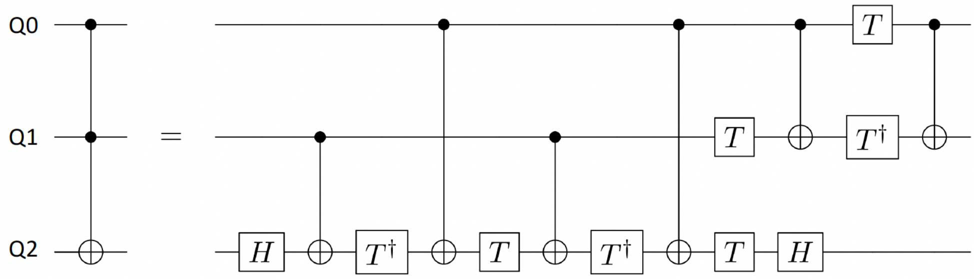
Exercise Set 4
Quantum Computation

Exercise 1 *Coding practice in qBraid: Implementation and tests with the Toffoli gate*

In Homework 3, you proved the following circuit identity:



The goal of this exercise is to "test" this identity using, first, a quantum simulator, and second, a noisy quantum simulator/computer. Please see the accompanying Jupyter notebook (Workbook Week 4) for details.

Exercise 2 *Square-root of the NOT gate*

The aim of the present exercise is to compute, for a given one-qubit gate U, a corresponding gate V such that $V^2 = U$ in the particular case where $U = X$ (the NOT gate). Here is first a description of the generic procedure.

First observe that since a one-qubit gate U is a 2×2 unitary matrix ($UU^\dagger = U^\dagger U = I$), it is in particular a *normal* matrix satisfying $UU^\dagger = U^\dagger U$. The *spectral theorem* then asserts that such a U is *unitarily diagonalizable*, i.e., there exists Λ a 2×2 diagonal matrix (with possibly complex entries) and W another 2×2 unitary matrix, such that $U = W \Lambda W^\dagger$.

In order to compute Λ and W, it suffices to compute the two solutions to the eigenvalue-eigenvector equation:

$$U w^{(i)} = \lambda_i \, w^{(i)}, \quad i = 0, 1$$

with the added constraint that $(w^{(i)})^\dagger w^{(j)} = \delta_{i,j}$. Then $\Lambda = \mathrm{diag}(\lambda_0, \lambda_1)$ and $W = (w^{(0)}, w^{(1)})$, i.e., $w^{(i)}$ is the i-th column of W.

Finally, consider $V = W \sqrt{\Lambda} \, W^\dagger$, where $\sqrt{\Lambda} = \mathrm{diag}(\sqrt{\lambda_0}, \sqrt{\lambda_1})$, with square roots being taken in the complex plane $\mathbb{C}$ (! two options for each of them !). You can check that $V^2 = U$.

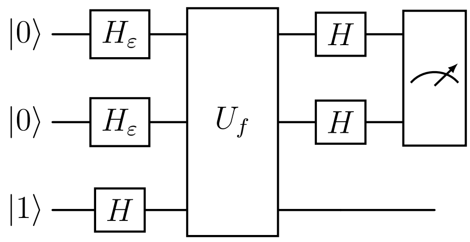
(a) Compute the 2×2 matrices Λ and W corresponding to $U = X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

(b) Deduce an explicit expression for a matrix V such that $V^2 = X$. Check now directly that $V^2 = X$.

(c) Is V also unitary? Justify your answer.

Exercise 3 *Deutsch-Josza's algorithm with noisy Hadamard gates*

Let us consider Deutsch's problem with $n = 2$. The aim of the algorithm is to decide whether $f : \{0,1\}^2 \to \{0,1\}$ is constant or balanced by using the following circuit:



where the Hadamard gates H_ε (with $0 \leq \varepsilon \leq 1$) are defined as

$$\begin{cases} H_\varepsilon |0\rangle = \sqrt{\frac{1+\varepsilon}{2}}\, |0\rangle + \sqrt{\frac{1-\varepsilon}{2}}\, |1\rangle \\ H_\varepsilon |1\rangle = \sqrt{\frac{1-\varepsilon}{2}}\, |0\rangle - \sqrt{\frac{1+\varepsilon}{2}}\, |1\rangle \end{cases}$$

(a) Verify that H_ε is unitary, for any $0 \leq \varepsilon \leq 1$.

(b) Compute the probability that the output state of the (first two qubits of the) above circuit is equal to $(0, 0)$ when f is constant.

(c) In order to ensure an error probability no greater than δ, what is the (approximate) maximum value taken by the parameter ε? Check in particular the cases $\delta = 0.1$ and $\delta = 0.01$.