right) + \phi(r)$$

$$= \frac{1}{2}\left(\dot{r}^2 + (r\dot{\varphi})^2\right) + \phi(r) = E$$

or

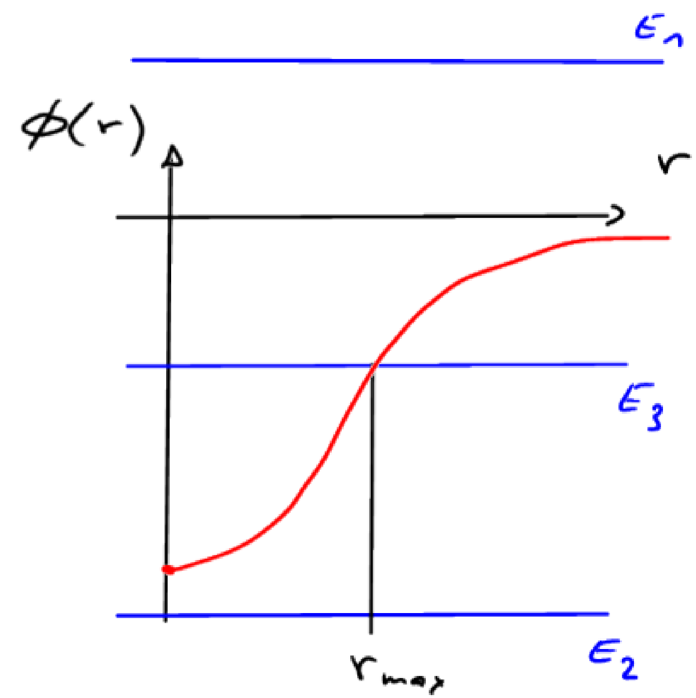
$$H(r,\varphi,p_r,p_\varphi) = \frac{1}{2}p_r^2 + \frac{1}{2}\frac{p_\varphi^2}{r^2} + \phi(r) = E$$

E (Energy) is conserved
as $\mathcal{L}$ is time independant

Radial orbits $\quad \dot{\varphi} = 0 \implies L = 0$

$$\begin{cases} \text{Equation of motion} : & \ddot{r} = -\dfrac{\partial \phi}{\partial r} \\ \text{Energy} : & E = \frac{1}{2}\dot{r}^2 + \phi(r) \end{cases}$$

3 cases



① $E > \phi(\infty) \implies \forall t,\ \dot{r}^2 > 0$

orbit not bounded

② $E < \phi(0) \implies$ impossible

③ $\phi(0) < E < \phi(\infty)$

$\exists\, r$ tq $\dot{r} = 0$ i.e $E = \phi(r)$

$r = r_{max}$

Non radial orbits $\quad r \neq 0 \quad \dot{\varphi} \neq 0 \quad L \neq 0$

$$\text{EOM} \begin{cases} \ddot{r} - r\dot{\varphi}^2 + \dfrac{\partial \phi}{\partial r} = 0 & (1) \\ \dfrac{d}{dt}\left(r^2 \dot{\varphi}\right) = 0 \end{cases}$$

replace t by φ

$$\frac{d}{dt} = \frac{d}{d\varphi}\dot{\varphi} = \frac{L}{r^2}\frac{d}{d\varphi}$$

(1) becomes

$$\frac{L^2}{r^2}\frac{d}{d\varphi}\left(\frac{1}{r^2}\frac{dr}{d\varphi}\right) - \frac{L^2}{r^3} = -\frac{\partial \phi}{\partial r}$$

use $u = \dfrac{1}{r}$

$$\boxed{\frac{d^2 u}{d\varphi^2} + u = \frac{1}{L^2 u^2}\frac{\partial \phi}{\partial r}\left(\frac{1}{u}\right)}$$

No analytical general solution

Radial energy equation

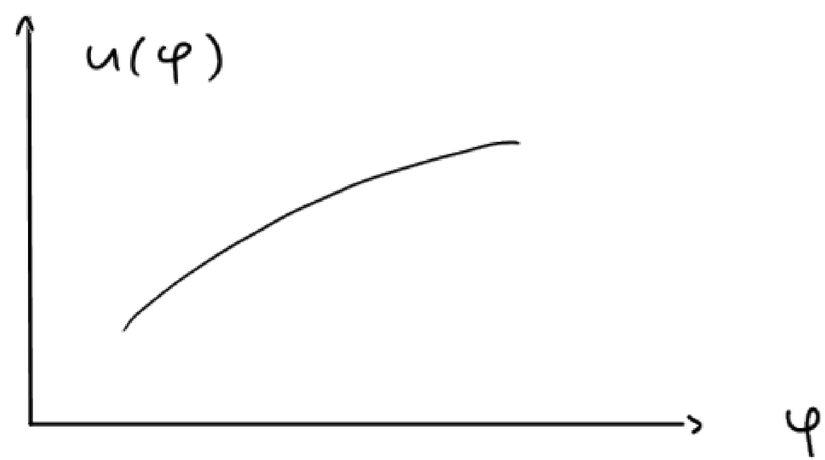
From the energy $\boxed{E = \frac{1}{2}\left(\dot{r}^2 + (r\dot{\varphi})^2\right) + \phi(r)}$

1) multiply by $\frac{2}{L^2}$

2) use $u = \frac{1}{r}$ and $\frac{d}{dt} = \frac{L}{r^2}\frac{d}{d\varphi}$

we get

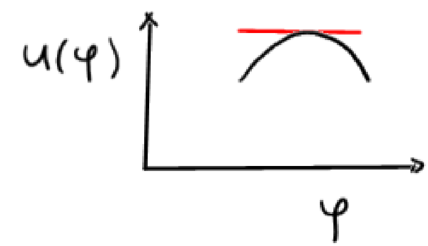
$$\boxed{\left(\frac{du}{d\varphi}\right)^2 + u^2 + \frac{2\phi(\frac{1}{u})}{L^2} = \frac{2E}{L^2}}$$



Orbit properties

Minimal radius

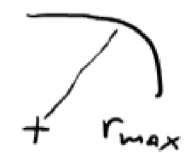
As $L \neq 0$, the orbit cannot cross the center there must be a <u>minimal radius</u>

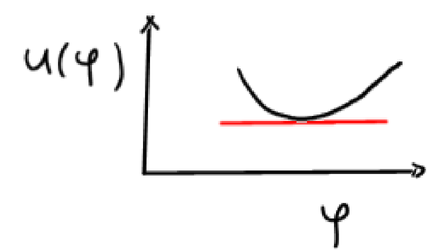


$$\forall \ \varphi \quad \text{such that} \quad \frac{du}{d\varphi} = 0$$

Maximal radius

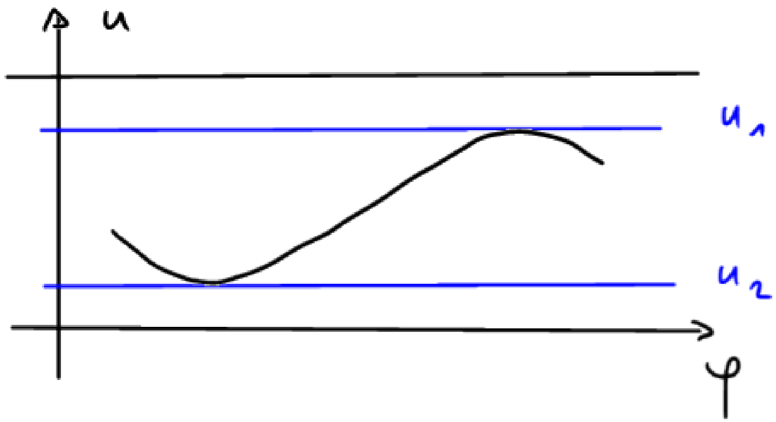
If the orbit is bounded there must be a <u>maximal radius</u>

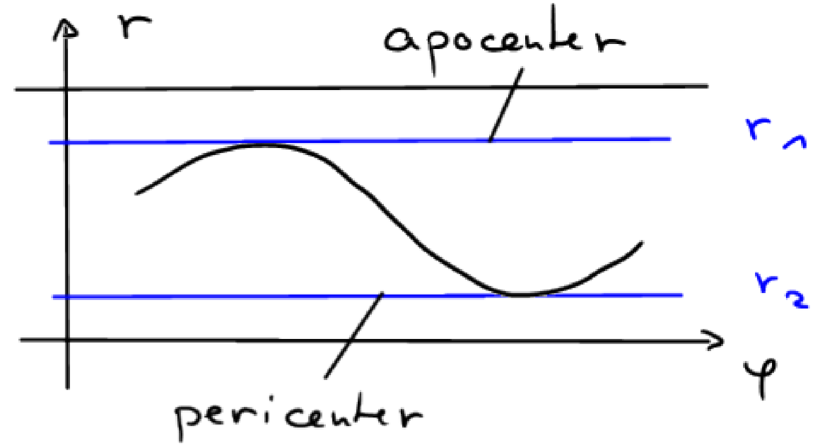




$$\forall \ \varphi \quad \text{such that} \quad \frac{du}{d\varphi} = 0$$

$$\boxed{\text{For} \quad \frac{du}{d\varphi} = 0 \qquad u^2 = \frac{2\left[E - \phi(1/u)\right]}{L^2}}$$

