

# Potential Theory

end of the 2<sup>nd</sup> part

# Outlines

## Spherical systems:

- Rotation curves

## Examples of spherical models:

- “Potential based” models
- “Density based” models

## Axisymmetric models for disk galaxies

- “Potential based” models
- Potential of flattened systems
- Potential of infinite thin (razor-thin) disks
- “Potential based” razor-thin disks models
- Potential of infinite thin (razor-thin) disks from homoeoids
- Potential of infinite thin (razor-thin) disks from homoeoids : applications

## Idealized but useful models

- the infinite wire, the infinite slab
- infinite slab with oscillatory surface density, tightly wound spiral

# **Rotation curves (for spherical models)**





## Spherical systems : circular speed, circular velocity

Speed of a test particle in a circular orbit in the potential  $\phi(r)$  at a radius  $r$ :

$$\vec{g}_s = \vec{a}_c$$


$\vec{a}_c$  : centripetal acceleration

$$\frac{v_c^2}{r}$$

$\vec{g}_s$  : gravity acceleration (spec force)

$$-\frac{GM(r)}{r^2} = -\frac{\partial\phi}{\partial r}$$

$$v_c^2 = \frac{GM(r)}{r}$$

$$v_c^2 = r \frac{\partial\phi}{\partial r}$$

$$[v_c^2] : \frac{\text{erg}}{\text{s}}$$

as  $\phi$

$\equiv$  specific energy

$$GM(r) = r^2 \frac{\partial\phi}{\partial r}$$

## Velocity composition

Note:  $V_0^2$  scale with the mass ( $M(r)$ ): it is thus the "important" quantity (spec. energy)

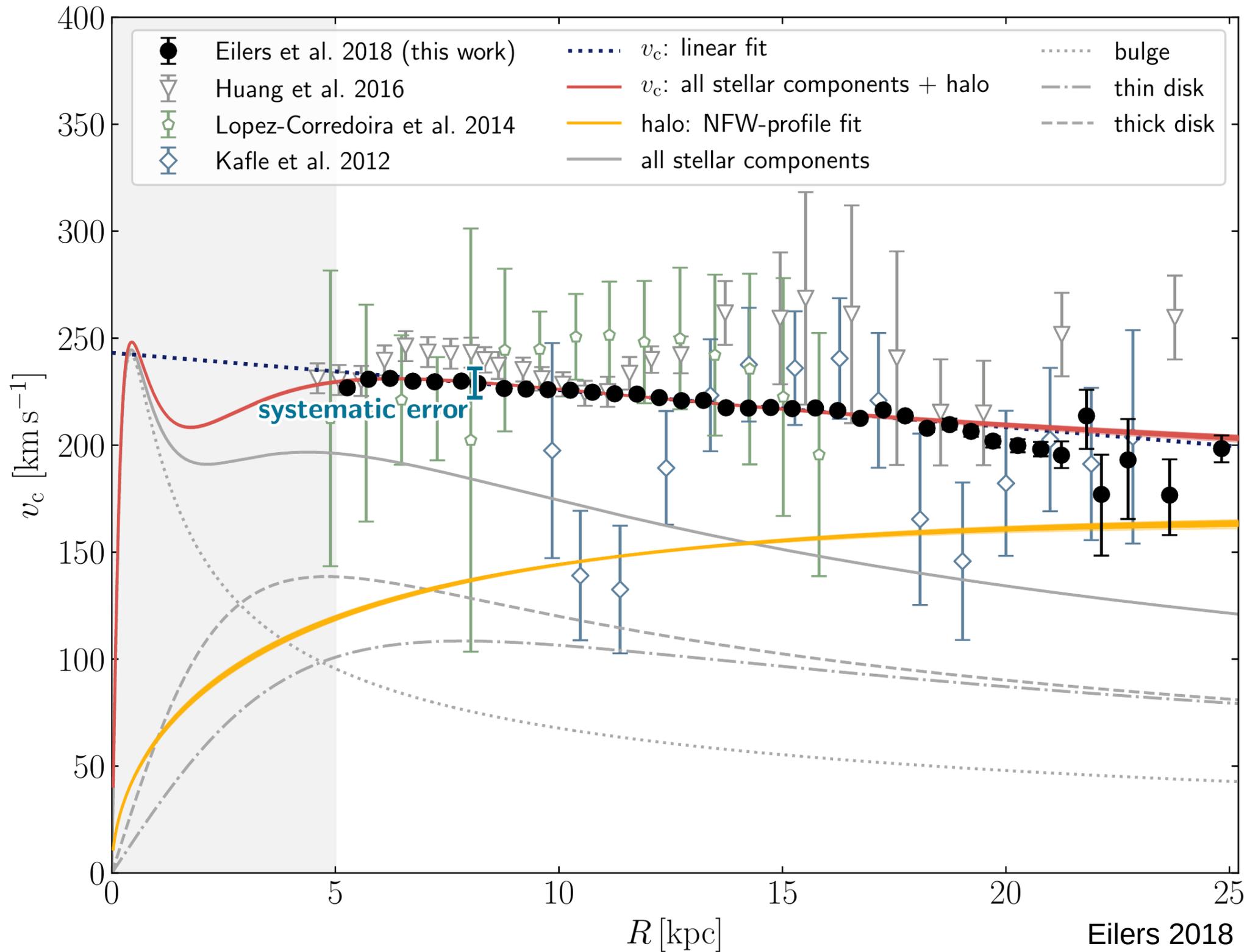
Multi-components system: ex: bulge + stellar halo + DM halo

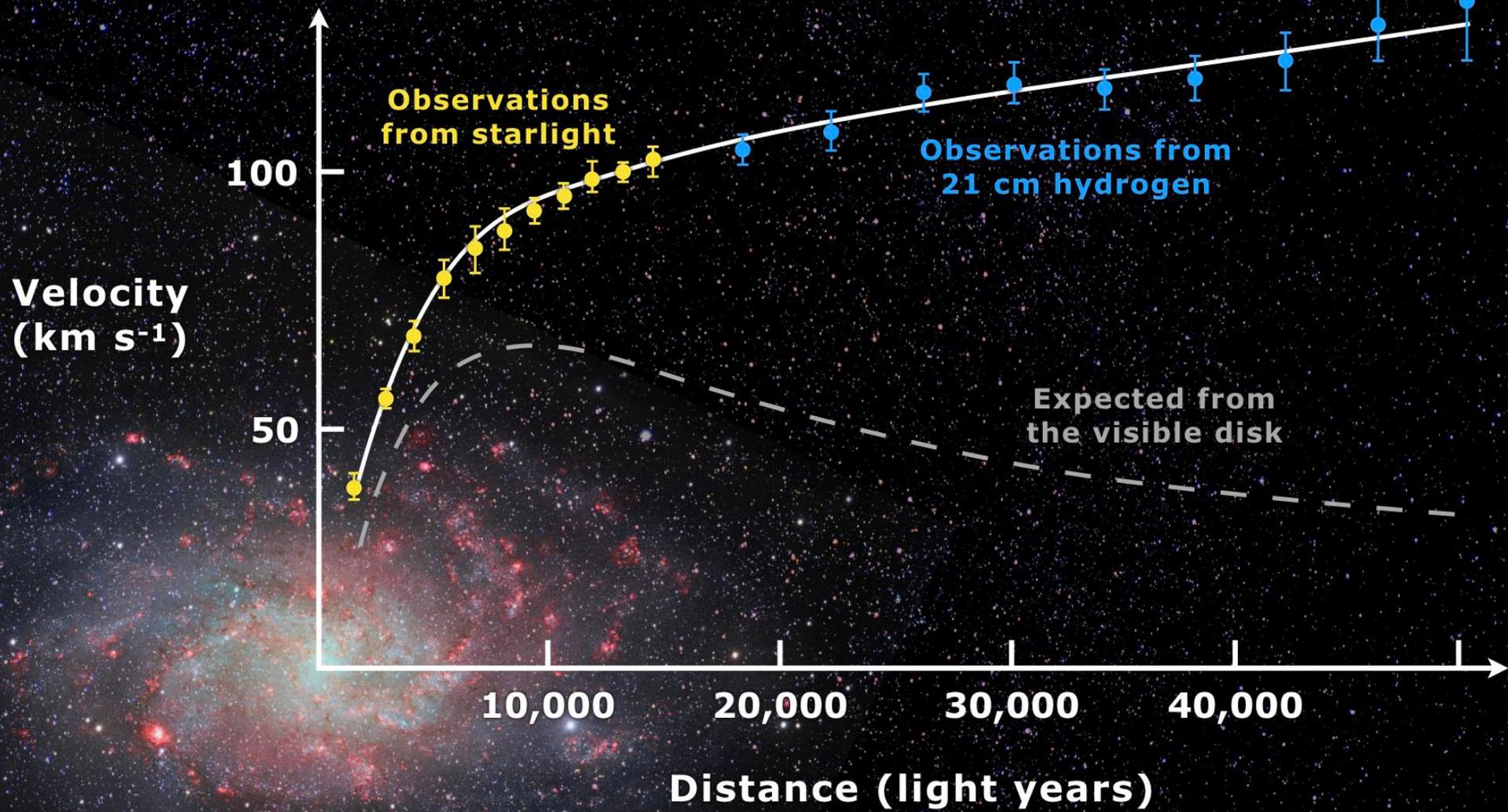
$$\left\{ \begin{array}{l} \rho_B(r) \quad , \quad M_B(r) \quad , \quad \phi_B(r) \quad \rightarrow \quad V_{c,B}(r) \\ \rho_H(r) \quad , \quad M_H(r) \quad , \quad \phi_H(r) \quad \rightarrow \quad V_{c,H}(r) \\ \rho_{DM}(r) \quad , \quad M_{DM}(r) \quad , \quad \phi_{DM}(r) \quad \rightarrow \quad V_{c,DM}(r) \end{array} \right.$$

$$V_{c,tot}^2 = \frac{GM_{tot}(r)}{r} = \frac{G}{r} \sum_i M(r)$$

$$V_{c,tot}^2 = \sum_i V_{c,i}^2$$

$V_c^2 \sim$  energy: extensive quantity





Period of the circular orbit

$$T(r) = \frac{2\pi r}{v_c(r)} = \frac{2\pi \sqrt{r^3}}{\sqrt{GM(r)}} = 2\pi \sqrt{\frac{r}{\frac{\partial \phi}{\partial r}}}$$

Circular frequency (angular frequency)

$$\Omega(r) = \frac{2\pi}{T(r)} = \sqrt{\frac{GM(r)}{r^3}} = \sqrt{\frac{1}{r} \frac{\partial \phi}{\partial r}}$$

## Potential energy

from  $W = - \int f(\vec{x}) \vec{x} \cdot \vec{\nabla} \phi(\vec{x}) d^3\vec{x}$

$$W = -4\pi G \int_0^{\infty} \rho(r) M(r) r dr$$

# Spherical systems : useful relations

	$\rho(r)$	$\Phi(r)$	$M(r)$	$\frac{d\Phi}{dr}$
$\rho(r)$	$\rho(r)$	$\frac{1}{4\pi G} \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d\Phi}{dr} \right)$	$\frac{1}{4\pi r^2} \frac{dM(r)}{dr}$	$\frac{1}{4\pi G} \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d\Phi}{dr} \right)$
$\Phi(r)$	$-\frac{GM(r)}{r} - 4\pi G \int_r^\infty dr' r' \rho(r')$	$\Phi(r)$	$-G \int_r^\infty dr' \frac{M(r')}{r'^2}$	$-\int_r^\infty dr' \frac{d\Phi}{dr}$
$M(r)$	$4\pi \int_0^r dr' r'^2 \rho(r')$	$\frac{r^2}{G} \frac{d\Phi}{dr}$	$M(r)$	$\frac{r^2}{G} \frac{d\Phi}{dr}$
$\frac{d\Phi}{dr}$	$\frac{4\pi G}{r^2} \int_0^r dr' r'^2 \rho(r')$	$\frac{d\Phi}{dr}$	$\frac{GM(r)}{r^2}$	$\frac{d\Phi}{dr}$

Poisson in spherical coordinates

$$\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d\Phi}{dr} \right) = 4\pi G \rho(r)$$

Mass inside a radius r

$$M(r) = 4\pi \int_0^r dr' r'^2 \rho(r')$$

Potential in spherical coordinates

$$\Phi(r) = -\frac{GM(r)}{r} - 4\pi G \int_r^\infty \rho(r') r' dr'$$

Gradient of the potential in spherical coordinates

$$\frac{d\Phi(r)}{dr} = \frac{GM(r)}{r^2}$$

# **Examples of Spherical models**

**“Potential based”  
models**

# Point mass

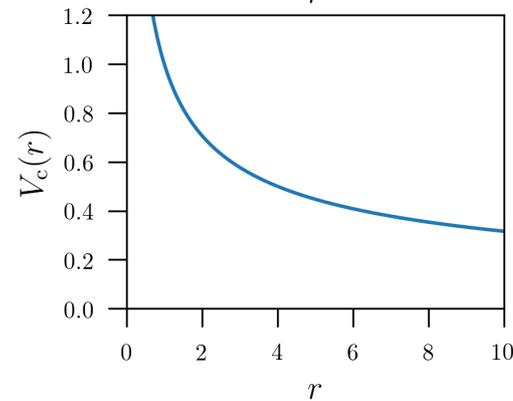
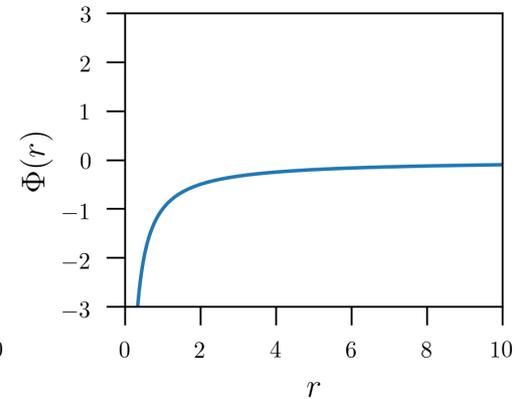
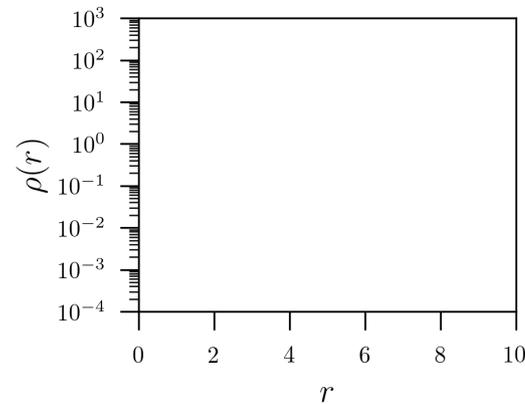
$$\Phi(r) = -\frac{GM}{r}$$

