

Astrophysics IV: Stellar and galactic dynamics

Solutions**Problem 1:**

From Poisson's equation in spherical coordinates we get:

$$\nabla^2\Phi = 4\pi G\rho$$

$\nabla^2\Phi$ written in spherical coordinates, and considering a spherical potential we get:

$$\nabla^2\Phi = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial\Phi}{\partial r} \right)$$

one then obtains

$$\nabla^2\Phi = \frac{3GMb^2}{(r^2 + b^2)^{5/2}}$$

and finally:

$$\rho = \frac{3M}{4\pi b^3} \left(1 + \frac{r^2}{b^2} \right)^{-5/2}$$

Problem 2:

a) Point mass:

$$V_c^2(r) = \frac{GM}{r}$$

b) Homogeneous sphere of radius a :

$$V_c^2(r) = \begin{cases} \frac{GMr^2}{3} & \text{if } r < a \\ \frac{GM}{r} & \text{if } r \geq a \end{cases}$$

c) Plummer-Schuster potential:

$$V_c^2(r) = \frac{GMr^2}{(r^2 + a^2)^{3/2}}$$

d) Miyamoto-Nagai potential:

$$V_c^2(R) = \frac{GMR^2}{[R^2 + (a + b)^2]^{3/2}}$$

Problem 3:

We are still in the plane $z = 0$ (where the rotation curves are defined.) With the parametrization:

$$\begin{aligned} h_R &= a + b \\ h_z &= b \end{aligned}$$

the circular velocity of the Miyamoto-Nagai potential can be written:

$$V_c^2(R) = \frac{GM R^2}{(R^2 + h_R^2)^{3/2}}$$

which is obviously independent of the scale height h_z . This parametrization is more telling than the a, b one: it shows how a Miyamoto-Nagai system has a circular velocity independent of the flattening of the potential. The two extremes are:

- spherical symmetry: $a = 0 \implies h_R = h_z = b$,
- thin disk: $b = 0 \implies h_R = a, h_z = 0$.

The rotation in the plane $z = 0$ is the same for these two extreme cases since V_c^2 is independent of h_z .

Problem 4:

From Poisson's equation in spherical coordinates we get:

$$\nabla^2 \Phi = 4\pi G \rho$$

$\nabla^2 \Phi$ written in spherical coordinates, and considering a spherical potential we get:

$$\nabla^2 \Phi = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \Phi}{\partial r} \right)$$

a lot of straight-forward algebra follows, but finally we get

$$\rho = \frac{v_s^2}{4\pi G r_s^2} \frac{1}{(r/r_s)(1 + r/r_s)^2}$$

The circular velocity also follows simply:

$$\begin{aligned} v_c^2 &= r \frac{\partial \Phi}{\partial r} = r v_s^2 \left[-\frac{1}{\frac{r}{r_s} \left(1 + \frac{r}{r_s}\right)} + \frac{\ln\left(1 + \frac{r}{r_s}\right)}{r_s \left(\frac{r}{r_s}\right)^2} \right] = v_s^2 \left[\frac{\ln\left(1 + \frac{r}{r_s}\right)}{\frac{r}{r_s}} - \frac{1}{\left(1 + \frac{r}{r_s}\right)} \right] \\ &= v_s^2 \left[\frac{\left(1 + \frac{r}{r_s}\right) \ln\left(1 + \frac{r}{r_s}\right) - \frac{r}{r_s}}{\frac{r}{r_s} \left(1 + \frac{r}{r_s}\right)} \right] = v_s^2 \frac{(r_s + r) r_s \ln\left(1 + \frac{r}{r_s}\right) - r r_s}{r (r_s + r)} \end{aligned}$$

Problem 5:

As per problem 4, the isochrone ρ is straightforward to derive, taking the form:

$$\rho = M \left[\frac{3(b+a)a^2 - r^2(b+3a)}{4\pi(b+a)^3 a^3} \right] \quad \text{with} \quad a \equiv \sqrt{b^2 + r^2}$$

The circular velocity is

$$v_c^2 = \frac{GM r^2}{(b+a)^2 a}$$

Problem 6:

Let's define a unit surface on the disk, corresponding to a mass Σ , which is then the surface density. Defining a slab enclosing the unit surface and making its thickness tend to a vanishing value ($\varepsilon \rightarrow 0$, see Fig. 1), the surface integral reduces to twice the gradient of the potential:

$$4\pi G \Sigma = \int d^2\mathbf{S} \nabla\Phi_K = 2 \frac{\partial\Phi_K}{\partial z}$$

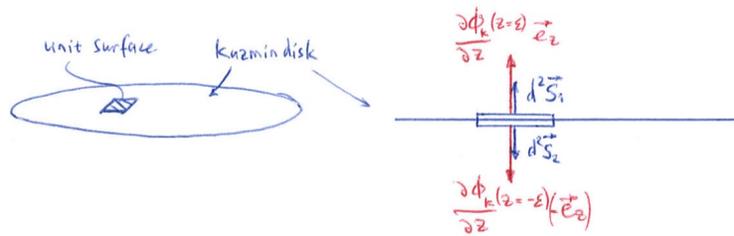


Figure 1: The Kuzmin disk with the unit surface (left) and seen edge-on (right), with the 2ε thick slab, on the surface of which the integration is made.

We have

$$\begin{aligned} \frac{\partial\Phi_K}{\partial z} &= \frac{\partial}{\partial z} \left[-GM [R^2 + (a + |z|)^2]^{-1/2} \right] \\ &= GM [R^2 + (a + |z|)^2]^{-3/2} (a + |z|) \end{aligned}$$

With $|z| \rightarrow 0$, we then have:

$$\begin{aligned} 4\pi G \Sigma_K &= 2 \frac{\partial\Phi_K}{\partial z} = 2aGM [R^2 + a^2]^{-3/2} \\ \Rightarrow \Sigma_K &= \frac{aM}{2\pi (R^2 + a^2)^{3/2}} \end{aligned}$$

Problem 7:

The velocity curve may be obtained from the formula (see course: result from a razor-thin homeoid since we cannot use Gauss law here):

$$v_c^2(R) = -4G \int_0^R da \frac{a}{\sqrt{R^2 - a^2}} \frac{d}{da} \int_a^\infty dR' \frac{R'\Sigma(R')}{\sqrt{R'^2 - a^2}} \quad (1)$$

Replacing $\Sigma(R')$ using the Mestel's surface density we get:

$$\begin{aligned} \int_a^\infty dR' \frac{R'\Sigma(R')}{\sqrt{R'^2 - a^2}} &= \frac{v_0^2}{2\pi G} \int_a^\infty dR' \frac{1}{\sqrt{R'^2 - a^2}} \\ &= \frac{v_0^2}{2\pi G} \int_a^{R_{\max}} dR' \frac{1}{\sqrt{(R'/a)^2 - 1}} \frac{1}{a} \\ &= \frac{v_0^2}{2\pi G} \int_a^{R_{\max}} dR' \frac{d}{dR'} (\operatorname{arccosh}(R/a)) \\ &= \frac{v_0^2}{2\pi G} [\operatorname{arccosh}(R_{\max}/a) - \operatorname{arccosh}(1)] \\ &= \frac{v_0^2}{2\pi G} \operatorname{arccosh}(R_{\max}/a) \end{aligned} \quad (2)$$

The derivative with respect to a of this latter result writes:

$$\begin{aligned} \frac{d}{da} \left(\frac{v_0^2}{2\pi G} \operatorname{arccosh}(R_{\max}/a) \right) &= \frac{v_0^2}{2\pi G} \frac{d}{da} \operatorname{arccosh}(R_{\max}/a) \\ &= -\frac{v_0^2}{2\pi G} \frac{R_{\max}}{\sqrt{R_{\max}^2 - a^2}} \frac{1}{a} \end{aligned} \quad (3)$$

which, in the limit $R_{\max} \rightarrow \infty$ gives:

$$-\frac{v_0^2}{2\pi G a} \quad (4)$$

This leads to the circular velocity:

$$\begin{aligned} v_c^2(R) &= \frac{2v_0^2}{\pi} \int_0^R da \frac{1}{\sqrt{R^2 - a^2}} \\ &= v_0^2 \end{aligned} \quad (5)$$