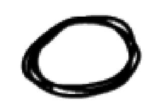




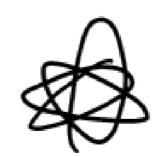
Notes

- if $u_1 = u_2$: periodic orbit

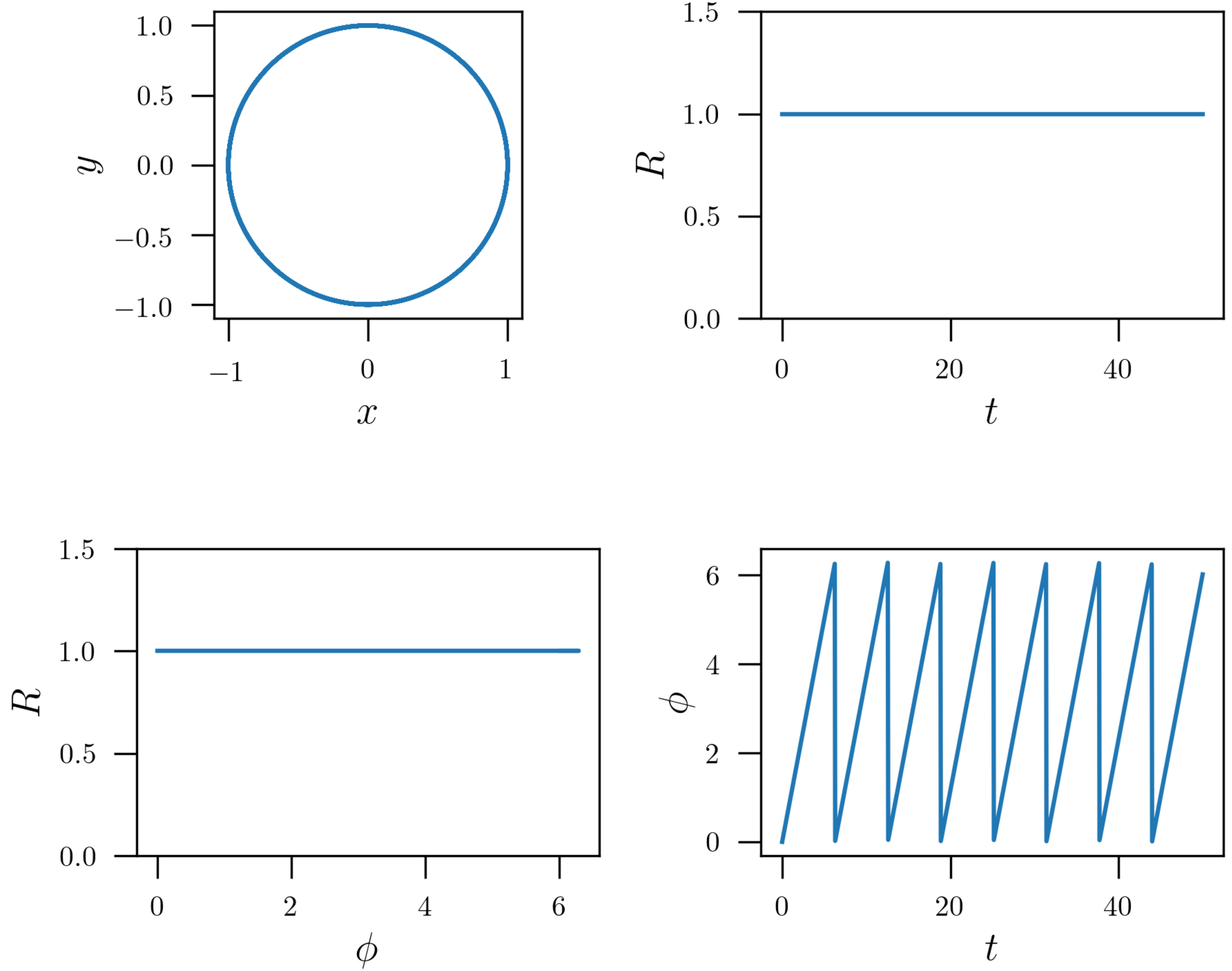
- if $u_1 \cong u_2$: orbit with a small eccentricity



- if $u_1 \gg u_2$: orbit eccentricity is nearly 1



Plummer



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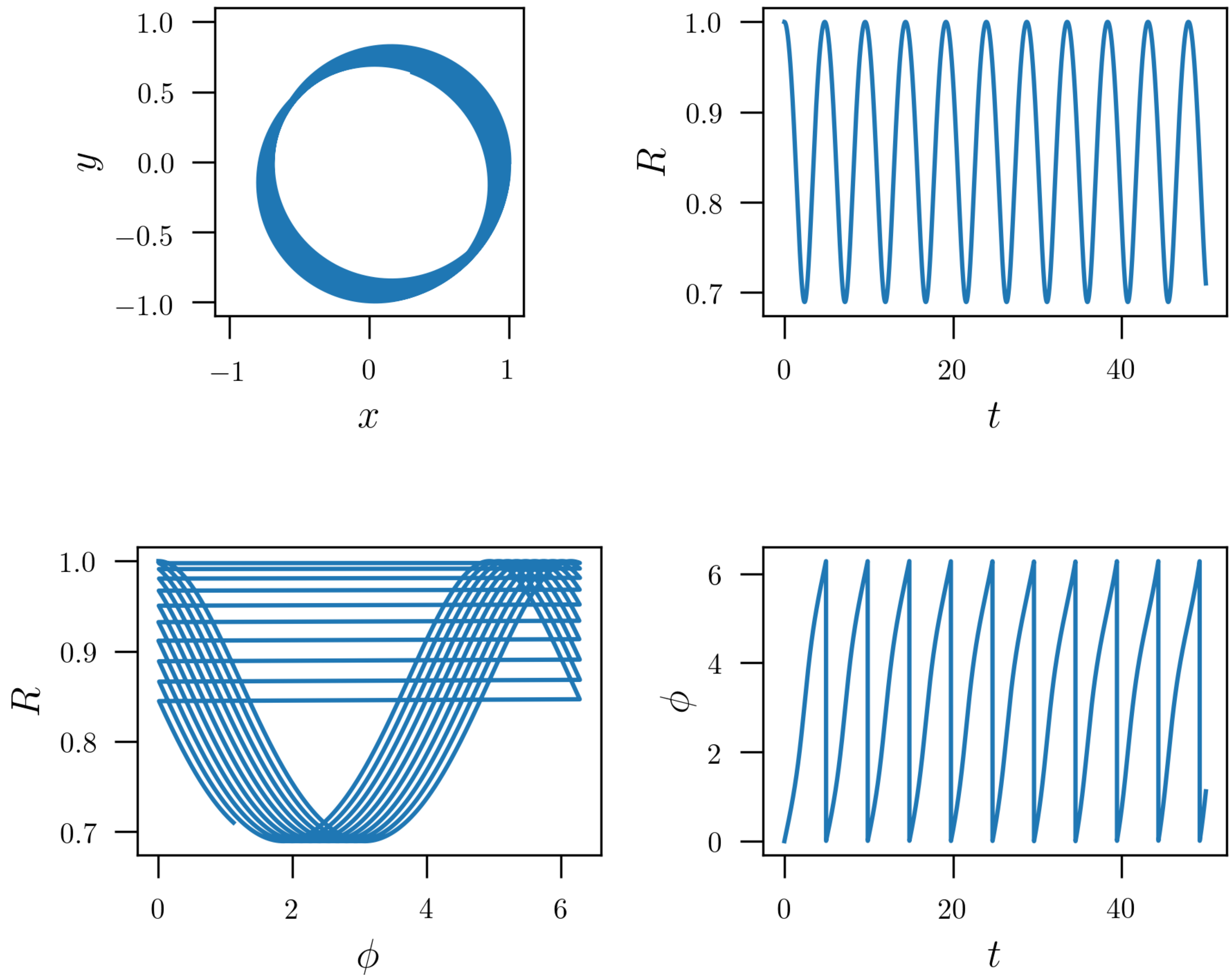
Plummer