$$\rho(r) = \frac{M\delta(0)}{4\pi r^2}$$

$$M(r) = M$$

$$V_c^2(r) = \frac{GM}{r}$$

$$T(r) = 2\pi\sqrt{\frac{r^3}{GM}}$$



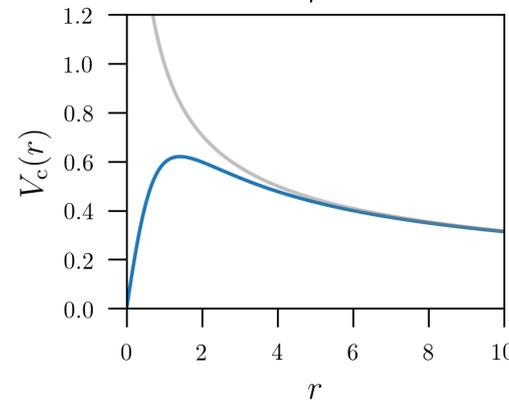
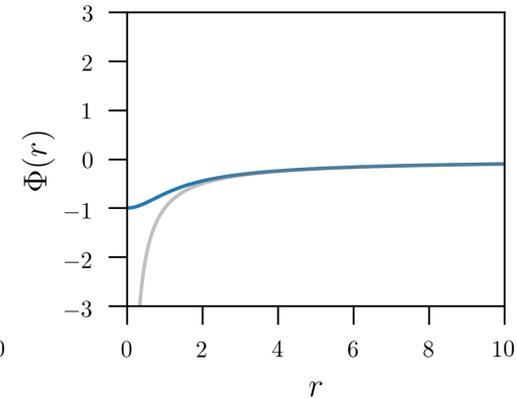
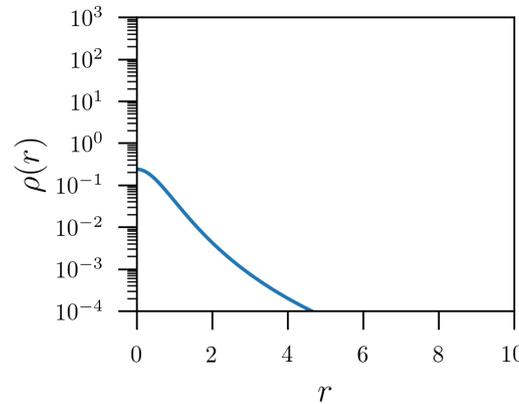
# Plummer model

$$\Phi(r) = -\frac{GM}{\sqrt{r^2 + b^2}}$$

$$\rho(r) = \frac{3M}{4\pi b^3} \left(1 + \frac{r^2}{b^2}\right)^{-5/2}$$

$$M(r) = \frac{Mr^3}{(r^2 + b^2)^{3/2}}$$

$$V_c^2(r) = \frac{GMr^2}{(r^2 + b^2)^{3/2}}$$



**EXERCICE**

- Globular clusters, dwarf spheroidal galaxies

# Isochrone potential

$$\Phi(r) = -\frac{GM}{b + \sqrt{r^2 + b^2}}$$

$$\rho(r) = M \frac{3(b + \sqrt{b^2 + r^2})(b^2 + r^2) - r^2(b + 3\sqrt{b^2 + r^2})}{4\pi(b + \sqrt{b^2 + r^2})^3(b^2 + r^2)^{3/2}}$$

$$M(r) = \frac{Mr^3}{\sqrt{b^2 + r^2}(b + \sqrt{b^2 + r^2})^2}$$

$$V_c^2(r) = \frac{GMr^2}{\sqrt{b^2 + r^2}(b + \sqrt{b^2 + r^2})^2}$$

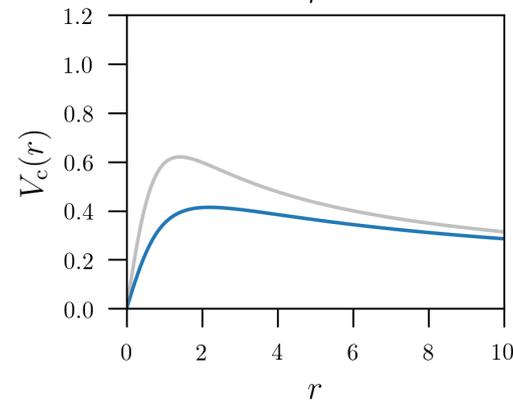
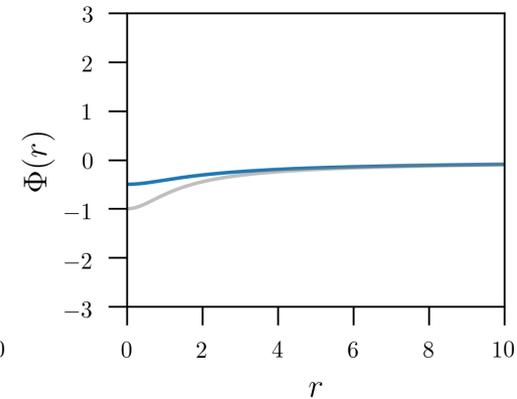
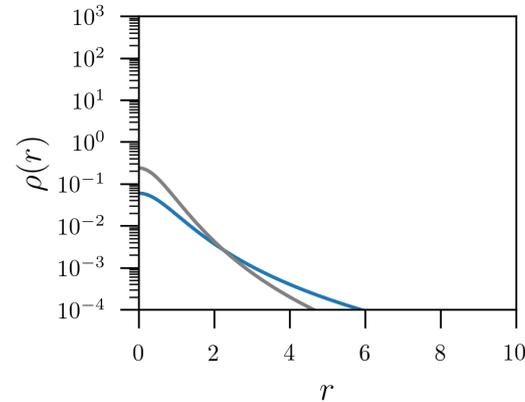
# Isochrone potential

$$\Phi(r) = -\frac{GM}{b + \sqrt{r^2 + b^2}}$$

$$\rho(r) = M \frac{3(b + \sqrt{b^2 + r^2})(b^2 + r^2) - r^2}{4\pi(b + \sqrt{b^2 + r^2})^3(b^2 -$$

$$M(r) = \frac{Mr^3}{\sqrt{b^2 + r^2}(b + \sqrt{b^2 + r^2})^2}$$

$$V_c^2(r) = \frac{GMr^2}{\sqrt{b^2 + r^2}(b + \sqrt{b^2 + r^2})^2}$$



Orbits are analytical !

# **Examples of Spherical models**

**“Density based”  
models**

# Homogeneous sphere

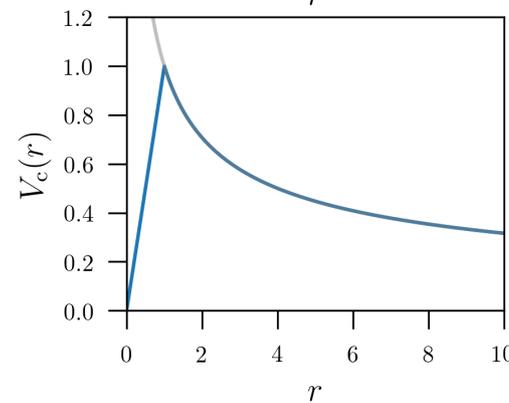
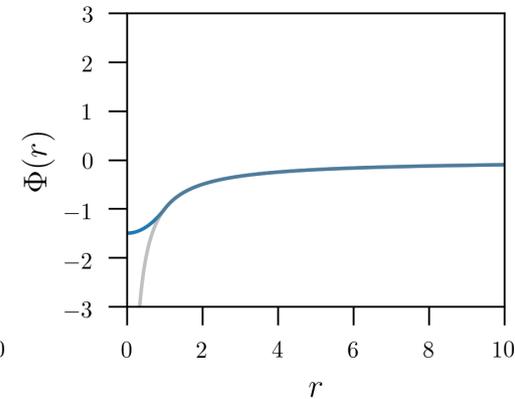
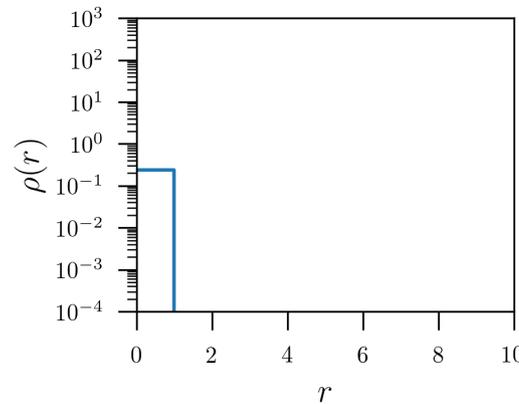
$$\rho(r) = \begin{cases} \rho & r < R \\ 0 & r > R \end{cases}$$

$$M(r) = \begin{cases} \frac{4}{3}\pi r^3 \rho_0 & r < R \\ \frac{4}{3}\pi R^3 \rho & r > R \end{cases}$$

$$\Phi(r) = \begin{cases} -2\pi G\rho \left(R^2 - \frac{1}{3}r^2\right) & r < R \\ -4\pi G\rho R^3 / (3r) & r > R \end{cases}$$

$$V_c^2(r) = \begin{cases} \frac{4}{3}\pi G\rho_0 r^2 & r < R \\ \frac{4}{3}\pi G\rho_0 \frac{R^3}{r} & r > R \end{cases}$$

$$T(r) = \begin{cases} \sqrt{\frac{3\pi}{G\rho_0}} & r < R \\ \sqrt{\frac{3\pi}{G\rho_0 R^3}} r^{3/2} & r > R \end{cases}$$



$$\frac{d^2 r}{dt^2} = -\frac{d\Phi(r)}{dr} = -\frac{GM(r)}{r^2} = -\frac{4}{3}\pi\rho_0 r = -\omega^2 r$$

Harmonic oscillator !

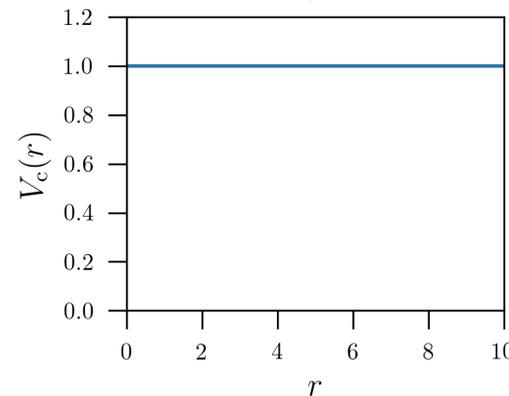
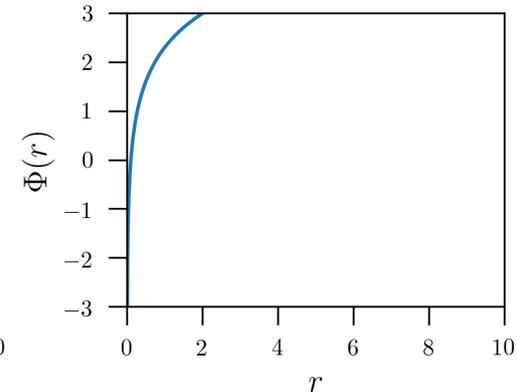
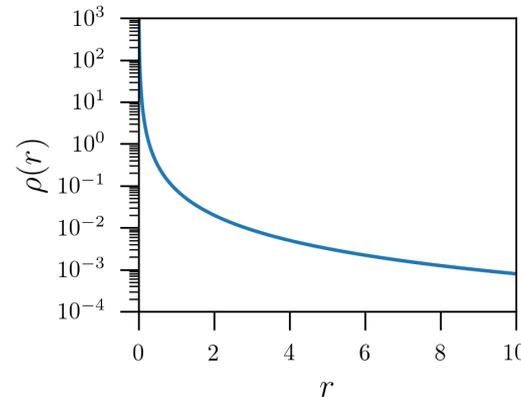
# Isothermal sphere

$$\rho(r) = \rho_0 \frac{a^2}{r^2}$$

$$\Phi(r) = 4\pi G \rho_0 a^2 \ln\left(\frac{r}{a}\right)$$

$$M(r) = 4\pi \rho_0 a^2 r$$

$$V_c^2(r) = 4\pi G \rho_0 a^2$$



- often used for gravitational lens models
- But !
  - diverge towards the centre !
  - infinite mass !

# Pseudo-isothermal sphere

$$\rho(r) = \rho_0 \frac{a^2}{a^2 + r^2}$$

$$\Phi(r) = 4\pi G \rho_0 a^2 \left( \frac{1}{2} \ln(a^2 + r^2) + \frac{a}{r} \arctan\left(\frac{r}{a}\right) \right)$$

$$M(r) = 4\pi r \rho_0 a^2 \left( 1 - \frac{a}{r} \arctan\left(\frac{r}{a}\right) \right)$$

$$V_c^2(r) = 4\pi G \rho_0 a^2 \left( 1 - \frac{a}{r} \arctan\left(\frac{r}{a}\right) \right)$$

- Avoid the central divergence of the isothermal sphere
  - However, the mass is still not bounded

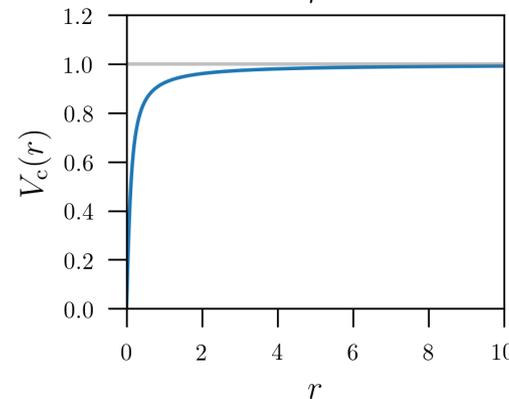
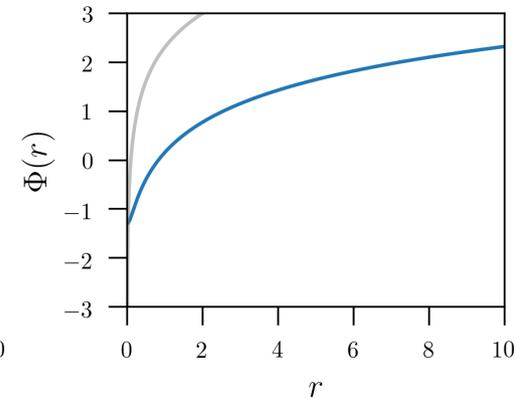
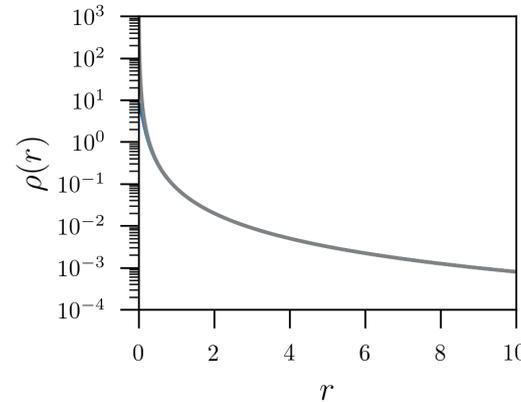
# Pseudo-isothermal sphere

$$\rho(r) = \rho_0 \frac{a^2}{a^2 + r^2}$$

$$\Phi(r) = 4\pi G \rho_0 a^2 \left( \frac{1}{2} \ln(a^2 + r^2) + \frac{a}{r} \arctan \frac{r}{a} \right)$$

$$M(r) = 4\pi r \rho_0 a^2 \left( 1 - \frac{a}{r} \arctan \left( \frac{r}{a} \right) \right)$$

$$V_c^2(r) = 4\pi G \rho_0 a^2 \left( 1 - \frac{a}{r} \arctan \left( \frac{r}{a} \right) \right)$$



- Avoid the central divergence of the isothermal sphere
  - However, the mass is still not bounded

# Generic two power density models

$$\rho(r) = \frac{\rho_0}{(r/a)^\alpha (1 + r/a)^{\beta-\alpha}}$$

- diverges at the center if  $\alpha \neq 0$

$$M(r) = 4\pi\rho_0 a^3 \int_0^{r/a} s \frac{s^{2-\alpha}}{(1+s)^{\beta-\alpha}}$$