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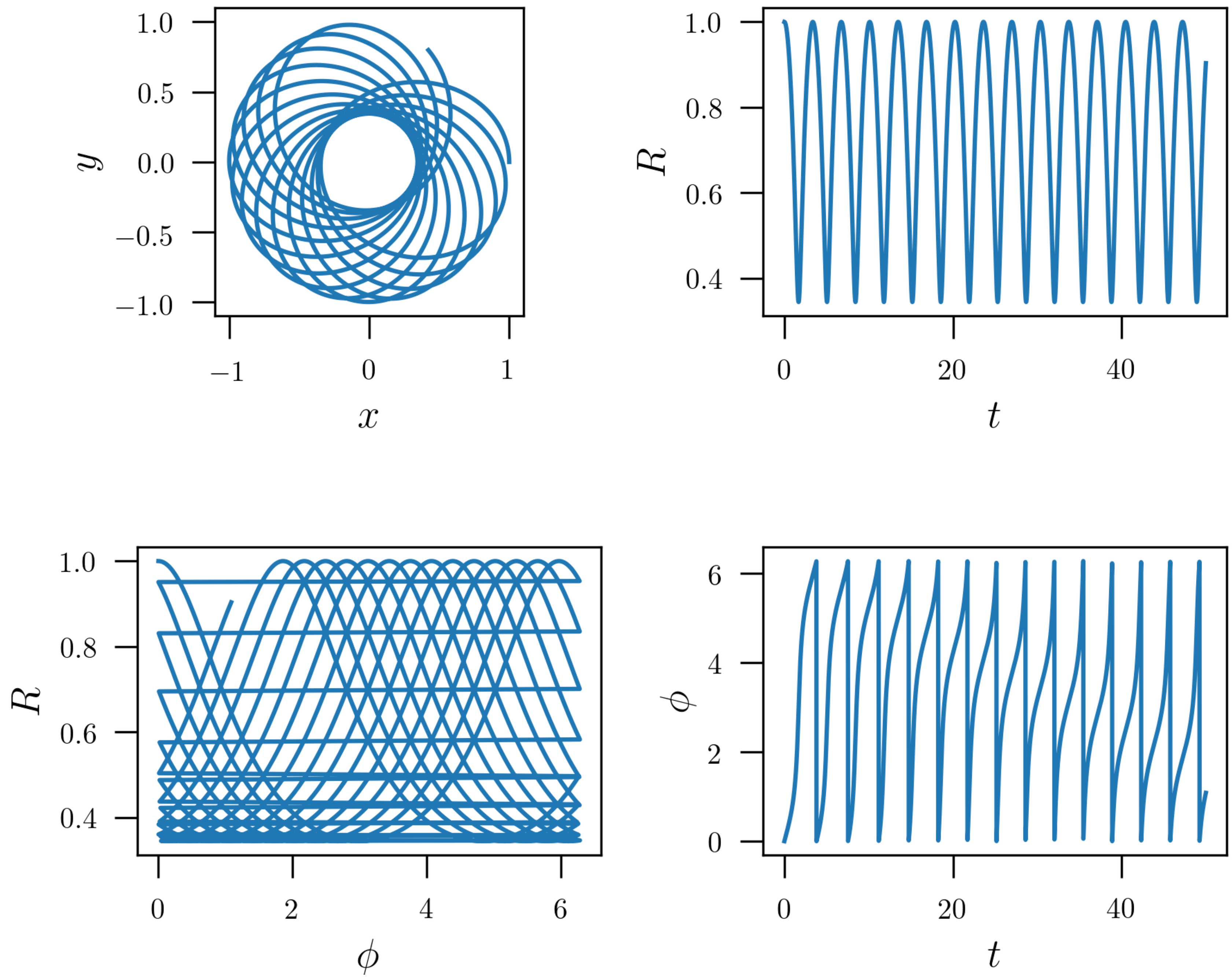
Plummer



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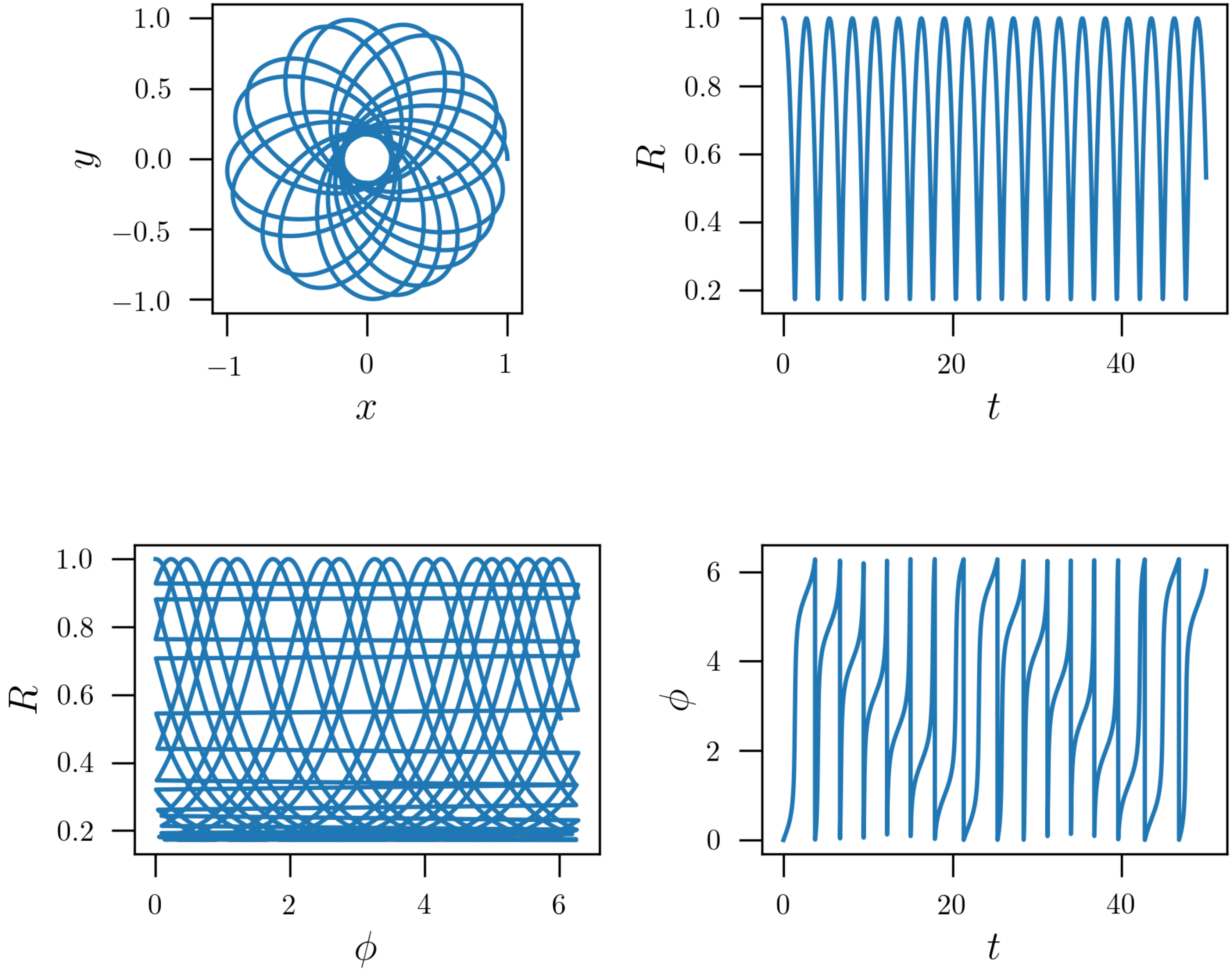
Plummer



Radial period

Time to travel from the apocenter to the pericenter

$$T_r = 2\int_{t_1}^{t_2} dt \underset{\substack{r(t) \\ dr = \frac{dr}{dt}dt}}{=} 2\int_{r_1}^{r_2} \frac{1}{\dot{r}}\, dr \qquad \begin{cases} r(t_1) = r_1 \\ r(t_2) = r_2 \end{cases}$$

From

$$E = \frac{1}{2}\left(\dot{r}^2 + (r\dot{\varphi})^2\right) + \phi(r) = \frac{1}{2}\dot{r}^2 + \frac{L}{2r^2} + \phi(r)$$

$$\dot{r}^2 = 2\left(E - \phi(r)\right) - \frac{L^2}{r^2}$$

$$\frac{dr}{dt} = \sqrt{2\left(E - \phi(r)\right) - \frac{L^2}{r^2}}$$

$$\frac{dt}{dr} = \frac{1}{\sqrt{2\left(E - \phi(r)\right) - \frac{L^2}{r^2}}}$$

$$\boxed{T_r = 2\int_{r_1}^{r_2} \frac{dr}{\sqrt{2\left(E - \phi(r)\right) - \frac{L^2}{r^2}}}}$$

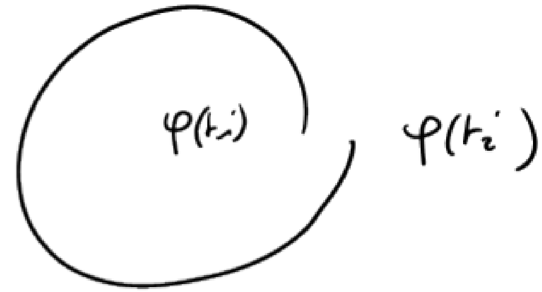
Increase of azimuth in a radial period

$$\Delta\varphi = 2\int_{\varphi_1}^{\varphi_2} d\varphi = 2\int_{t_1}^{t_2} \frac{d\varphi}{dt}\, dt = 2\int_{r_1}^{r_2} \frac{d\varphi}{dt}\, \frac{dt}{dr}\, dr$$

$$= 2\int_{r_1}^{r_2} \frac{L}{r^2}\, \frac{1}{\sqrt{2(\epsilon - \Phi(r)) - \frac{L^2}{r^2}}}\, dr$$

Azimuthal period

$$T_\varphi = \int_{t_1'}^{t_2'} dt \qquad \begin{cases} \varphi(t_1') = 0 \\ \varphi(t_2') = 2\pi \end{cases}$$



Using the radial period T_r and the azimuthal increase :

$$\boxed{T_\varphi = \frac{2\pi}{\Delta\varphi} \, T_r}$$

As in general $\dfrac{2\pi}{\Delta\varphi}$ is not a rational number

the orbit is not guarantee to be closed

Spherical Systems

Examples

Examples

① Kepler potential (potential of a mass point)

EXERCICE

$$\begin{cases} \phi(r) = -\dfrac{GM}{r} \\ \dfrac{\partial \phi}{\partial r}(r) = \dfrac{GM}{r^2} = GMu^2 \end{cases}$$

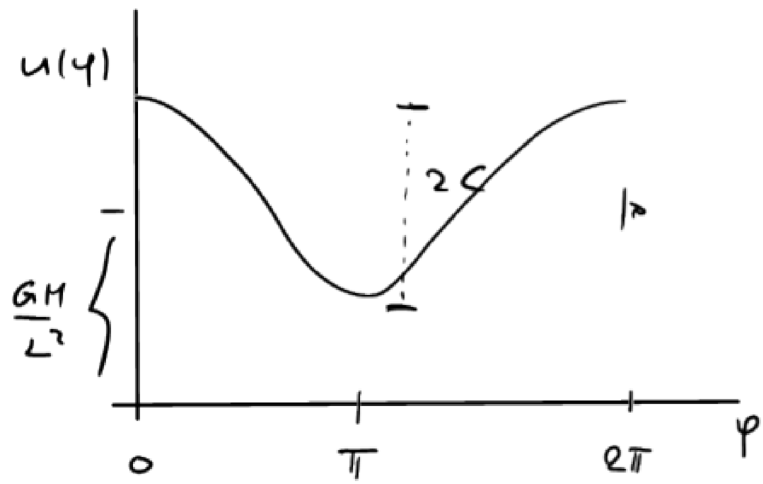
$$\boxed{\frac{d^2u}{d\varphi^2} + u = \frac{1}{L^2u^2}\frac{\partial \phi}{\partial r}\left(\frac{1}{u}\right)} \Rightarrow \boxed{\frac{d^2u}{d\varphi^2} + u = \frac{GM}{L^2}}$$

Harmonic equation,
with frequency 1

$$\boxed{\frac{d^2u}{d\varphi^2} + u = \frac{GM}{L^2}}$$

General solution

$$u(\varphi) = \underbrace{C}_{\text{free parameter}} \cos(\varphi - \underbrace{\varphi_0}_{\text{free parameter}}) + \frac{GM}{L^2}$$



period = 2π

In term of r

$$\boxed{r(\varphi) = \frac{1}{C\cos(\varphi - \varphi_0) + \frac{GM}{L^2}}}$$

Introducing

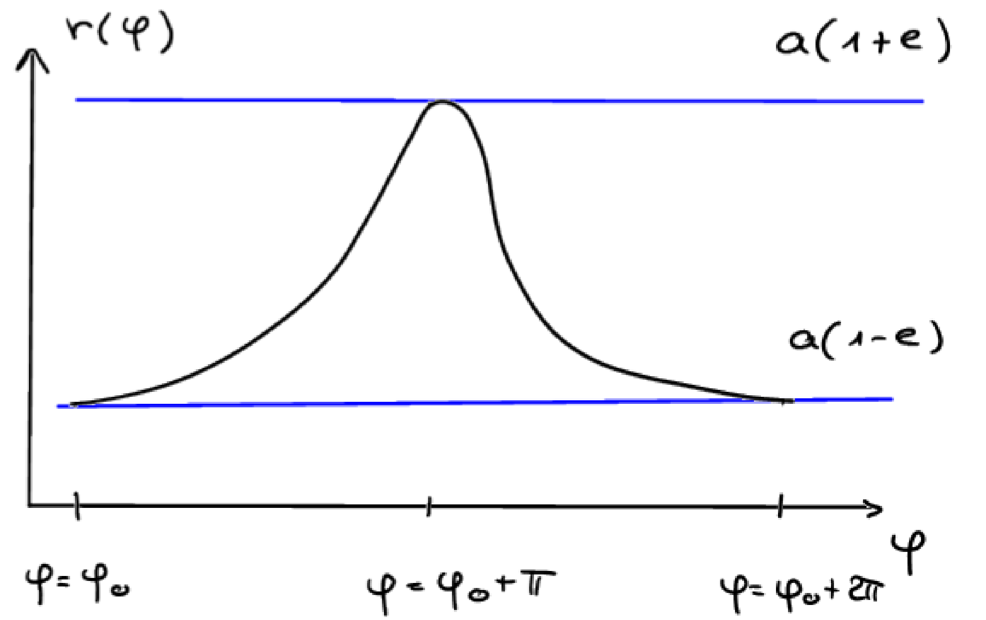
$$\begin{cases} e = \dfrac{CL^2}{GM} & \text{eccentricity} \\[2ex] a = \dfrac{L^2}{GM(1-e^2)} & \text{semi-major axis} \end{cases}$$

evaluate u and $\dfrac{du}{dt}$ for $\varphi = \varphi_0$ $\left(u(\varphi) \overset{\varphi=\varphi_0}{=} C + \dfrac{GM}{L^2} \quad \dfrac{du}{dt}(\varphi) \overset{\varphi=\varphi_0}{=} 0 \right)$

+ using $\dfrac{d^2u}{d\varphi^2} + u = \dfrac{1}{L^2u^2} \dfrac{\partial \phi}{\partial r}\left(\dfrac{1}{u}\right)$

$$\boxed{\begin{cases} r(\varphi) = \dfrac{a(1-e^2)}{1 + e\cos(\varphi - \varphi_0)} \\[2ex] E = -\dfrac{GM}{2a} \end{cases}}$$

↳ from the energy equation



Cases

$$\boxed{r(\varphi) = \frac{a(1-e^2)}{1 + e\cos(\varphi - \varphi_0)}}$$

$e \geq 1$

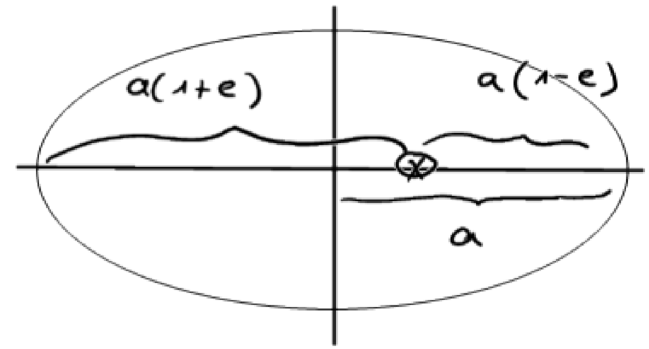
unbound orbit as $1 + e\cos(\varphi - \varphi_0)$ can be $= 0$

$\Rightarrow r \to \infty$

$e < 1$

bound orbit (ellipse)

EXERCICE



pericenter / apocenter

$$r_{min} = \frac{a(1-e^2)}{1+e} = a(1-e)$$

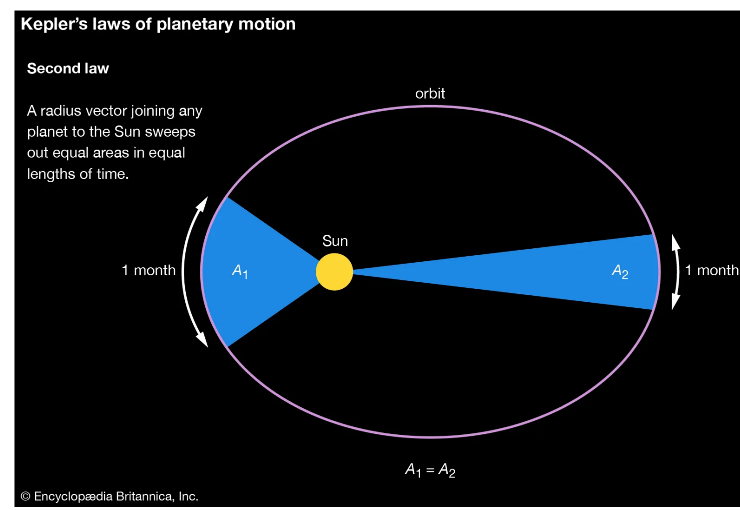
$$r_{max} = \frac{a(1-e^2)}{1-e} = a(1+e)$$

$e = 0$

$r_{min} = r_{max} = a$ (circular orbit)

Kepler laws (1609-1619) :