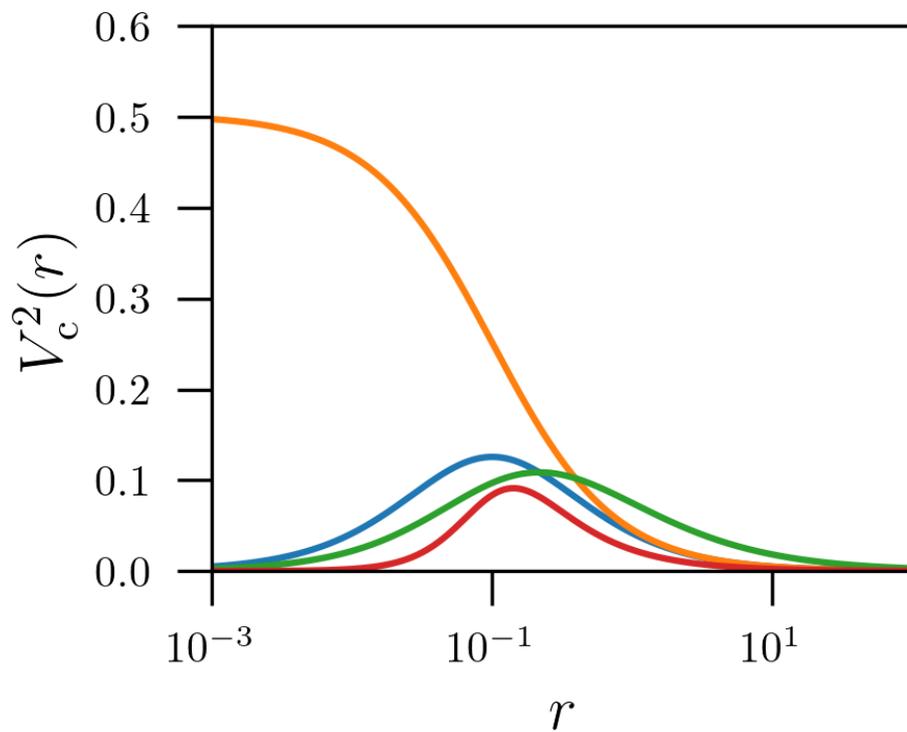
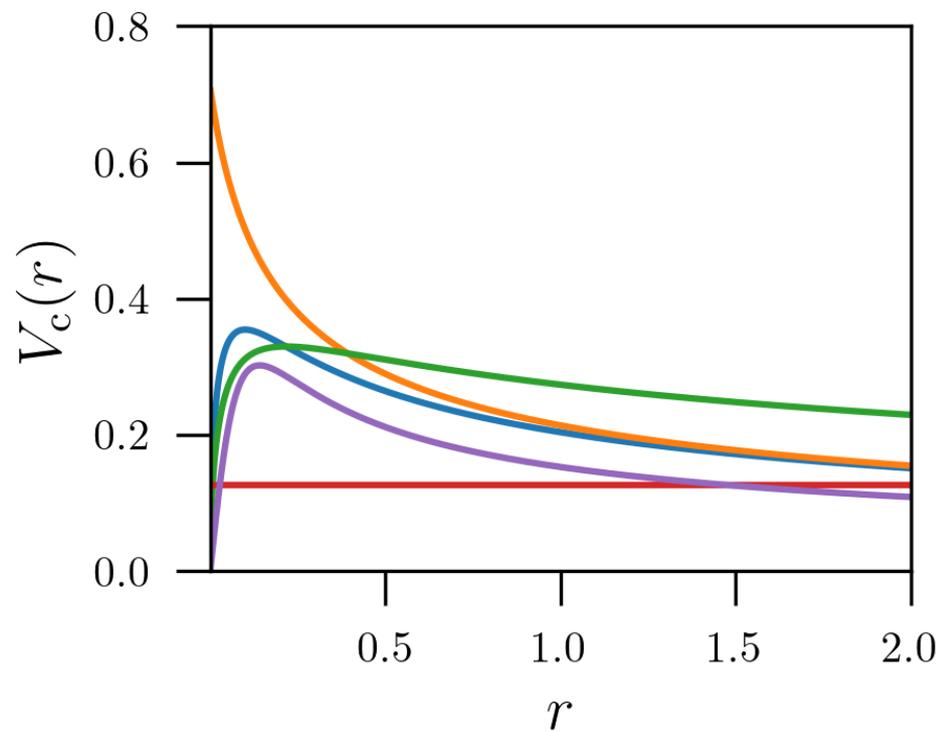
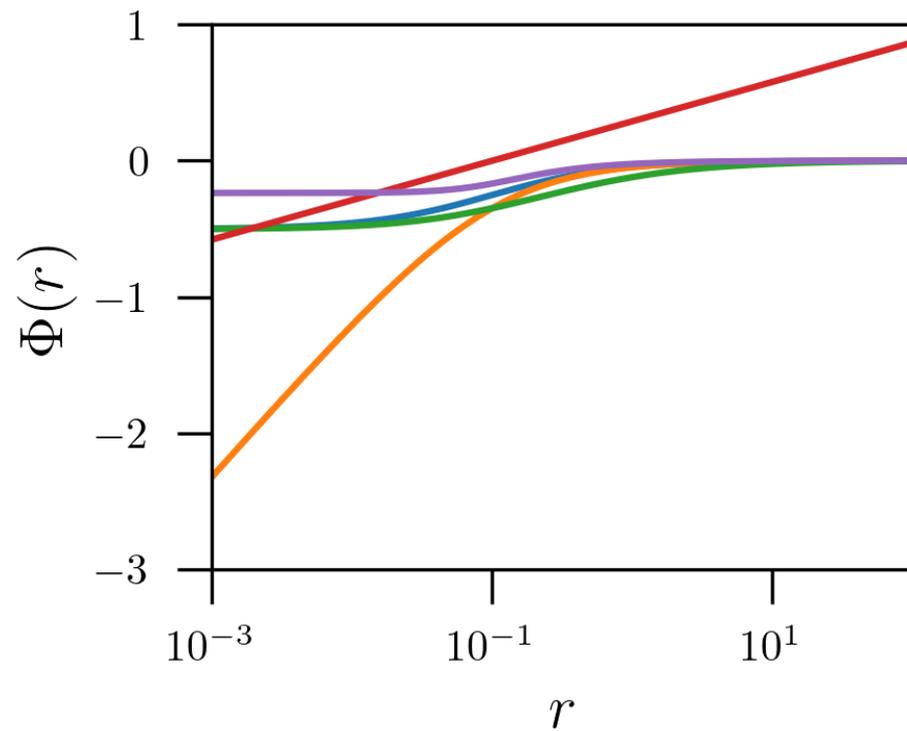
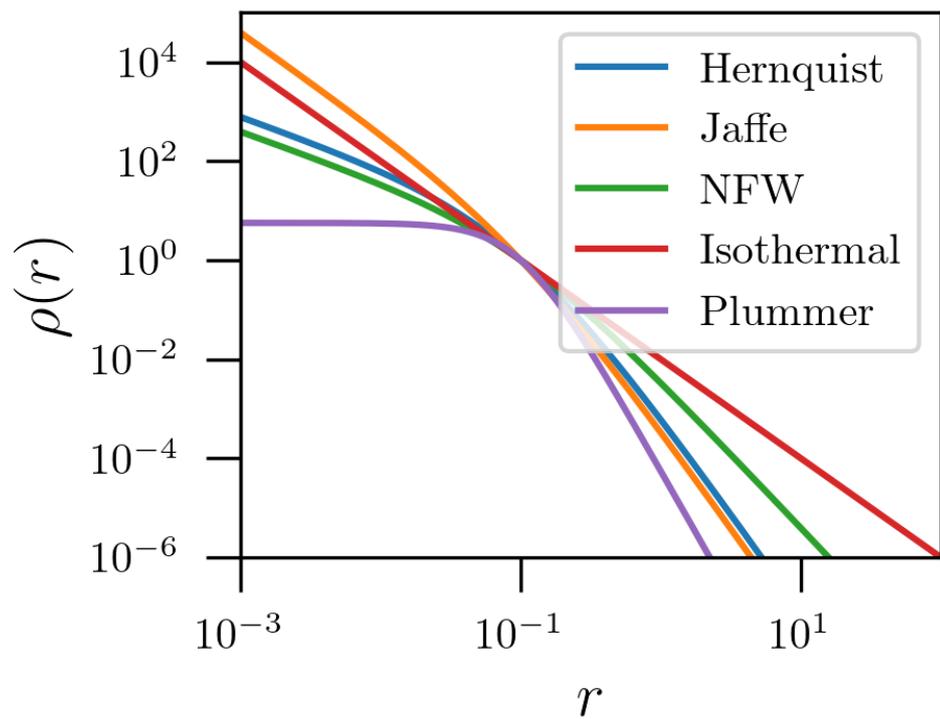
model name	inner slope $\alpha$	outer slope $\beta$	
Plummer	0	5	• globular clusters
Dehnen	any	4	
Hernquist	1	4	• bulges, elliptic. gal.
Jaffe	2	4	• elliptic. galaxies
NFW	1	3	• dark haloes

# Generic two power density model

$$M(r) = 4\pi\rho_0 a^3 \times \begin{cases} \frac{r/a}{1+r/a} & \text{(Jaffe)} \\ \frac{(r/a)^2}{2(1+r/a)^2} & \text{(Hernquist)} \\ \ln(1+r/a) - \frac{r/a}{1+r/a} & \text{(NFW)} \end{cases}$$

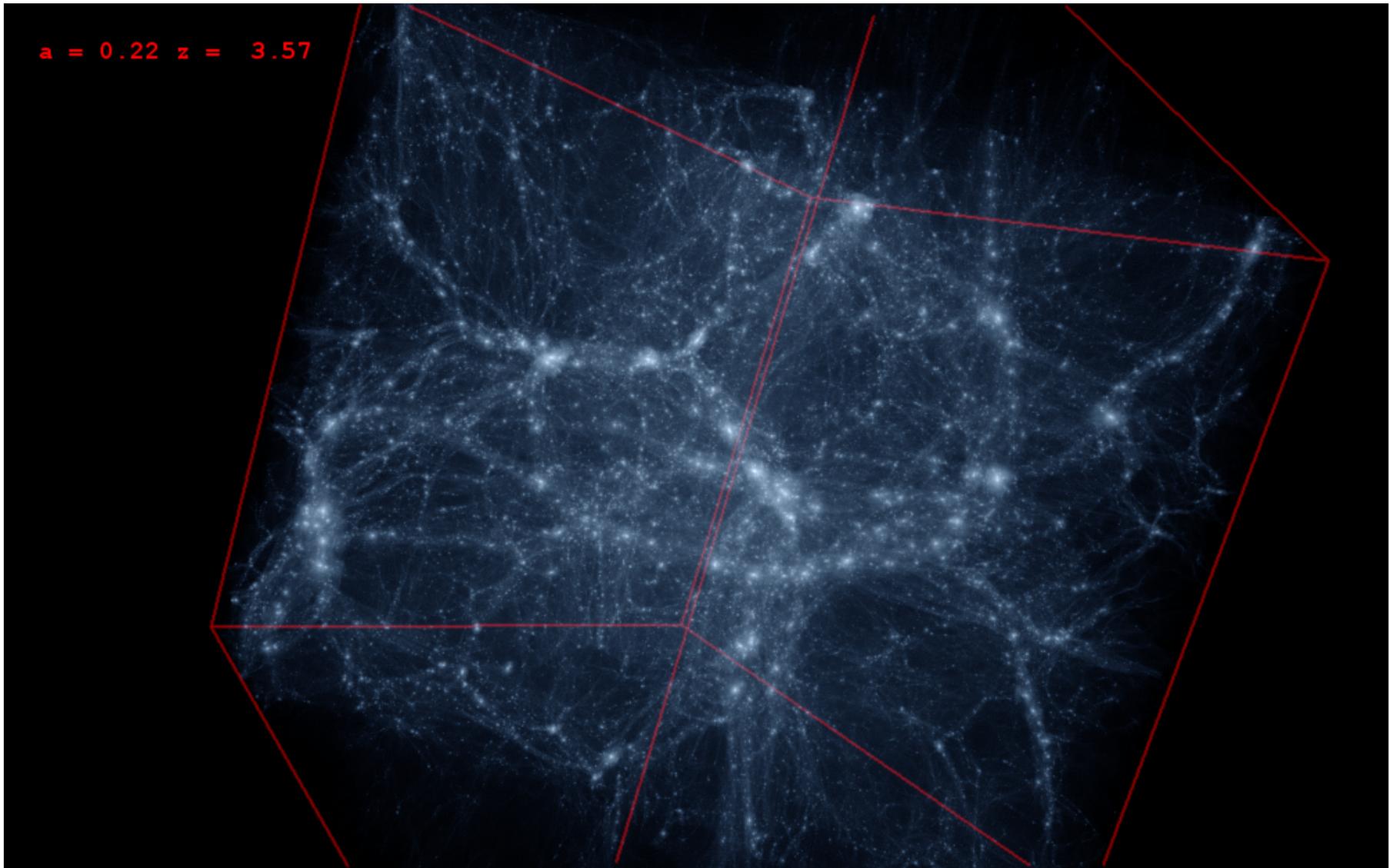
• diverges !!

$$\Phi(r) = -4\pi G\rho_0 a^2 \times \begin{cases} \ln(1+a/r) & \text{(Jaffe)} \\ \frac{1}{2(1+r/a)} & \text{(Hernquist)} \\ \frac{\ln(1+r/a)}{r/a} & \text{(NFW)} \end{cases}$$



# NFW (Navarro, Frenk & White 1995, 1996)

- Density profile that fit dark matter haloes formed in LCDM numerical simulations



# NFW (Navarro, Frenk & White 1995, 1996)

EXERCICE

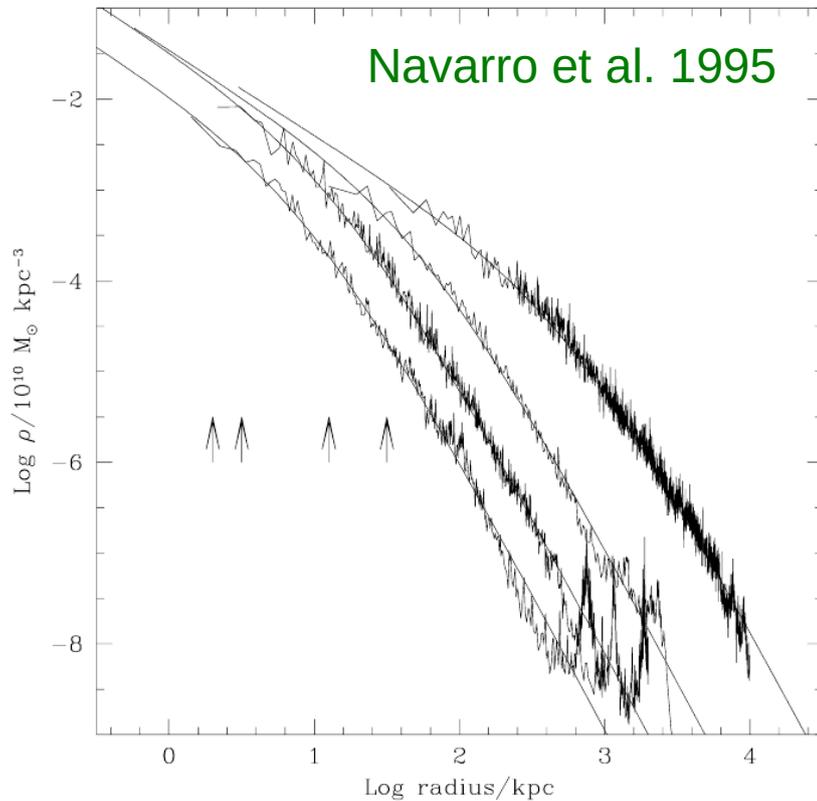


Fig. 3.— Density profiles of four halos spanning four orders of magnitude in mass. The arrows indicate the gravitational softening,  $h_g$ , of each simulation. Also shown are fits from eq.3. The fits are good over two decades in radius, approximately from  $h_g$  out to the virial radius of each system.