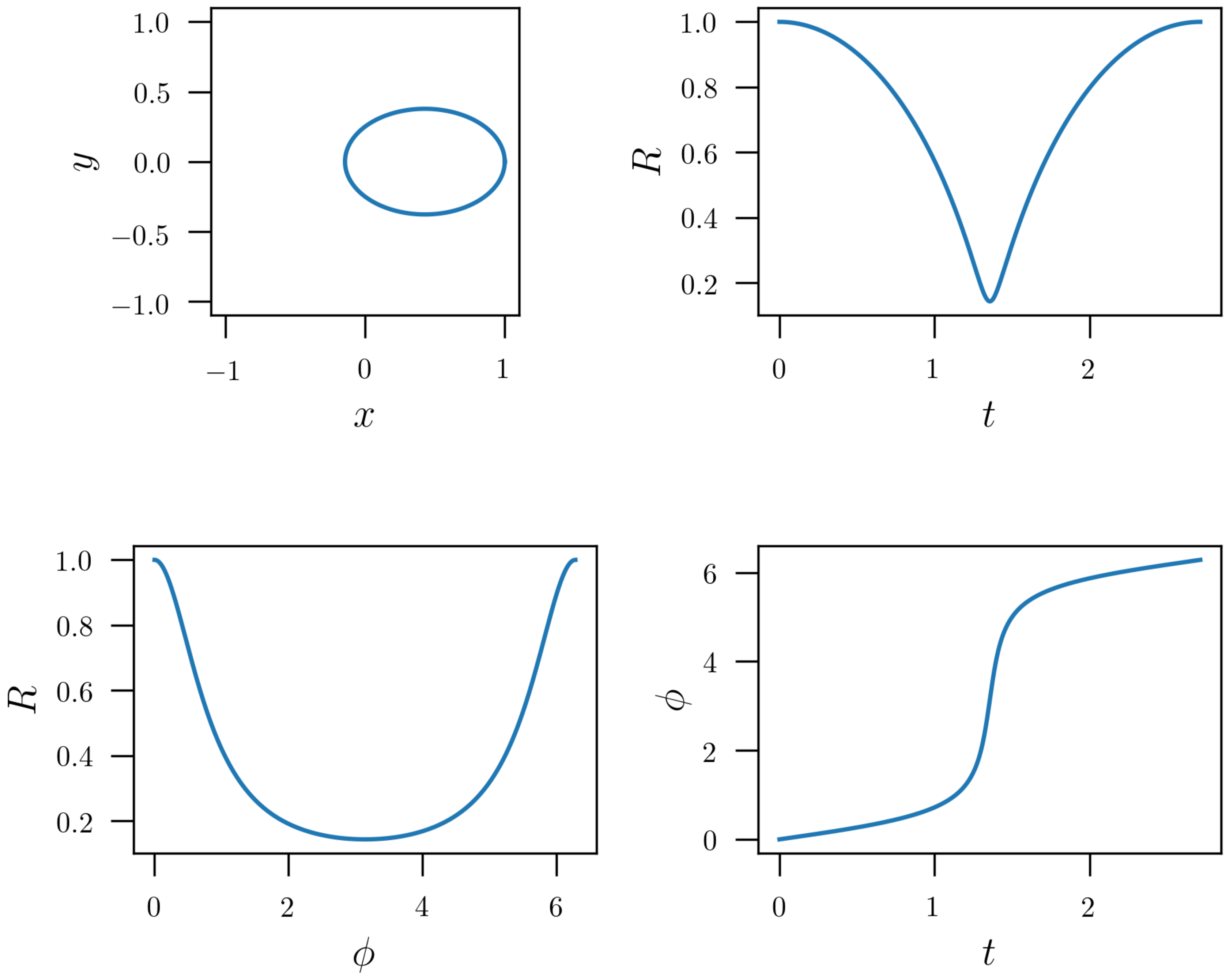
- The orbit of a planet is an ellipse with the Sun at one of the two foci.
- A line segment joining a planet and the Sun sweeps out equal areas during equal intervals of time.
- The square of a planet's orbital period is proportional to the cube of the length of the semi-major axis of its orbit.



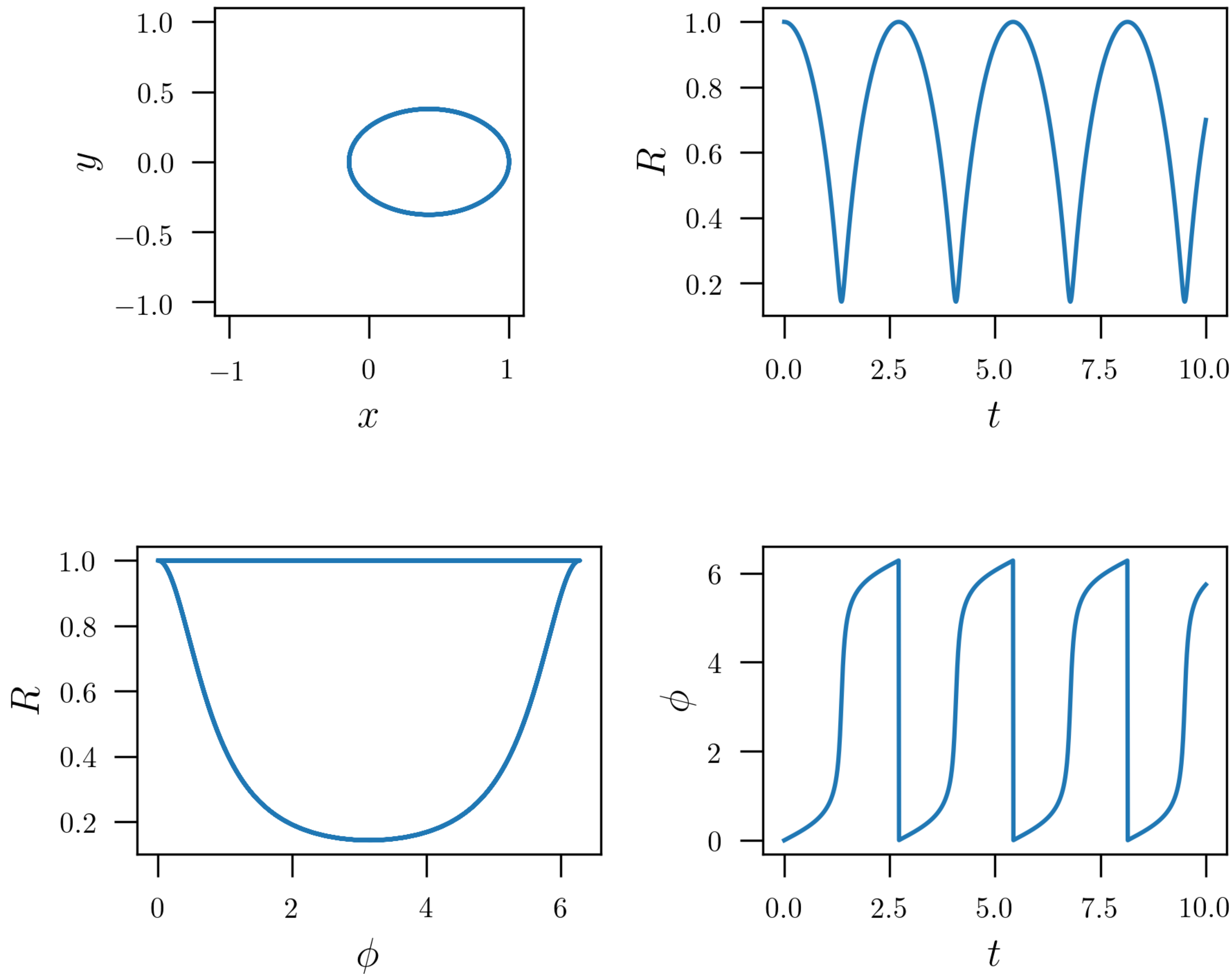
Radial and azimuthal periods:

$$T_r = T_\phi$$

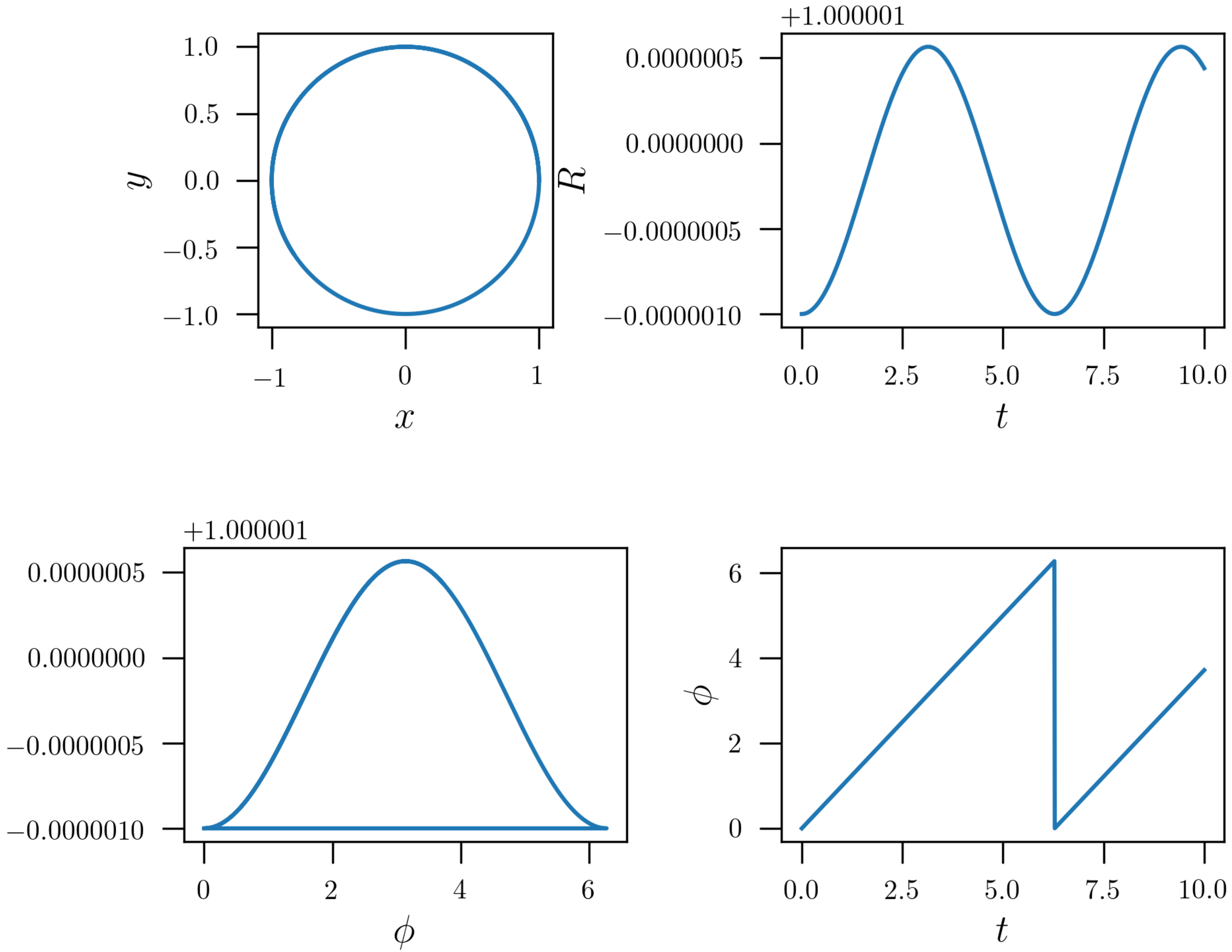
Keplerian orbits (point mass)



Keplerian orbits (point mass)



Keplerian orbits (point mass)



② Homogeneous sphere ρ_0, R_0 (Harmonic oscillations)

$$\phi(r) = \underbrace{-2\pi G\rho_0 R_0^2}_{\text{cte} \to 0} + \frac{2}{3}\pi G\rho_0 r^2$$

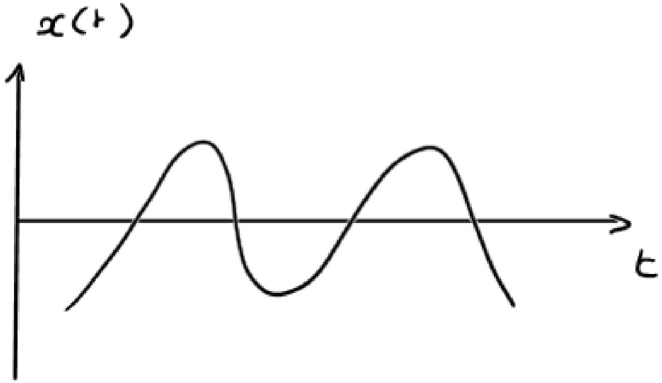
$$\phi(r) = \frac{1}{2}\Omega^2 r^2 \qquad \text{with } \Omega = \sqrt{\frac{4}{3}\pi G\rho_0}$$

Equations of motion (in carthesian coordinates)

$$L(x, y, \dot{x}, \dot{y}) = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\dot{y}^2 - \frac{1}{2}\Omega^2(x^2 + y^2)$$

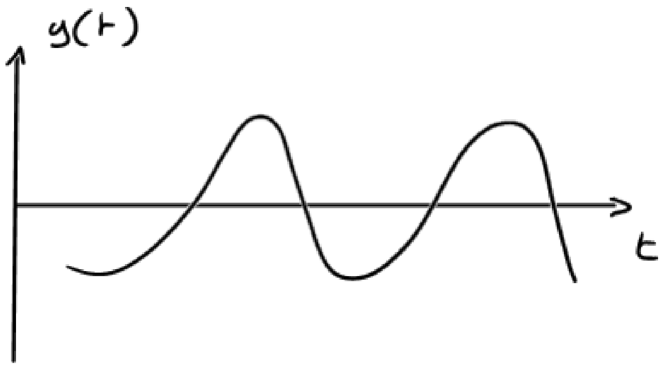
$$\begin{cases} \ddot{x} = -\Omega^2 x \\ \ddot{y} = -\Omega^2 y \end{cases} \qquad \begin{cases} x(t) = X\cos(\Omega t + \varepsilon_x) \\ y(t) = Y\cos(\Omega t + \varepsilon_y) \end{cases}$$

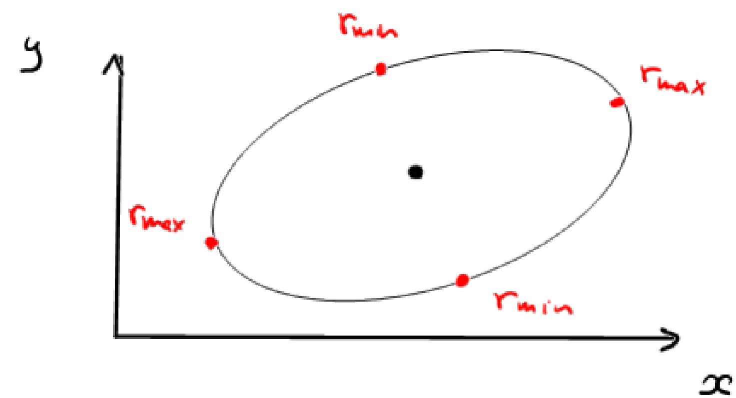
$X, Y, \varepsilon_x, \varepsilon_y$ constants fixed by the initial conditions



same period

$\Rightarrow$ closed orbits (ellipse)