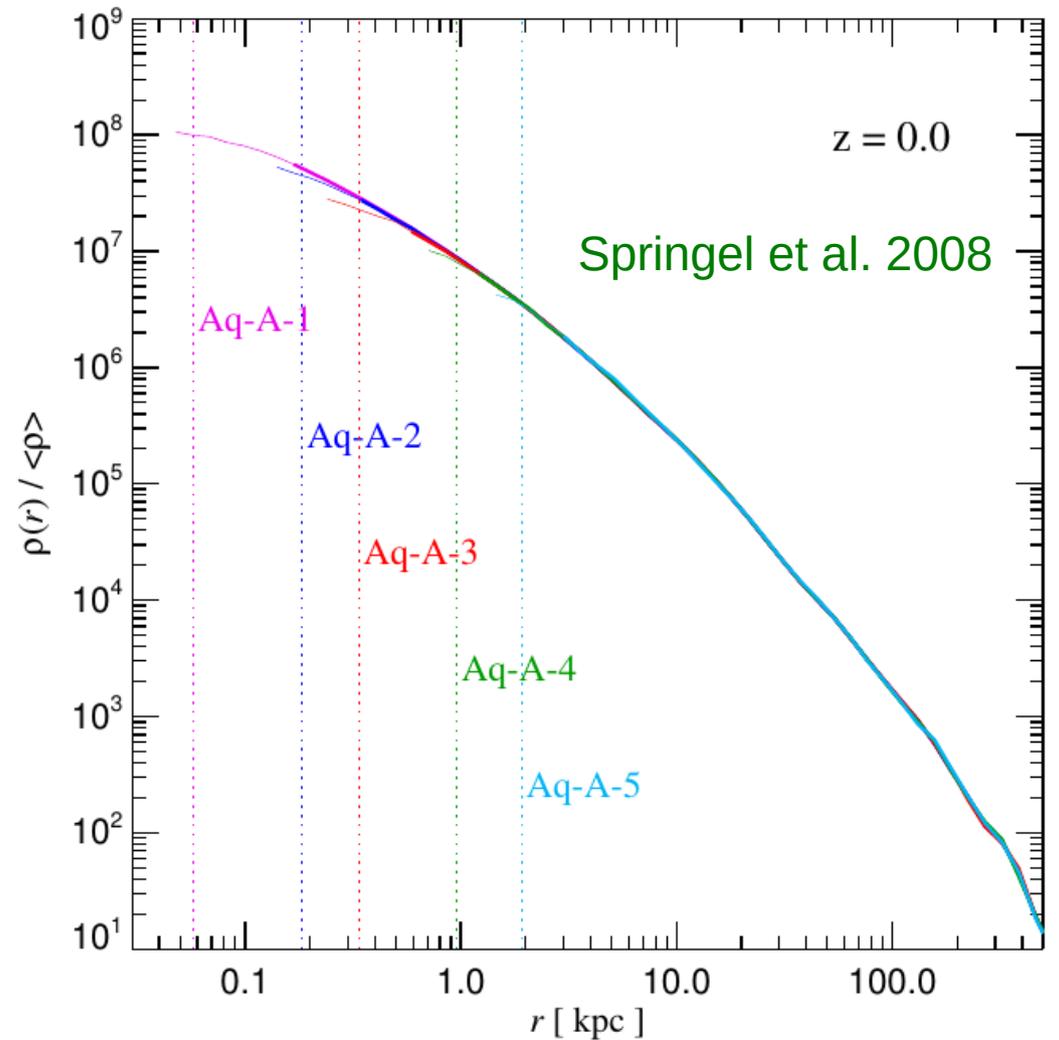


Figure 4. Spherically averaged density profile of the Aq-A halo at  $z = 0$ , at different numerical resolutions. Each of the pro-

# NFW (Navarro, Frenk & White 1995, 1996)

- Two parameters model (however, strongly correlated)

$$\rho(r) = \frac{\rho_0}{(r/r_s)(1 + r/r_s)^2} \equiv \frac{\delta_c \rho_c}{(r/r_s)(1 + r/r_s)^2}$$

- Definitions

- $r_{200}$  Radius, such that the mean density inside  $r_{200}$  is  $200 \times \rho_c$   $\bar{\rho} = \frac{M(r_{200})}{\frac{4}{3}\pi r_{200}^3} = 200 \rho_c$
- $M_{200}$  Mass inside  $r_{200}$
- Halo concentration  $c = \frac{r_{200}}{r_s}$   $\rho_c(t) = \frac{3 H^2(t)}{8 \pi G}$
- Mass inside  $r_{200}$

$$4\pi\delta_c\rho_cr_s^3 = \frac{M_{200}}{[\ln(1+c) - c/(1+c)]} \quad \delta_c = \frac{200}{3} \frac{c^3}{\left[\ln(1+c) - \left(\frac{c}{1+c}\right)\right]}$$

$$\rho(r) = \rho(r, c, M_{200})$$

# Potential Theory

## Axisymmetric models for disk galaxies

$$\rho(\vec{x}) = \rho(R, |z|)$$

$$R = \sqrt{x^2 + y^2}$$

# **Examples of axisymmetric models**

**“Potential based”  
models**

# Kuzmin disk

Kuzmin 1956

$$\Phi_K(R, z) = -\frac{GM}{\sqrt{R^2 + (a + |z|)^2}} = -\frac{GM}{\sqrt{R^2 + z^2 + a^2 + 2a|z|}}$$

Comparison with Plummer:

$$\Phi_P(R, z) = -\frac{GM}{\sqrt{R^2 + z^2 + a^2}}$$

# Kuzmin disk

Kuzmin 1956

$$\Phi_K(R, z) = -\frac{GM}{\sqrt{R^2 + (a + |z|)^2}} = -\frac{GM}{\sqrt{R^2 + z^2 + a^2 + 2a|z|}}$$

$$\Sigma_K(R) = \frac{aM}{2\pi(R^2 + a^2)^{3/2}}$$

$$V_{c,K}^2(R) = \frac{GM R^2}{(R^2 + a^2)^{3/2}}$$



Infinitely thin disk

Equivalent to the Plummer model

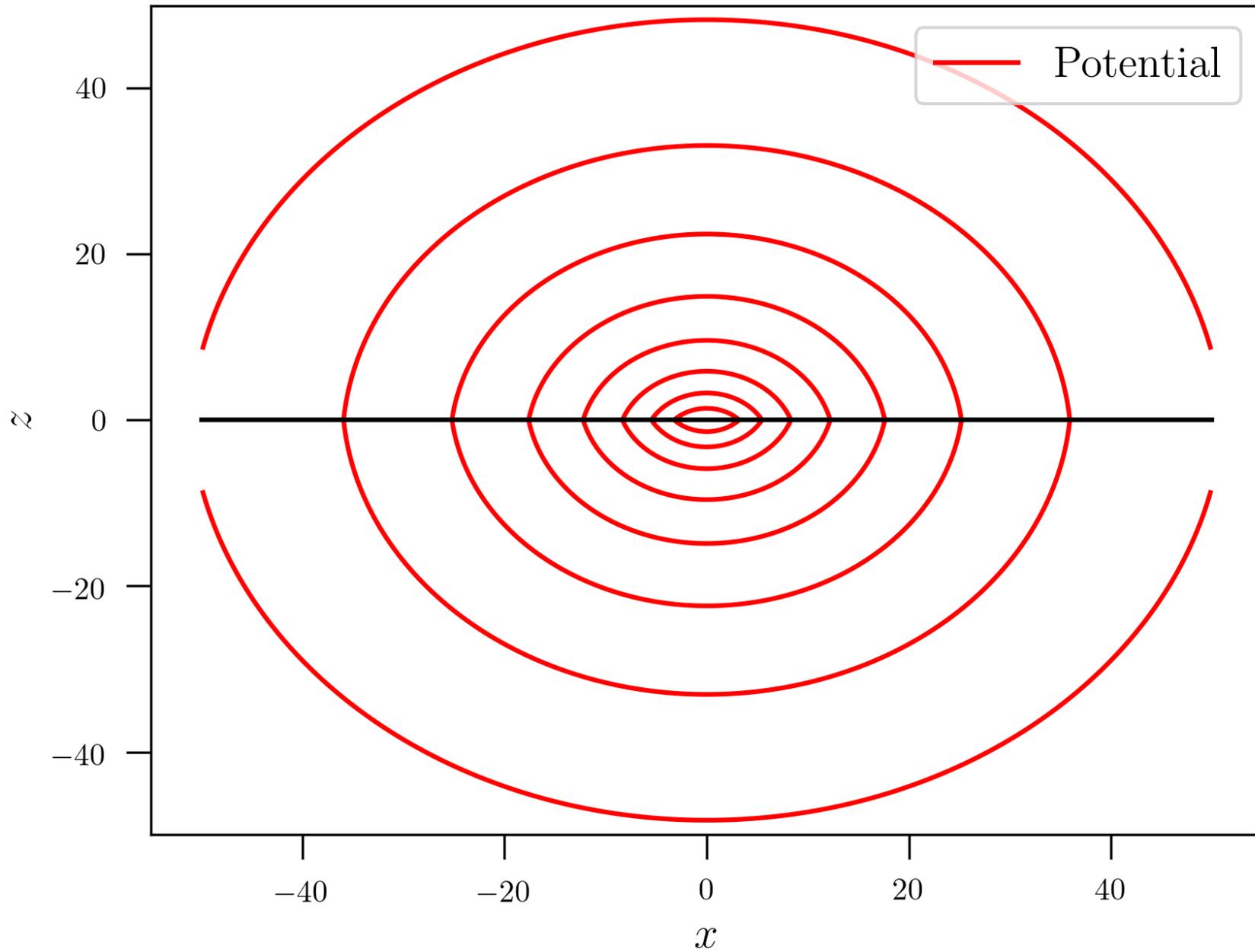
$$V_{c,P}^2(r) = \frac{GM r^2}{(r^2 + b^2)^{3/2}}$$

Note: for an axi-symmetric model, the circular velocity is computed in the plane  $z=0$ .

$$V_c^2(R) = R \frac{d\Phi(R, z=0)}{dR}$$

# Kuzmin disk

$a=3.0$



# Miyamoto-Nagai potential

Miyamoto & Nagai 1975

$$\Phi_{\text{MN}}(R, z) = - \frac{GM}{\sqrt{R^2 + (a + \sqrt{z^2 + b^2})^2}} \quad b=0 \rightarrow \text{Kuzmin}$$

$$\rho_{\text{MN}}(R, z) = \left( \frac{b^2 M}{4\pi} \right) \frac{aR^2 + (a + 3\sqrt{z^2 + b^2})(a + \sqrt{z^2 + b^2})^2}{[R^2 + (a + \sqrt{z^2 + b^2})^2]^{5/2} (z^2 + b^2)^{3/2}}$$

$$V_{c,\text{MN}}^2(R) = \frac{GM R^2}{(R^2 + (a + b)^2)^{3/2}}$$

Equivalent to the Plummer model

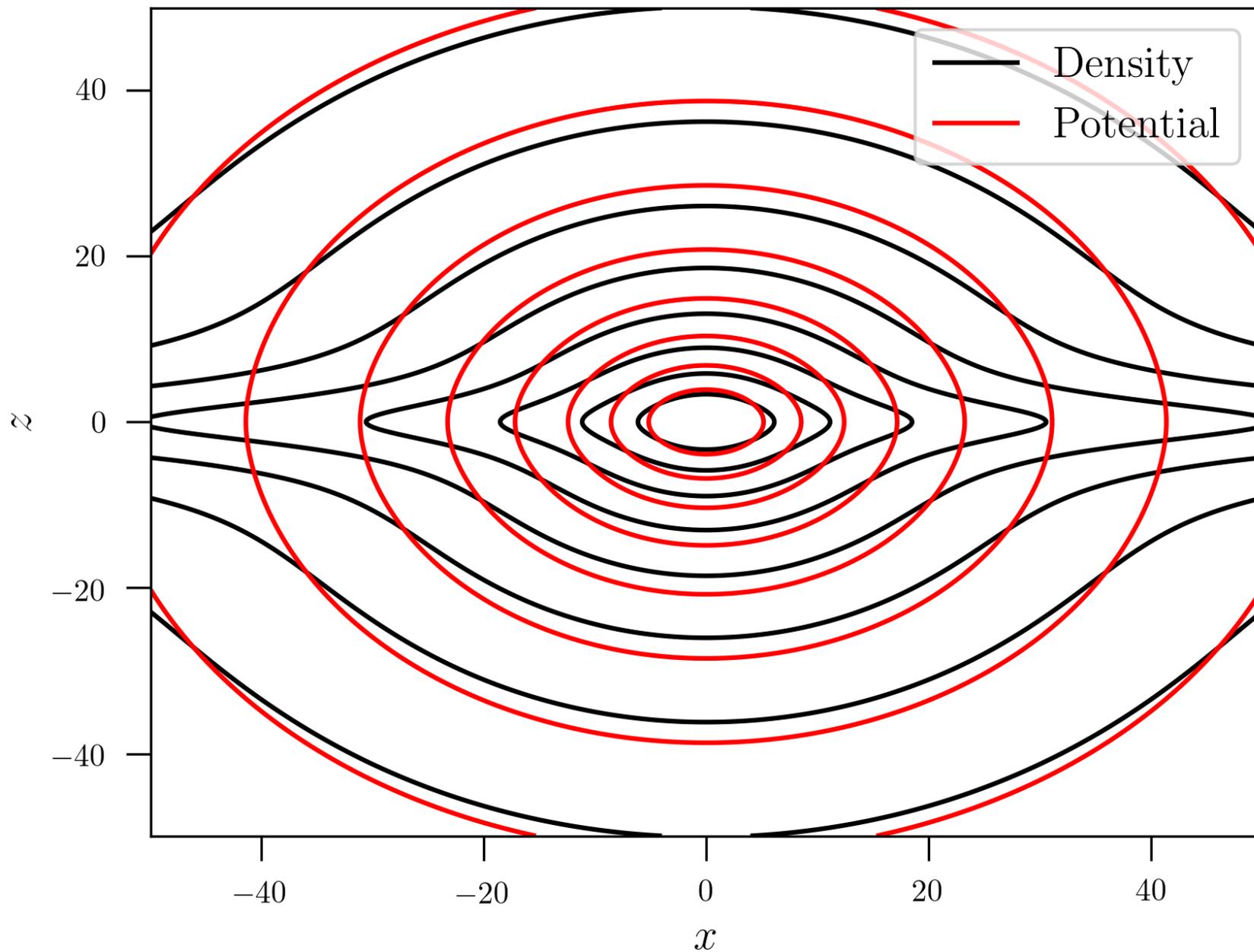
$$V_{c,\text{P}}^2(r) = \frac{GM r^2}{(r^2 + b^2)^{3/2}}$$