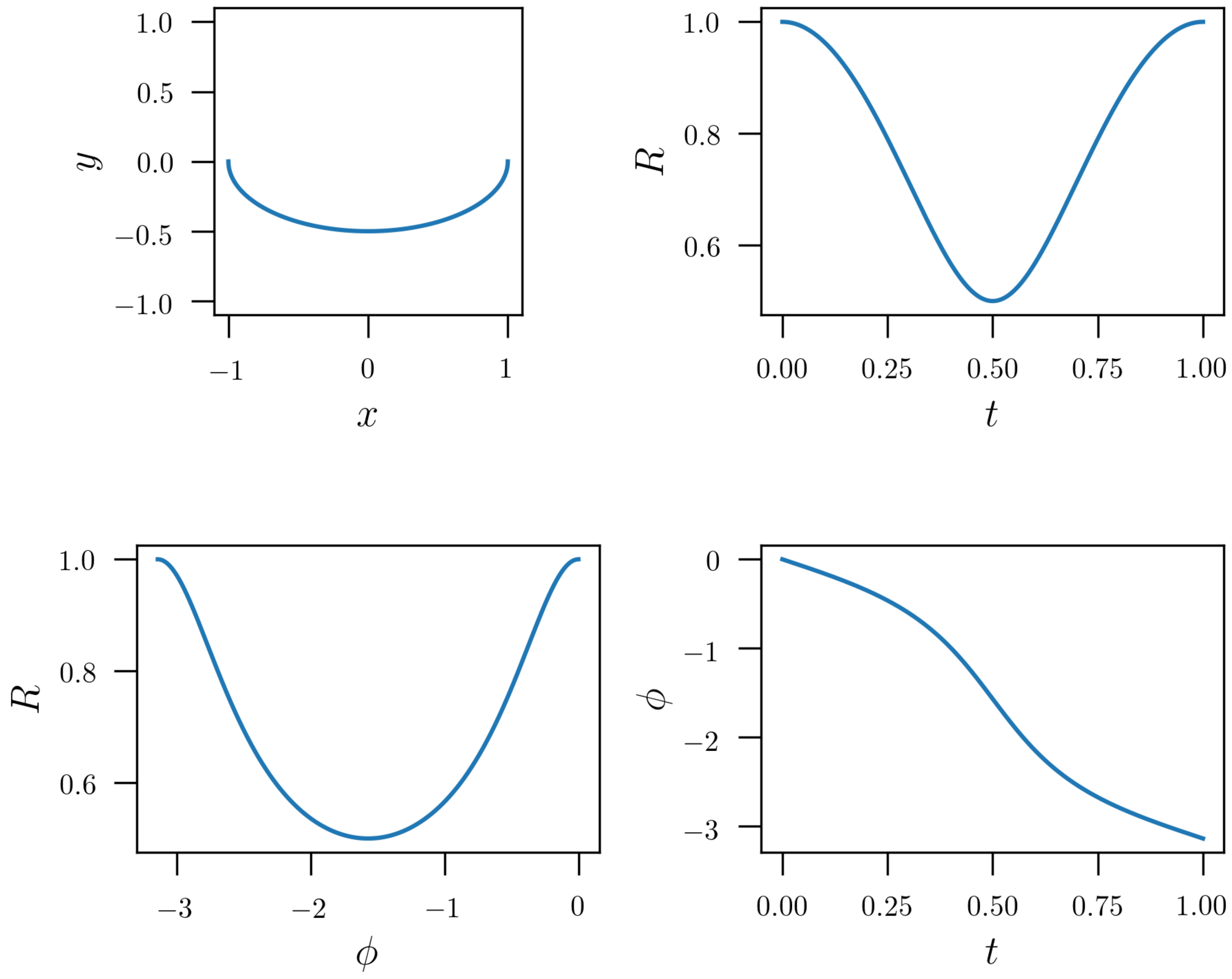




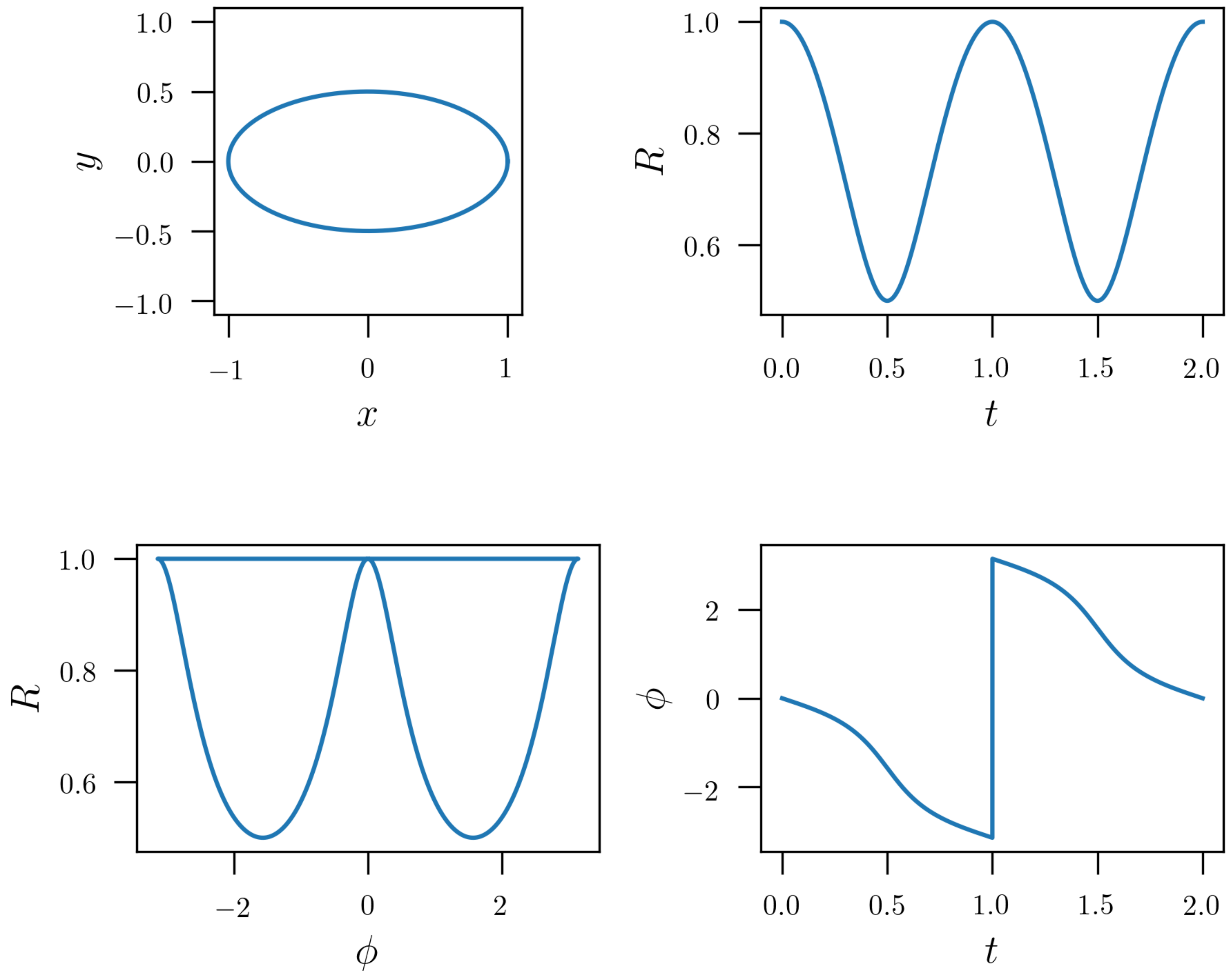
Periods

$$T_\varphi = \frac{2\pi}{\Omega} \qquad T_r = \frac{1}{2} T_\varphi = \frac{\pi}{\Omega}$$

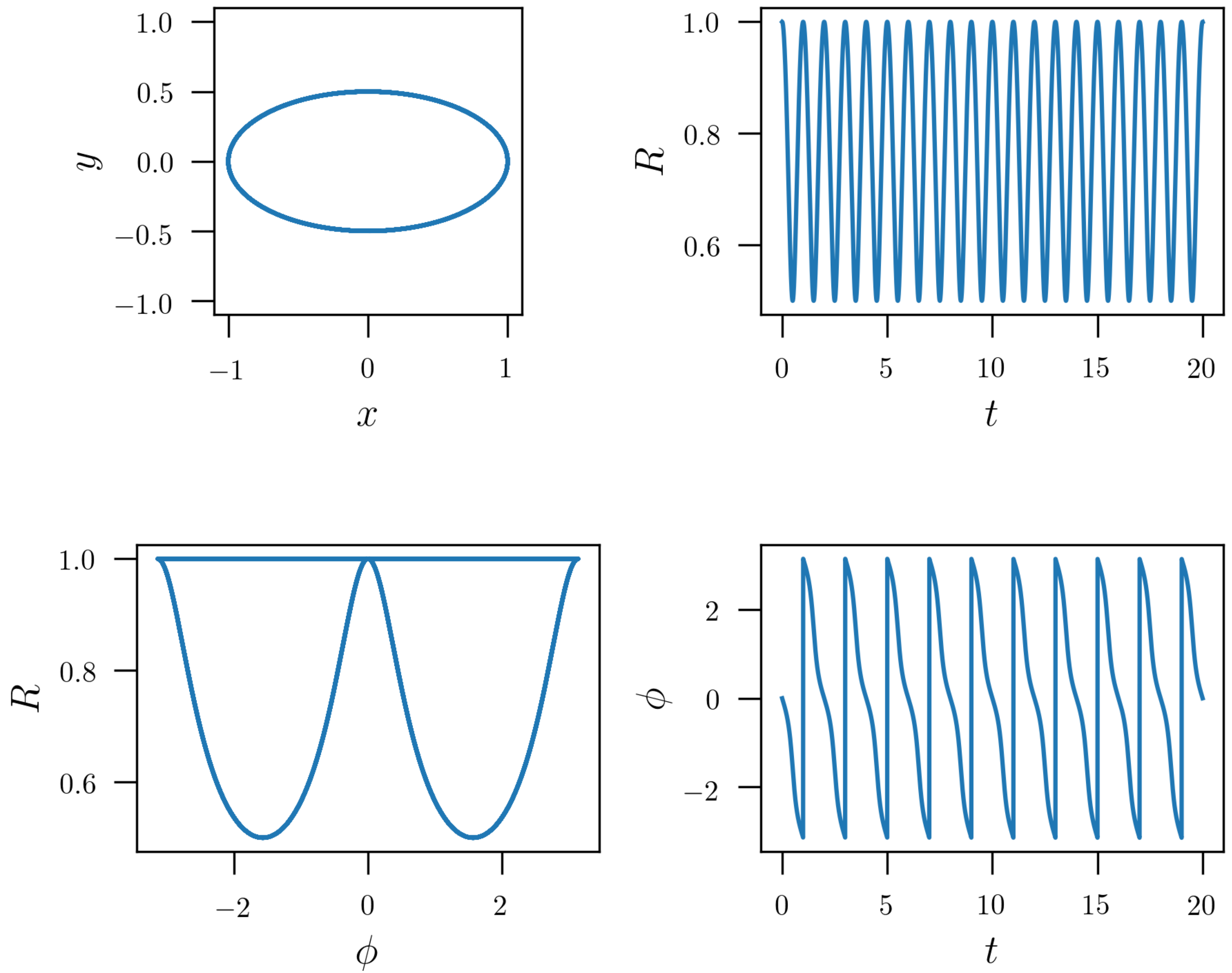
Homegeneous sphere (harmonic)



Homegeneous sphere (harmonic)



Homegeneous sphere (harmonic)



Isochrone potential

Good galaxy model that leads to analytical orbits

$$\boxed{\phi(r) = -\frac{GM}{b+\sqrt{b^2+r^2}}}$$

New variable

$$s = -\frac{GM}{b\,\phi(r)} = \frac{b+\sqrt{b^2+r^2}}{b} = 1+\sqrt{1+\frac{r^2}{b^2}}$$