**EXERCICE**

Better parametrisation :  
Revaz & Pfenniger 2004

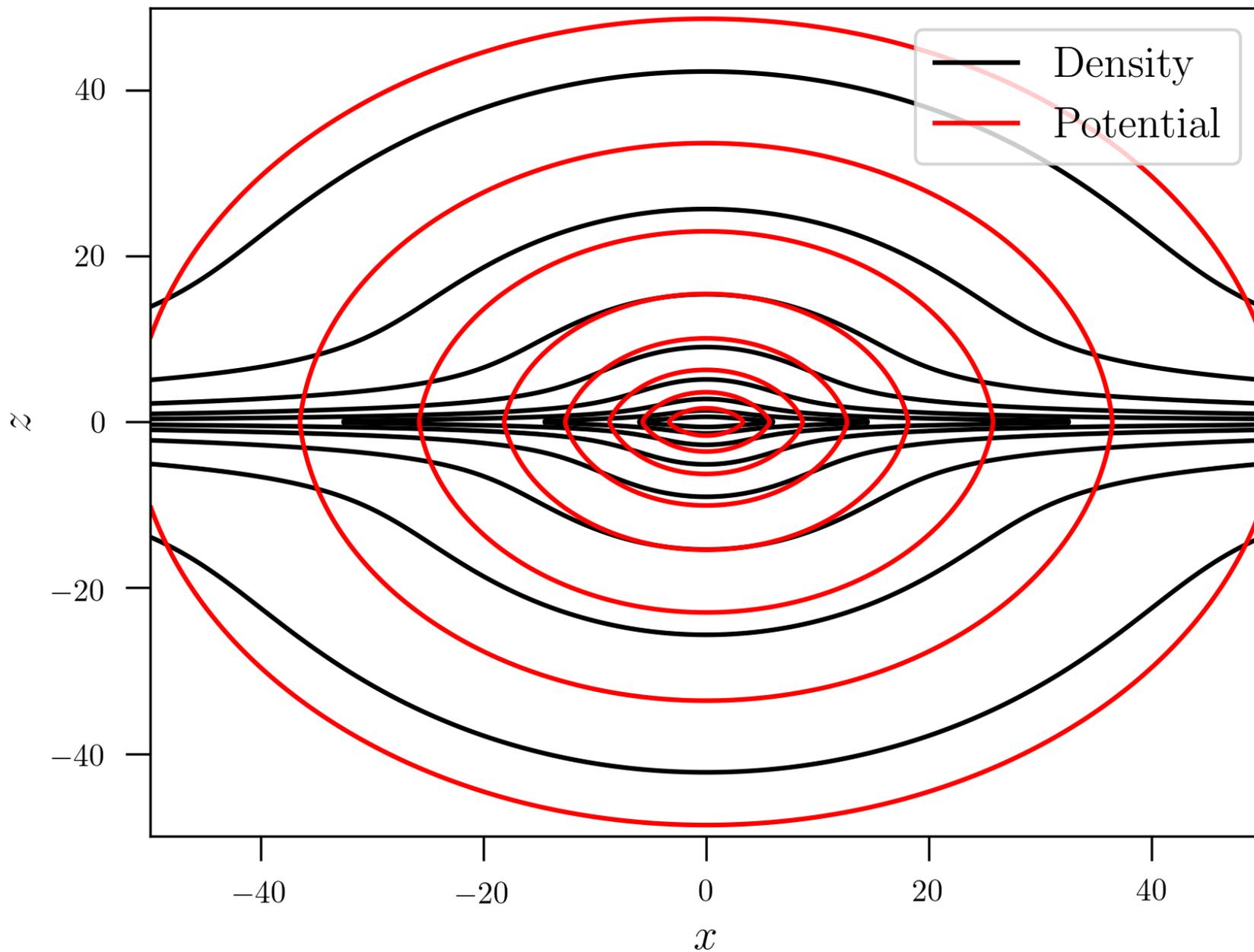
# Miyamoto-Nagai potential

$a=3.0$   $b=3.0$



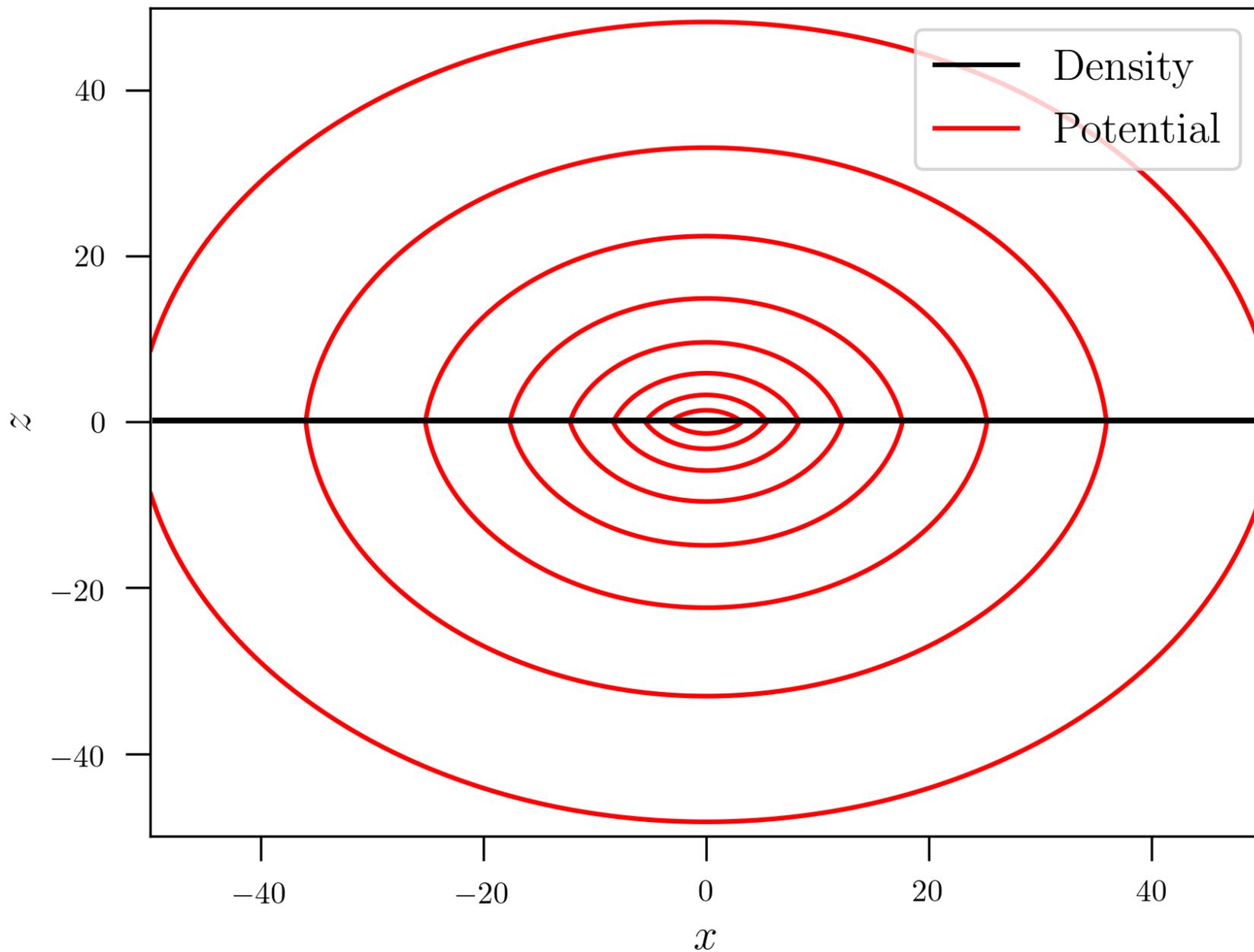
# Miyamoto-Nagai potential

$a=3.0$   $b=0.3$



# Miyamoto-Nagai potential

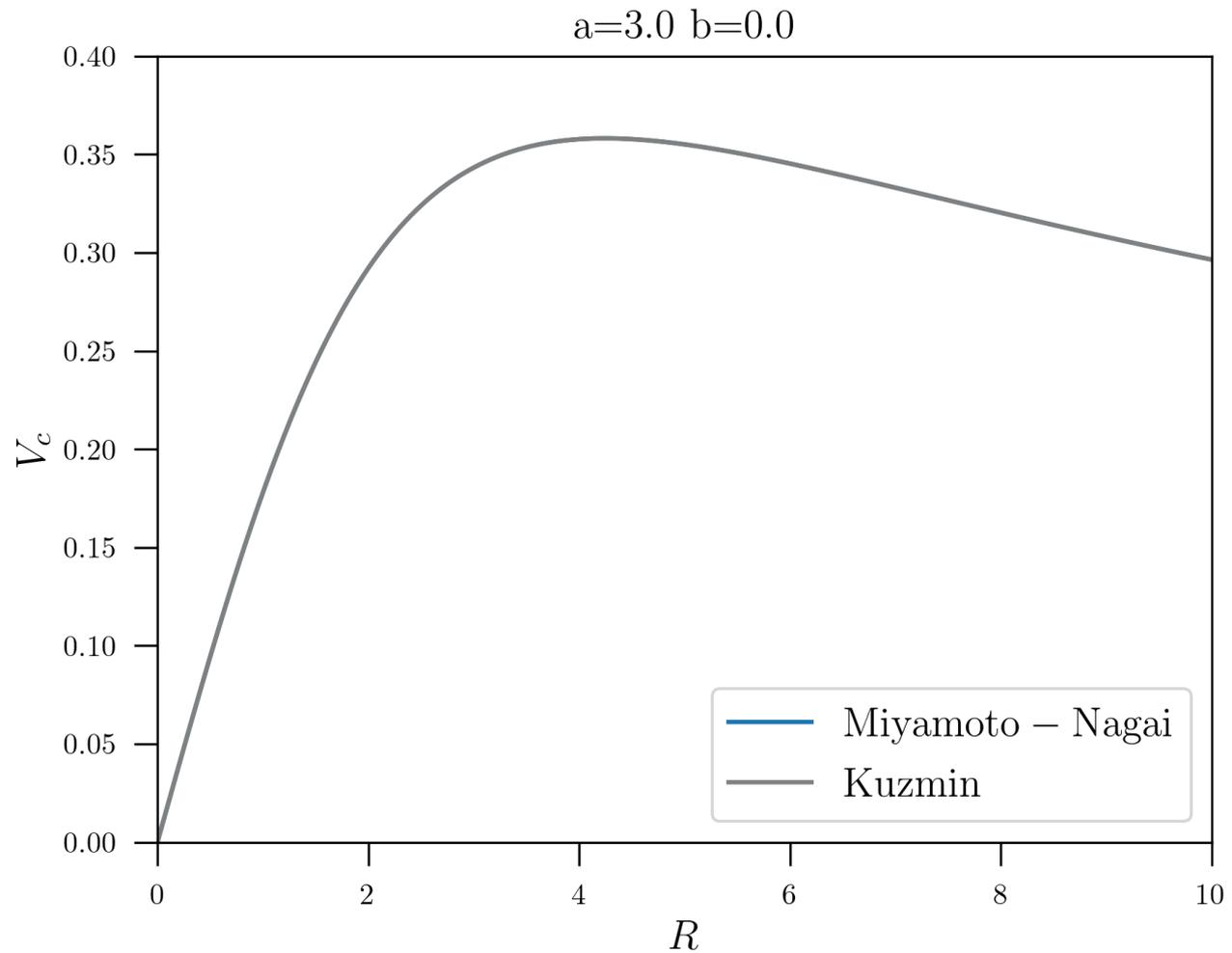
$a=3.0$   $b=0.0$



# Miyamoto-Nagai potential

Miyamoto & Nagai 1975

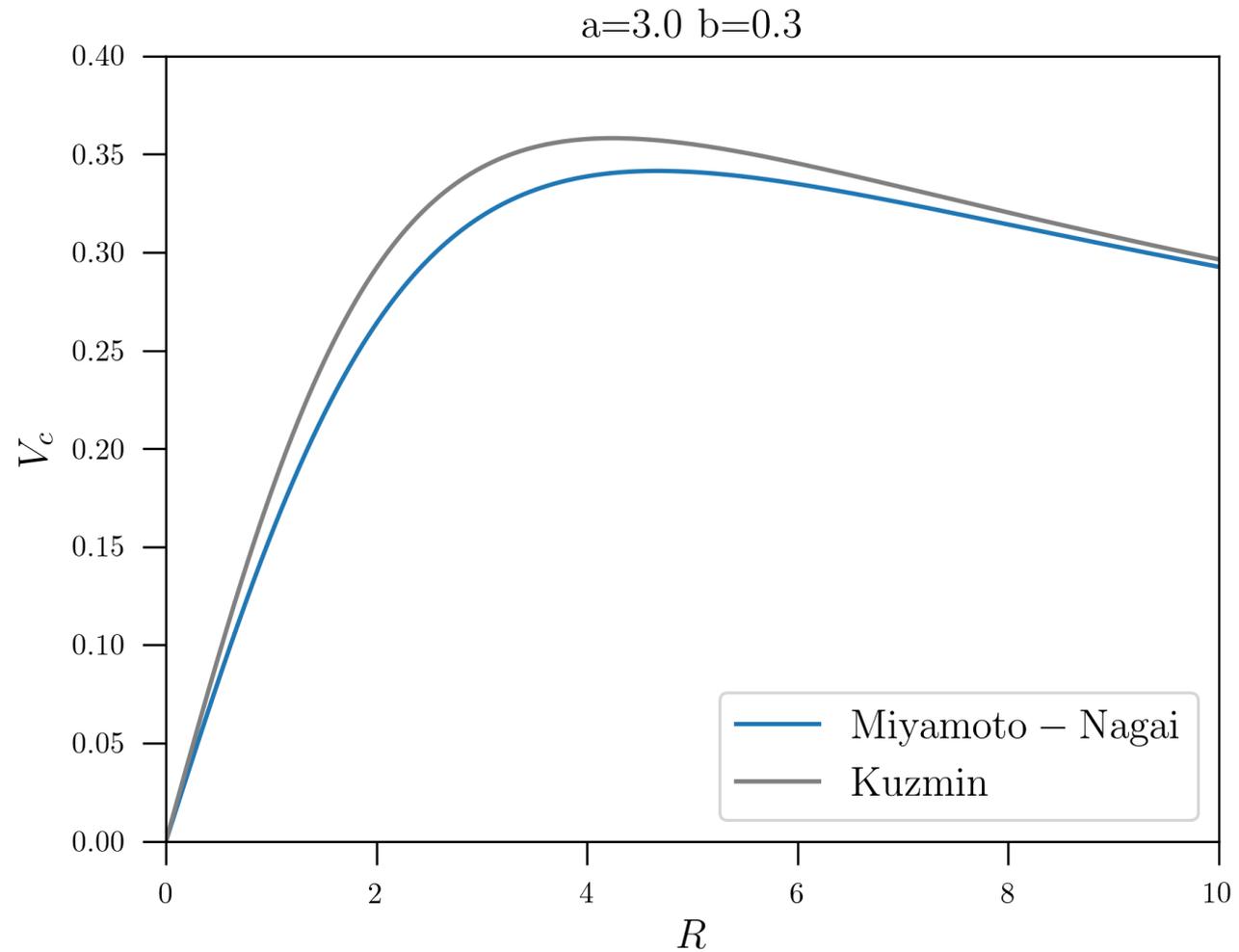
## Circular velocity rotation curve



# Miyamoto-Nagai potential

Miyamoto & Nagai 1975

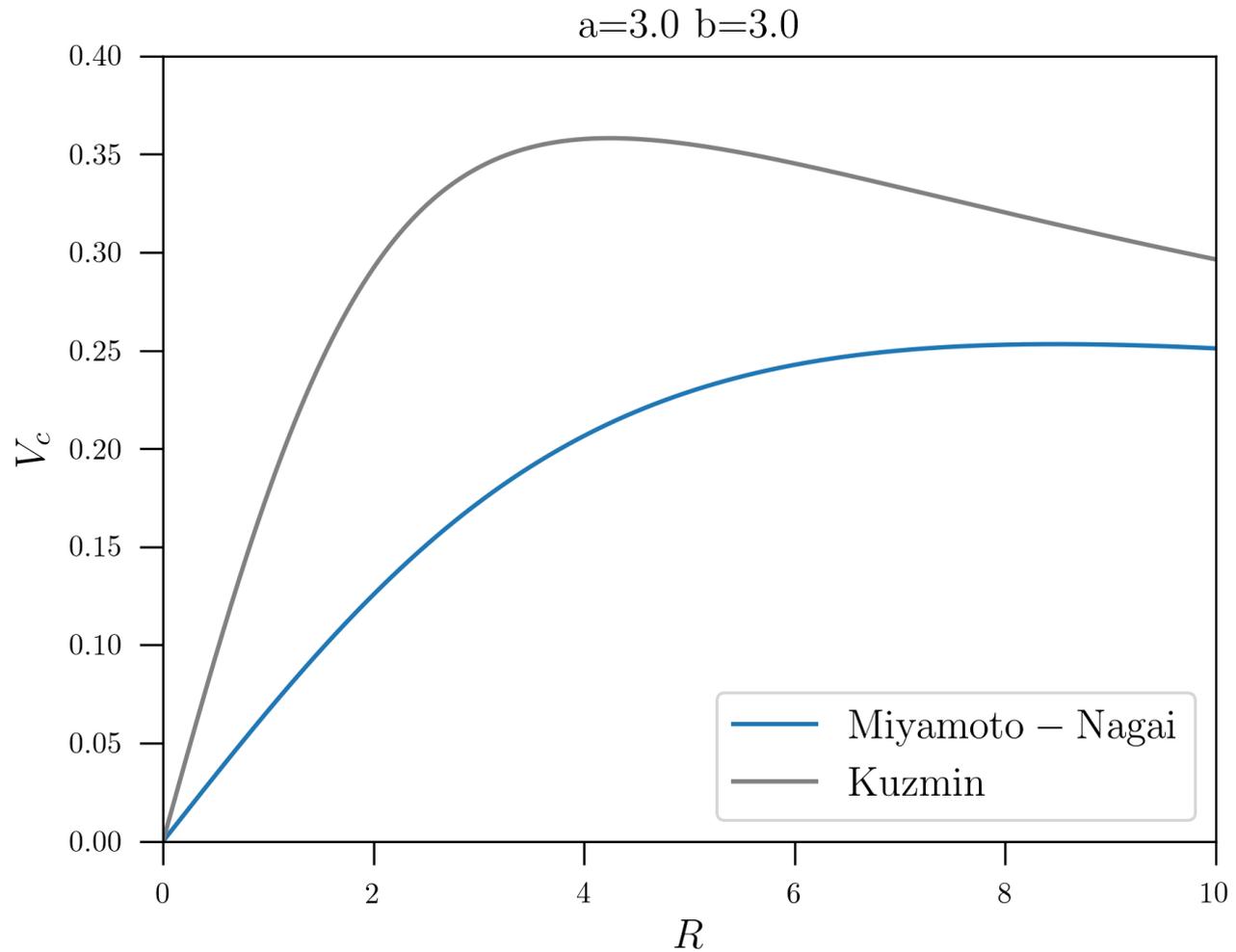
## Circular velocity rotation curve



# Miyamoto-Nagai potential

Miyamoto & Nagai 1975

## Circular velocity rotation curve



# Logarithmic potential

$$\Phi_{\log}(R, z) = \frac{1}{2} V_0^2 \ln \left( R_c^2 + R^2 + \frac{z^2}{q^2} \right) \quad \begin{array}{l} R_c=0 \text{ and } q=1 \\ \rightarrow \text{Isothermal sphere} \end{array}$$

$$\rho_{\log}(R, z) = \frac{V_0^2}{4\pi G q^2} \frac{(2q^2 + 1)R_c^2 + R^2 + (2 - 1/q^2)z^2}{(R_c^2 + R^2 + (z^2/q^2))^2}$$



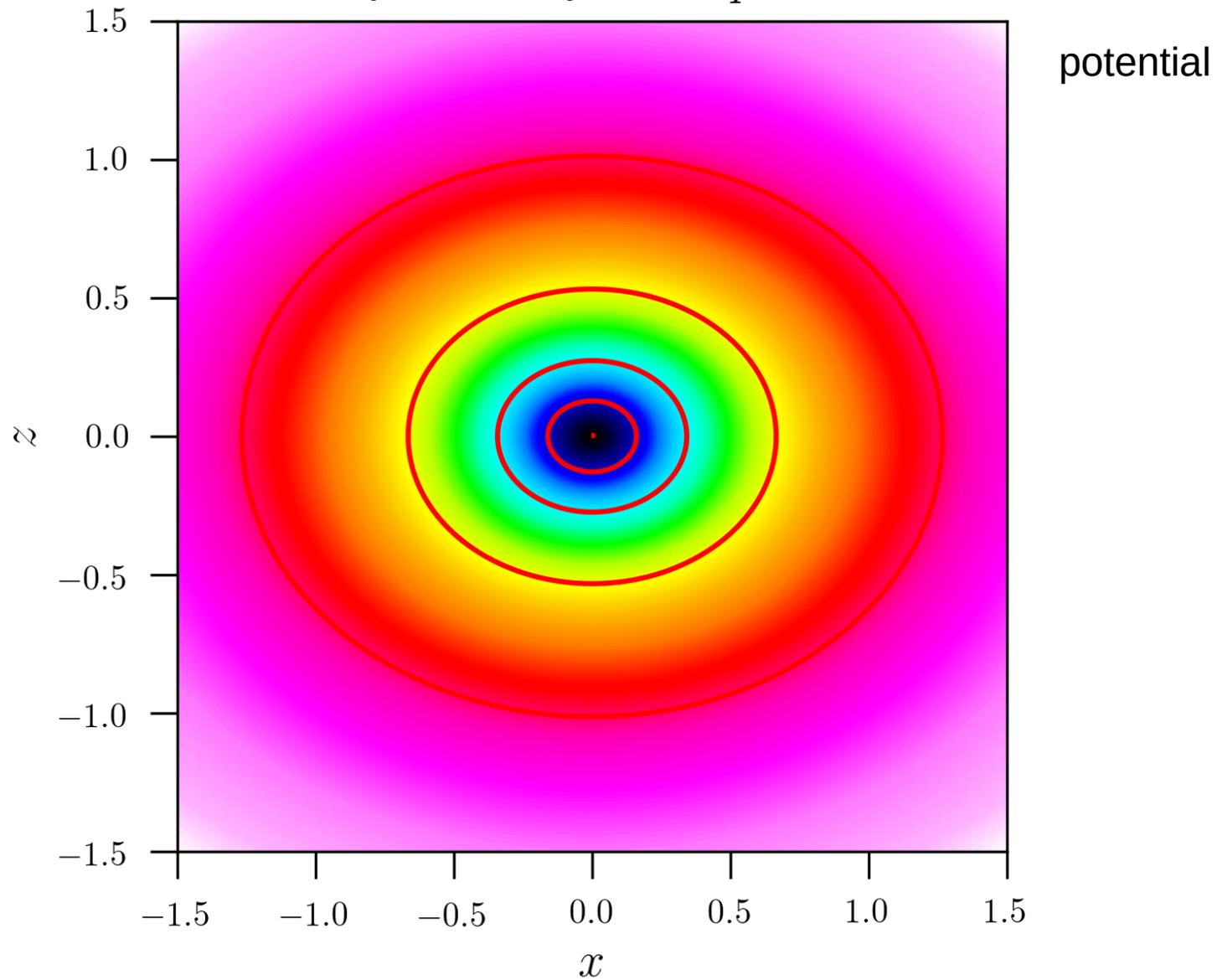
- negative for  $q < 1/\sqrt{2} \cong 0.707$

$$V_{c,\log}^2(R) = V_0^2 \frac{R^2}{R_c^2 + R^2}$$

- does not depend on  $q$
- flat rotation curve at large radius

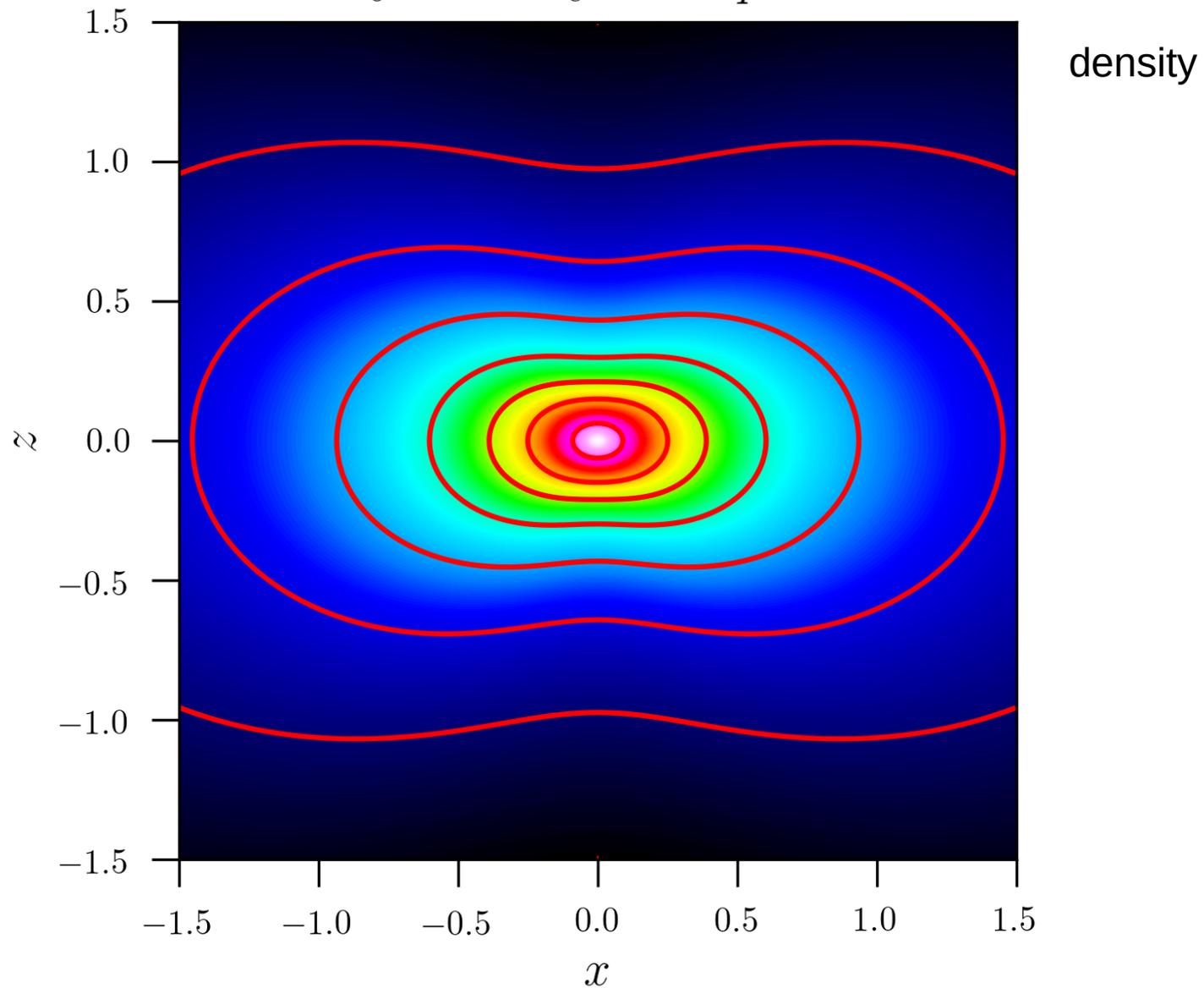
# Logarithmic potential

$$V_0=1.0 \quad R_c=0.1 \quad q=0.8$$



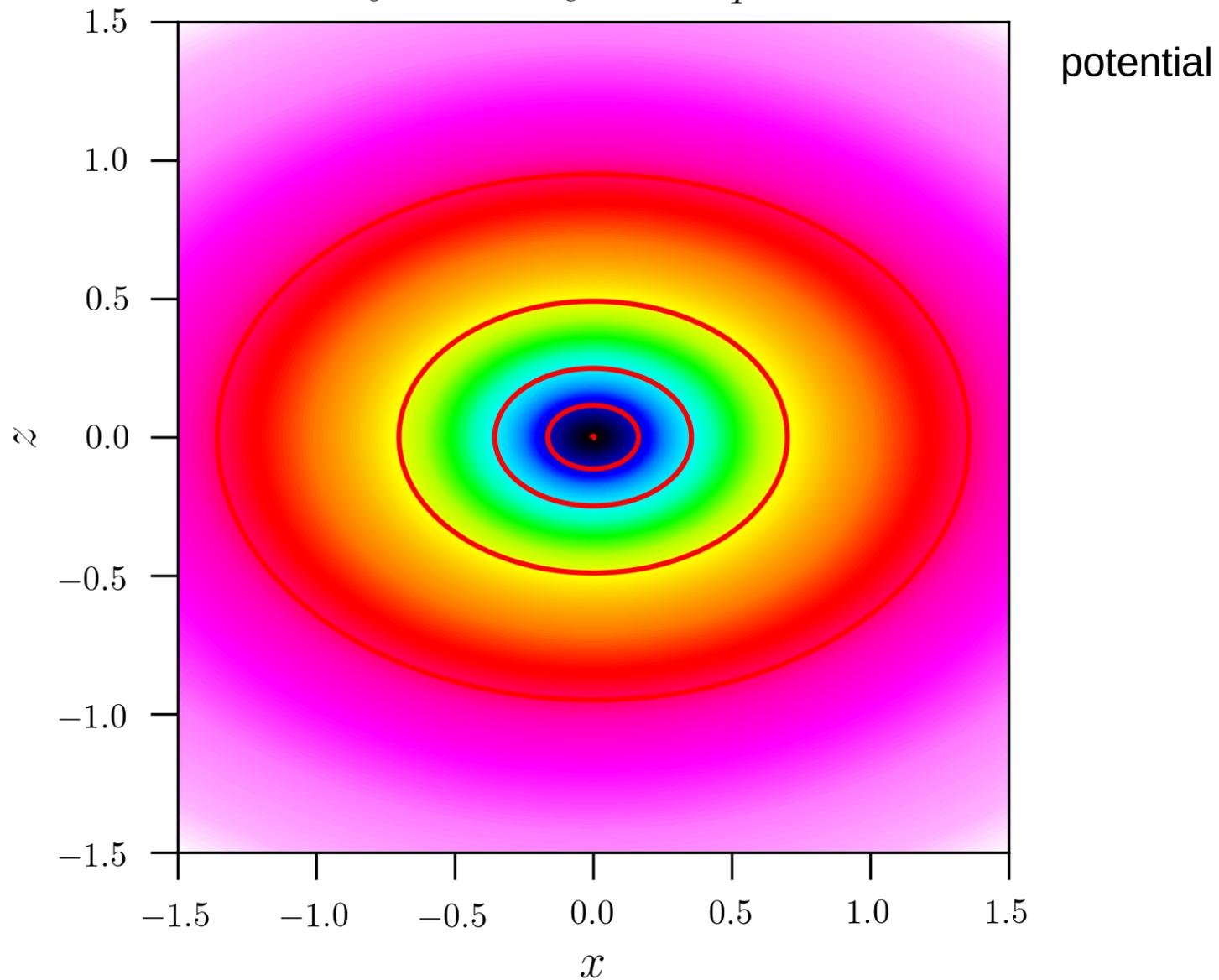
# Logarithmic potential

$$V_0=1.0 \quad R_c=0.1 \quad q=0.8$$



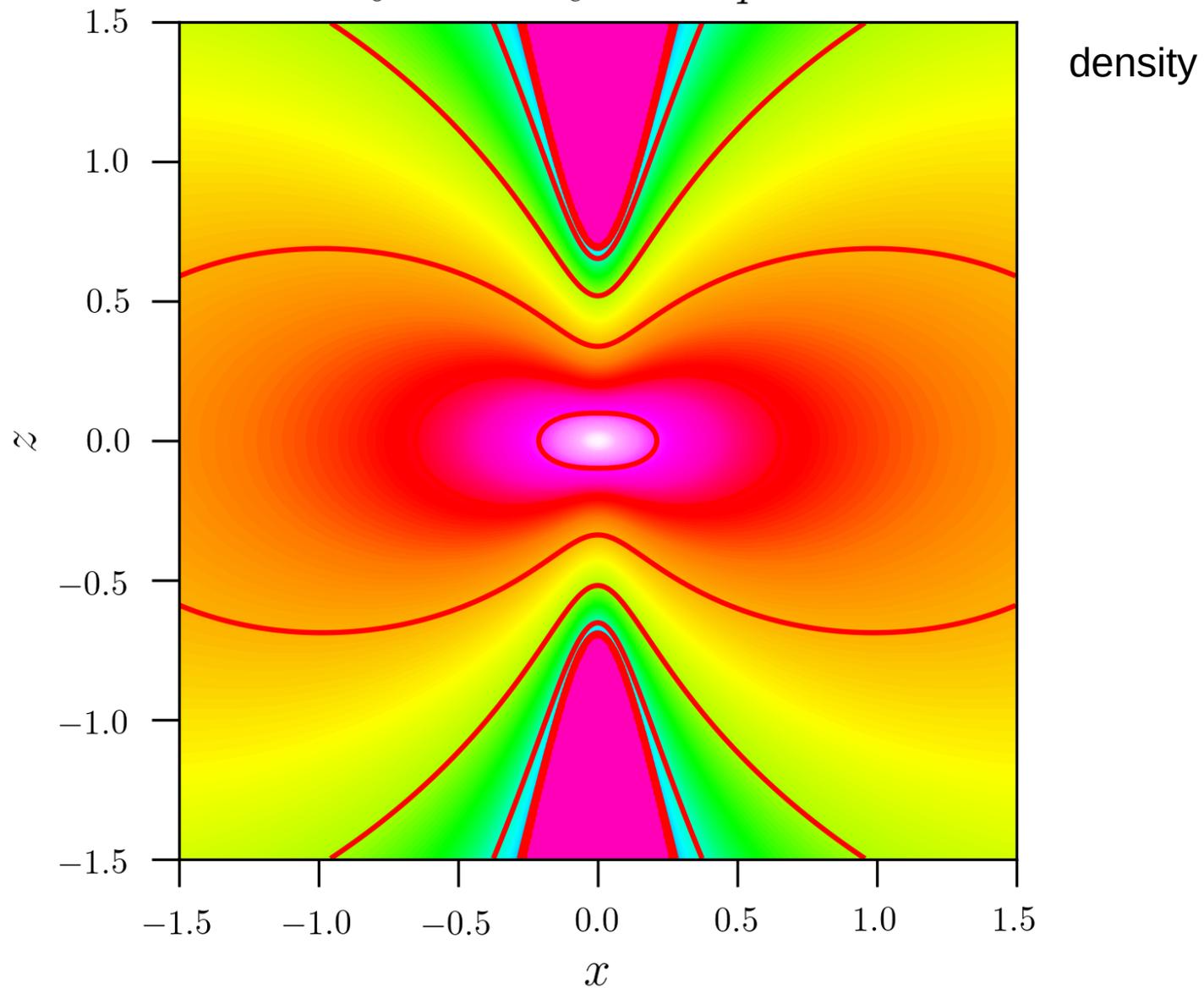
# Logarithmic potential

$$V_0=1.0 \quad R_c=0.1 \quad q=0.7$$



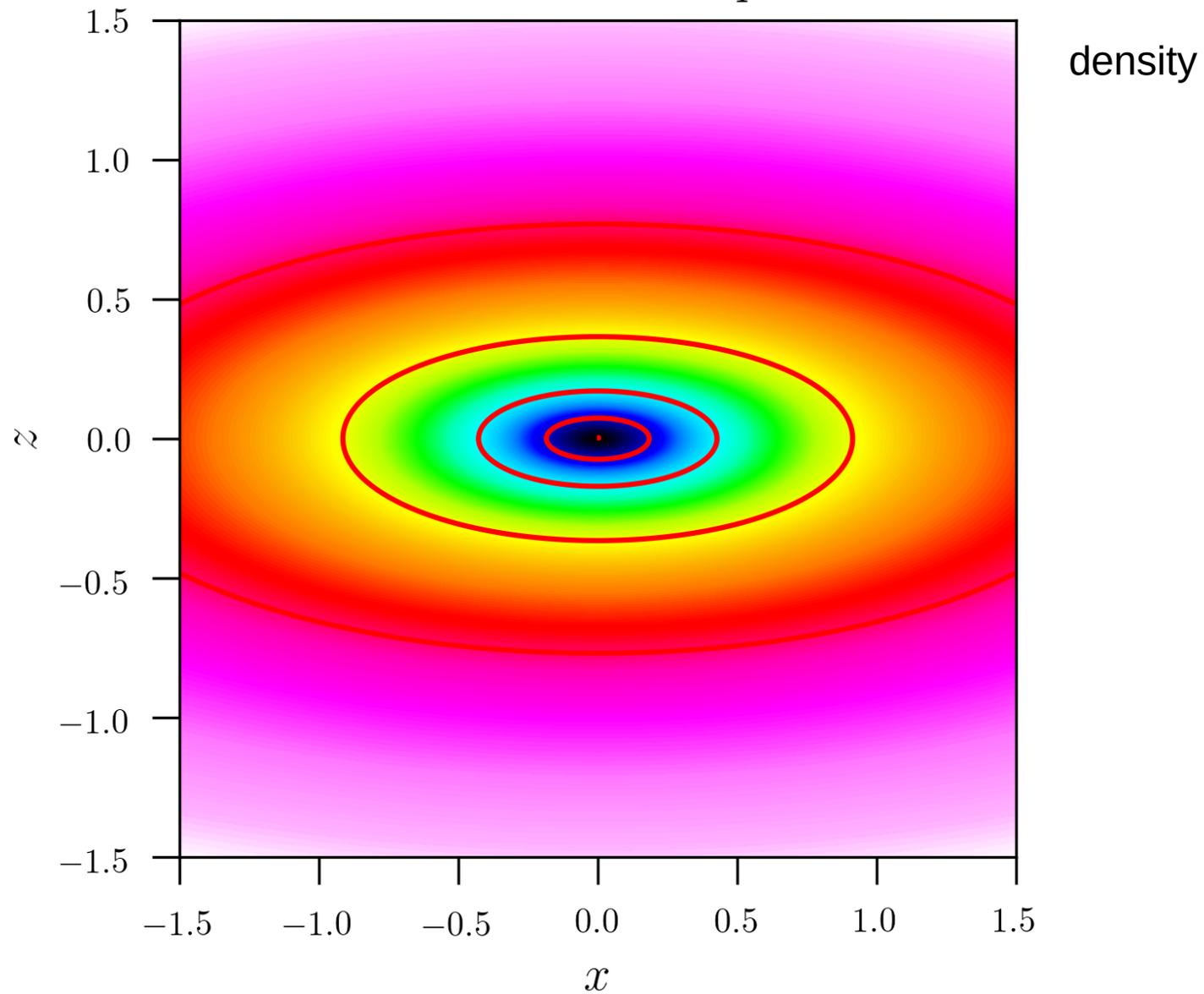
# Logarithmic potential

$$V_0=1.0 \quad R_c=0.1 \quad q=0.7$$



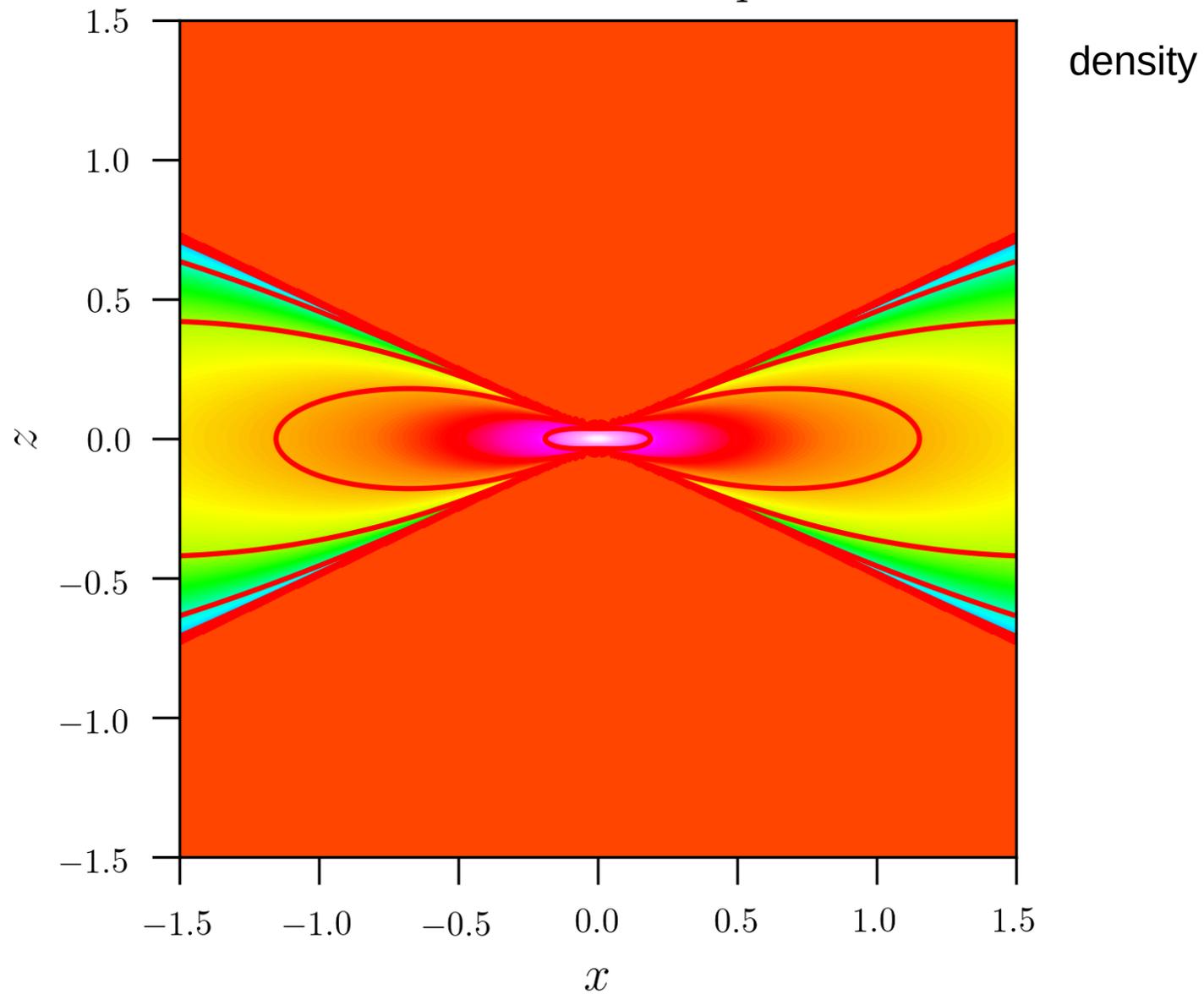
# Logarithmic potential

$$V_0=1.0 \quad R_c=0.1 \quad q=0.4$$



# Logarithmic potential

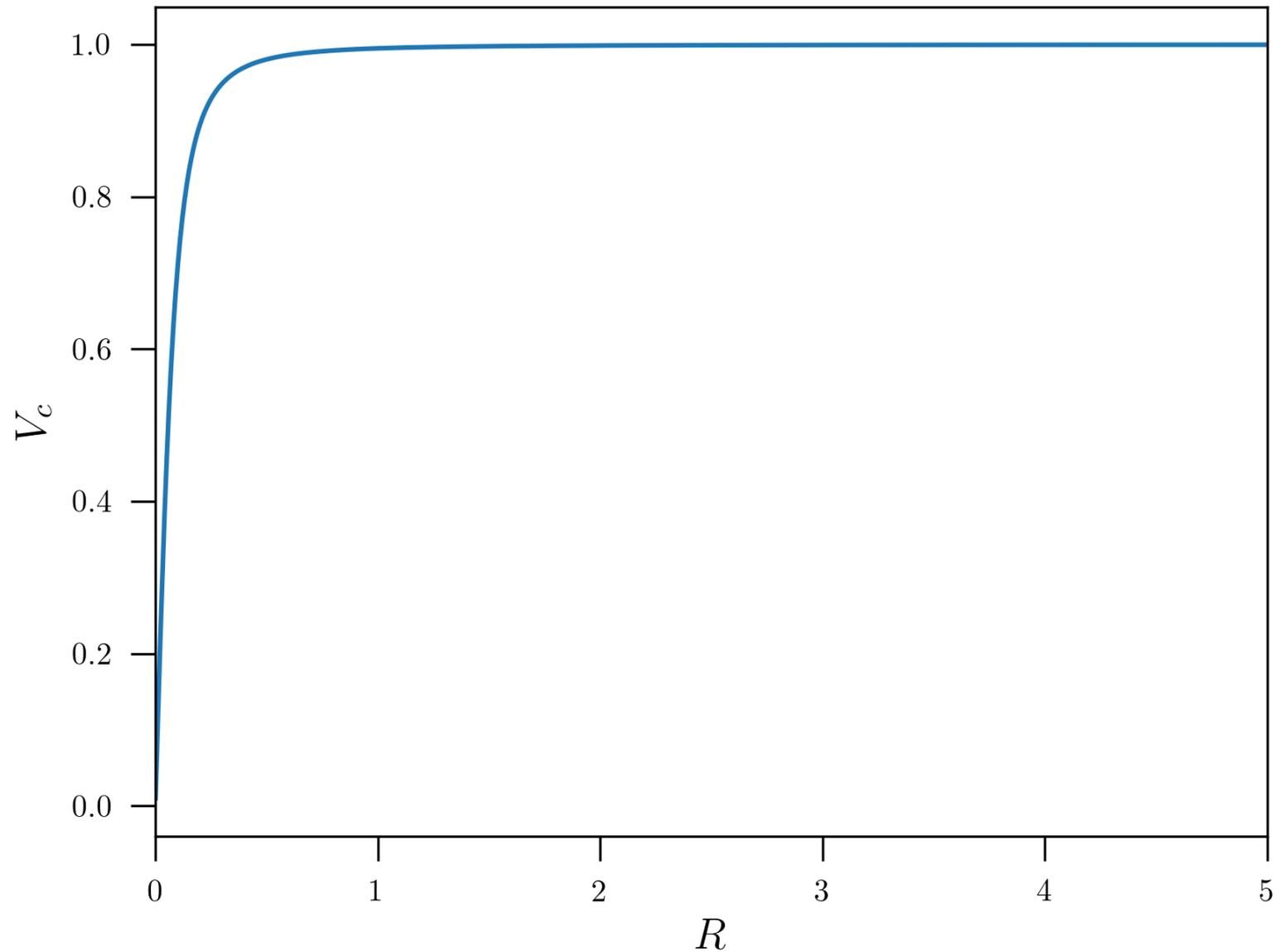
$$V_0=1.0 \quad R_c=0.1 \quad q=0.4$$



# Logarithmic potential

## Circular velocity rotation curve

$$V_0=1.0 \quad R_c=0.1 \quad q=0.8$$



# Potential Theory

## The potential of flattened systems

## Poisson Equation for very flattened axisymmetric systems

Aim : get  $\phi(R, z)$  from  $\rho(R, z)$

Poisson equation in cylindrical coord. for axisymmetric systems  $\frac{\partial}{\partial \phi} = 0$

$$\nabla^2 \phi(R, z) = \frac{1}{R} \frac{\partial}{\partial R} \left( R \frac{\partial \phi}{\partial R} \right) + \frac{\partial^2 \phi}{\partial z^2} = 4\pi G \rho(R, z)$$

What is the behaviour of the Poisson equation when the system get flatter and flatter ?