Henon 1959

solution of $s^2 - 2s - \frac{r^2}{b^2} = 0$

$$\Rightarrow \quad \frac{r^2}{b^2} = s^2\left(1-\frac{2}{s}\right)$$

We can write

$$\frac{ds}{dt} = \frac{ds}{dr}\frac{dr}{dt} \quad \Rightarrow \quad \boxed{s(t) = \int_{t_0}^{t} \frac{ds}{dr}\frac{dr}{dt}\,dt}$$

can be integrated

$$\frac{ds}{dt} = \frac{ds}{dr}\frac{dr}{dt} = \left(1+\frac{r^2}{b^2}\right)^{-\frac{1}{2}} \frac{r}{b^2}\sqrt{2(E-\phi) - \frac{L^2}{r^2}}$$

COMPLEMENT

Radial and azimuthal periods

$$T_r = 2\int_{r_1}^{r_2} \frac{dr}{\sqrt{2(E-\phi) - \frac{L^2}{r^2}}} \quad \text{and} \quad \Delta\varphi = 2L\int_{r_1}^{r_2} \frac{dr}{r^2\sqrt{2(E-\phi) - \frac{L^2}{r^2}}}$$

as $\frac{dr}{dt} = \sqrt{2(E-\phi) - \frac{L^2}{r^2}}$ $\qquad \underbrace{2(E-\phi) - \frac{L^2}{r^2}}_{(r-r_1)(r-r_2)} = 0$ solutions $r_1\, r_2$

We can re-write $\frac{dr}{\sqrt{2(E-\phi) - \frac{L^2}{r^2}}}$ in term of s

$$T_r = \frac{2b}{\sqrt{-2E}} \int_{s_1}^{s_2} ds \frac{s-1}{\sqrt{(s_2-s)(s-s_1)}}$$

COMPLEMENT

COMPLEMENT

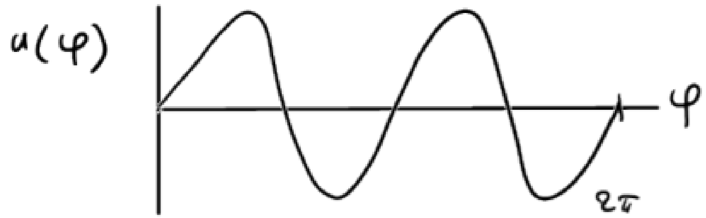
$$\left\{\begin{aligned} T_r &= \frac{2\pi GM}{(-2E)^{3/2}} \\ T_\psi &= \frac{4\pi GM}{(-2E)^{3/2}} \, \frac{\sqrt{L^2 + 4GMb}}{|L| + \sqrt{L^2 + 4GMb}} \end{aligned}\right.$$

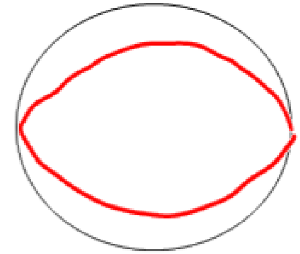
$\longrightarrow$ ⚠ independent of L (isochrone)

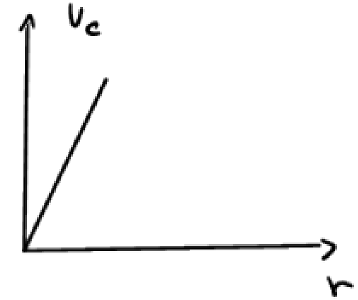
Important Remarks

Homogeneous sphere

$$\boxed{T_r = \frac{1}{2} T_\varphi}$$



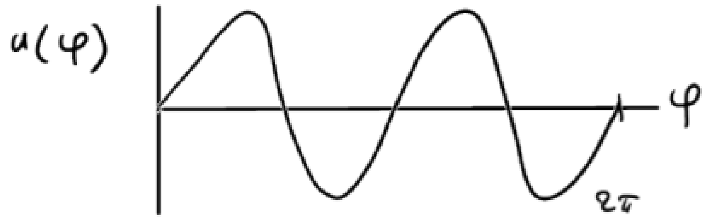


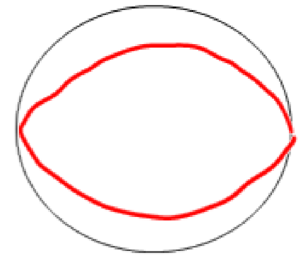


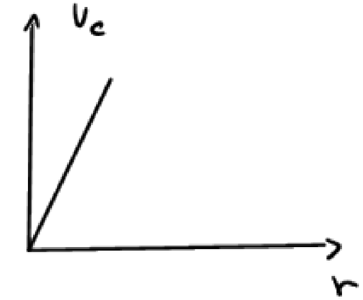
Important Remarks

Homogeneous sphere

$$\boxed{T_r = \frac{1}{2} T_\varphi}$$

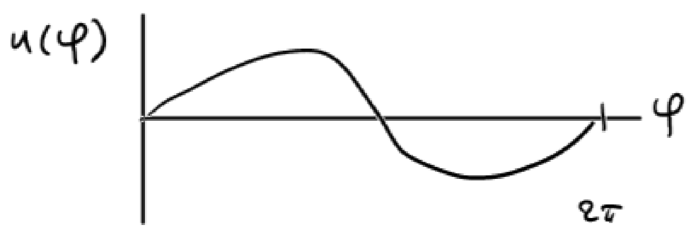


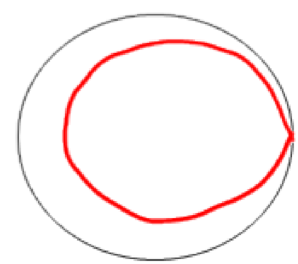




Keplerian potential

$$\boxed{T_r = T_\varphi}$$

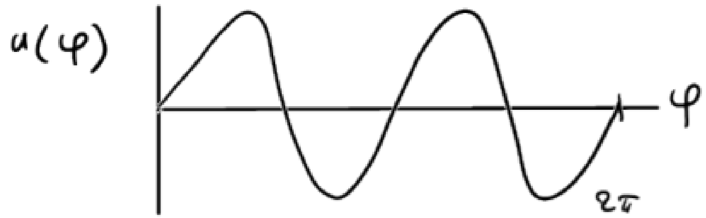


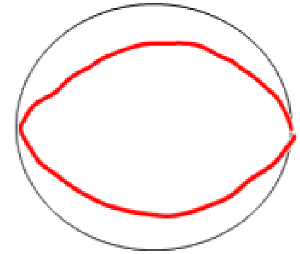


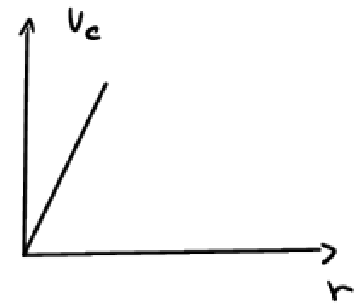
Important Remarks

Homogeneous sphere

$$\boxed{T_r = \frac{1}{2} T_\varphi}$$

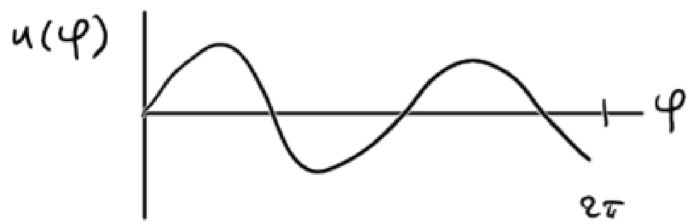


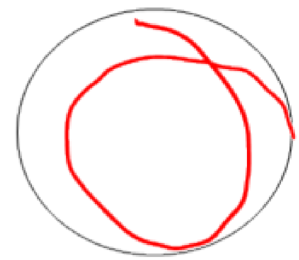




Galaxy

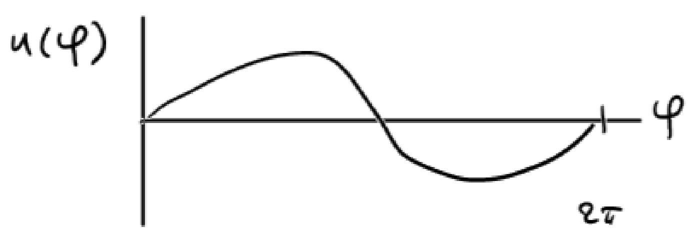
$$\boxed{\frac{1}{2} T_\varphi < T_r < T_\varphi}$$

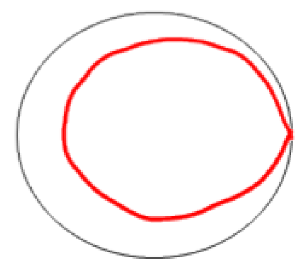


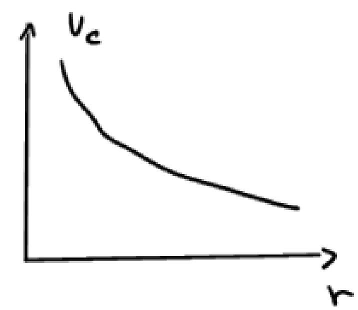


Keplerian potential

$$\boxed{T_r = T_\varphi}$$







The End