Example : Miyamoto-Nagai disk  $b \rightarrow 0$

$$1) \int_{MN} (R, z=0) \stackrel{b \rightarrow 0}{=} \frac{b^2 M}{4\pi} \frac{aR^2 + a^3}{(R^2 + a^2)^{5/2}} \frac{1}{b^3} \sim \frac{1}{b} \rightarrow \infty$$

$$2) \left. \frac{\partial \phi_{MN}}{\partial R} \right|_{z=0} = \left. \frac{\partial \phi_K}{\partial R} \right|_{z=0} = \frac{GM}{(R^2 + a^2)^{3/2}} \quad : \text{ does not diverge}$$

$$\Rightarrow \frac{1}{R} \frac{\partial}{\partial R} \left( R \frac{\partial \phi}{\partial R} \right) \quad : \text{ does not diverge}$$

$\Rightarrow$  becomes negligible compared to  $\rho$

Near  $z=0$  the Poisson equation becomes

$$\frac{\partial^2 \phi}{\partial z^2} = 4\pi G \rho(R, z)$$

The vertical variation of  $\phi$  depends only on the density  $\rho$  at that radius

## Solutions of the Poisson equation

$$\phi(R, z) = \phi_0(R, 0) + \phi_z(R, z)$$

"zero point"

"vertical dep."

1)  $\phi_0(R, 0)$  is obtained by assuming a "razor-thin" disk

$$\rho(R, z) \rightarrow \Sigma(R)$$

$$2) \phi_z(R, z) = 4\pi G \int_0^z dz' \int_0^{z'} dz'' \rho(R, z'')$$

We need a machinery to find  $\phi_0(R, 0)$  from  $\Sigma(R)$

# Potential Theory

**Surface density-based  
(razor-thin) disks**

# Kuzmin disk

Kuzmin 1956

$$\Phi_K(R, z) = - \frac{GM}{\sqrt{R^2 + (a + |z|)^2}}$$

Plummer based model

$$\Sigma_K(R) = \frac{aM}{2\pi(R^2 + a^2)^{3/2}}$$

Infinitely thin disk

$$V_{c,K}^2(R) = \frac{GM R^2}{(R^2 + a^2)^{3/2}}$$

Equivalent to the Plummer model

$$V_{c,P}^2(r) = \frac{GM r^2}{(r^2 + b^2)^{3/2}}$$

Note: for an axi-symmetric model, the circular velocity is computed in the plane  $z=0$ .

$$V_c^2(R) = \frac{1}{R} \frac{d\Phi(R, z=0)}{dR}$$

# Mestel disk

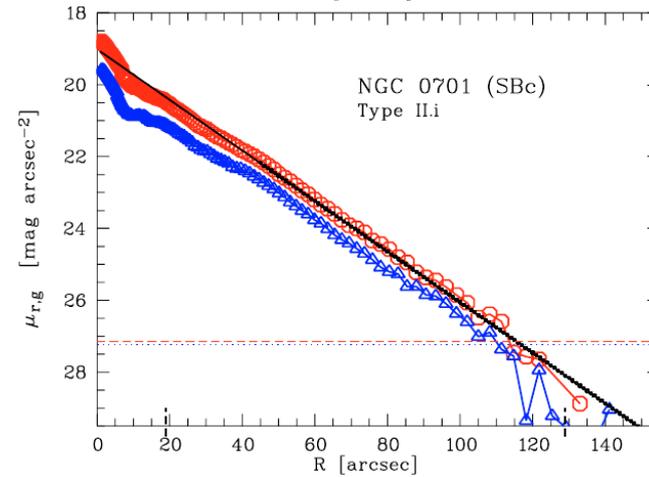
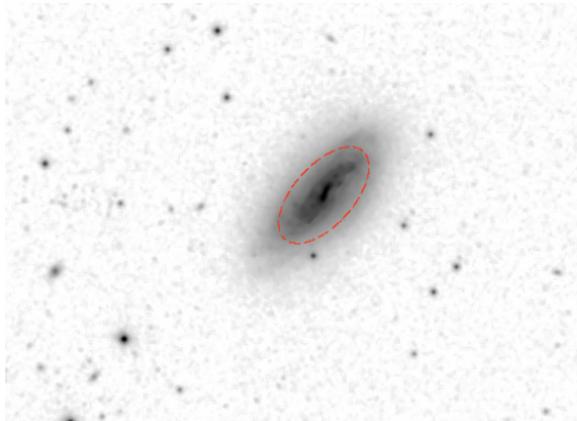
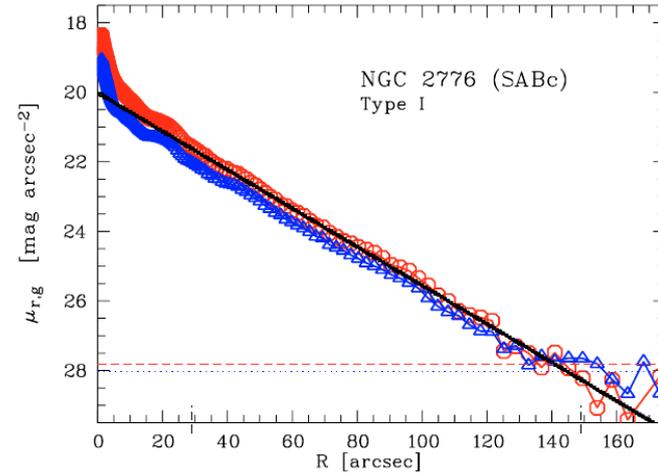
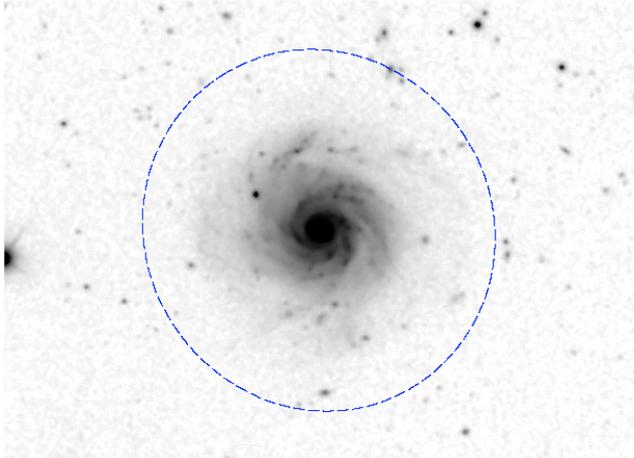
$$\Sigma(R) = \begin{cases} \frac{v_0^2}{2\pi GR} & (R < R_{\max}) \\ 0 & (R \geq R_{\max}) \end{cases}$$

“2D” version of the Isothermal sphere

$$\Phi(R, z) = ? \quad V_c(R) = ?$$

# Exponential disk

$$\Sigma(R) = \Sigma_0 e^{-R/R_d}$$



$$\Phi(R, z) = ? \quad V_c(R) = ?$$

Pohlen & Trujillo 2006  
See also Freeman 1970

# Potential Theory

**The potential of infinite thin  
(razor-thin) disks**

## Potential of zero-thickness (razor-thin) disks

Idea: Sum the contribution of a set of rings as we did for spherical models, summing shells

$$\Sigma(R) = \frac{M \delta(R-R_0)}{2\pi R_0} \quad \text{as } M = 2\pi \int_0^\infty \frac{M}{2\pi R_0} \delta(R-R_0) R dR$$

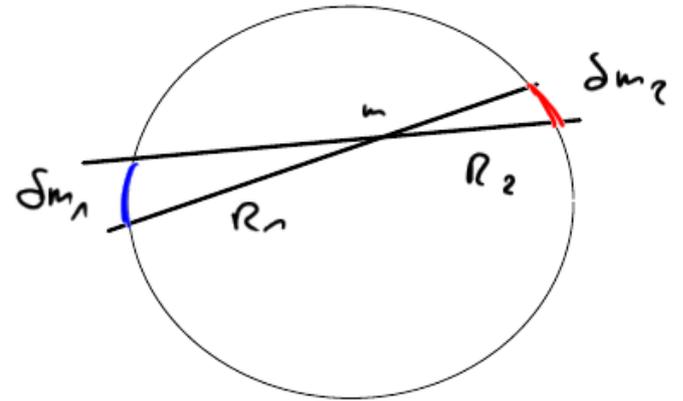
## Potential of a ring

no Newton theorem ! ☹️

$$\bullet \delta m_1 = \Sigma \cdot R_1 d\theta dR$$

$$\bullet \delta m_2 = \Sigma \cdot R_2 d\theta dR$$

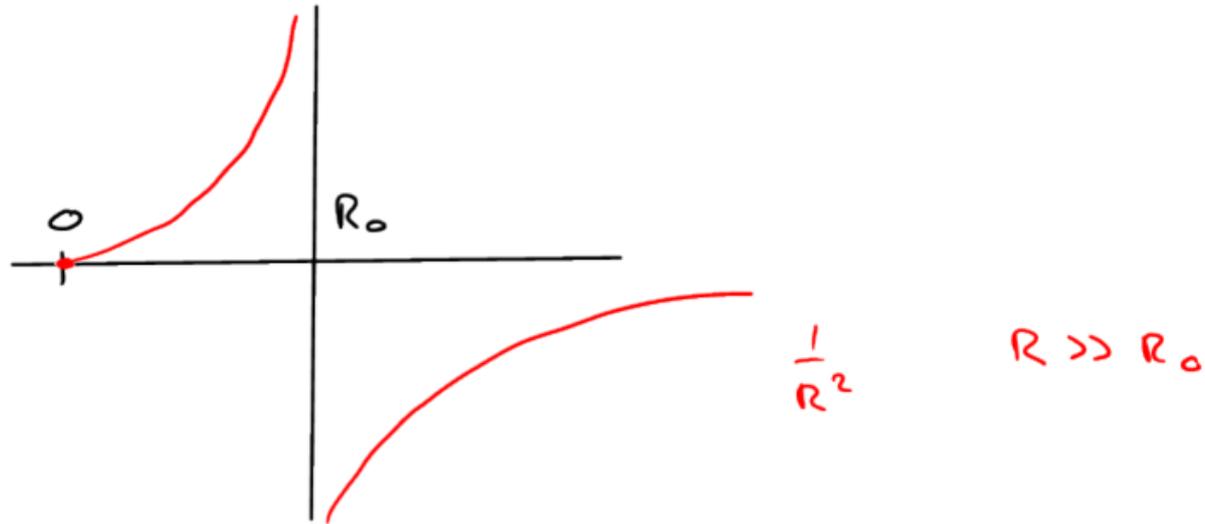
$$\delta F_1 = \frac{G m \Sigma d\theta dR}{R_1} \neq \frac{G m \Sigma d\theta dR}{R_2} = \delta F_2$$



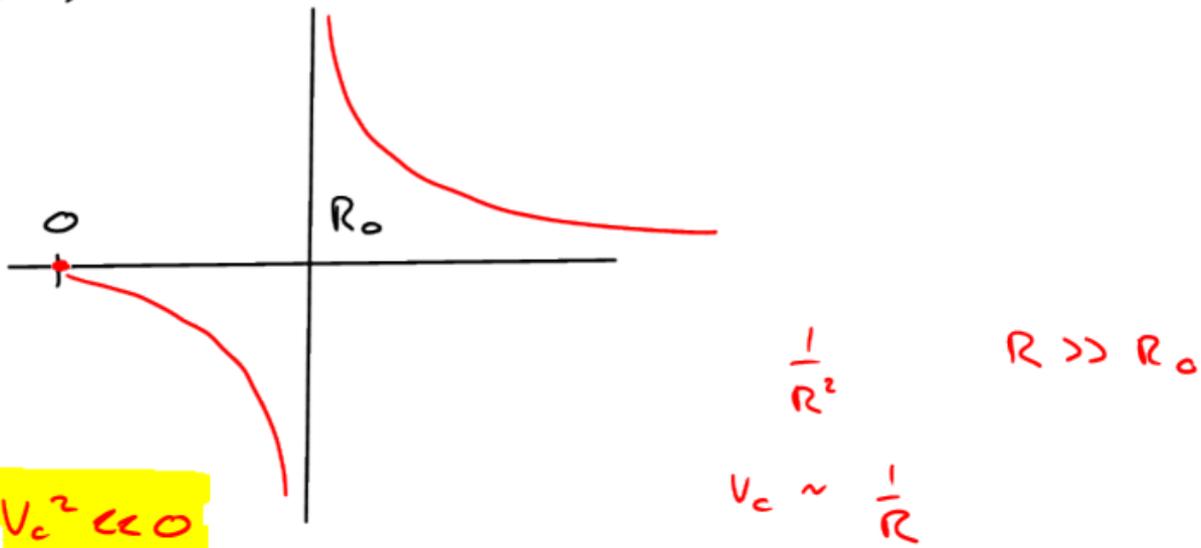
# Estimation of the gravitational field / circular velocity

---

$g(R)$



$$V_c^2 = R \frac{\partial \phi}{\partial R} = -g(R)R$$



# Potential / force of a ring

Lass & Blitzer 1982

$$\phi(R, z) = - \frac{2 GM K(k)}{\pi \sqrt{(R_0 + R)^2 + z^2}}$$

$= \infty$   
 $k=1$   
 $R_0 = R$

$$\left( k^2 = \frac{4R_0 R}{(R + R_0)^2 + z^2} \right)$$

$$g(R, z) = - \frac{GM}{R \pi \sqrt{(R_0 + R)^2 + z^2}} \left[ k \frac{R^2 - R_0^2 - z^2}{4(1-k)R R_0} E(k) + K(k) \right]$$

with •  $K(m)$  : complete elliptic integral of first kind

$$K(m) = \int_0^{\pi/2} [1 - m^2 \sin^2(t)]^{-1/2} dt$$

•  $E(m)$  : complete elliptic integral of second kind

$$E(m) = \int_0^{\pi/2} [1 - m^2 \sin^2(t)]^{+1/2} dt$$

# Potential / force of a ring

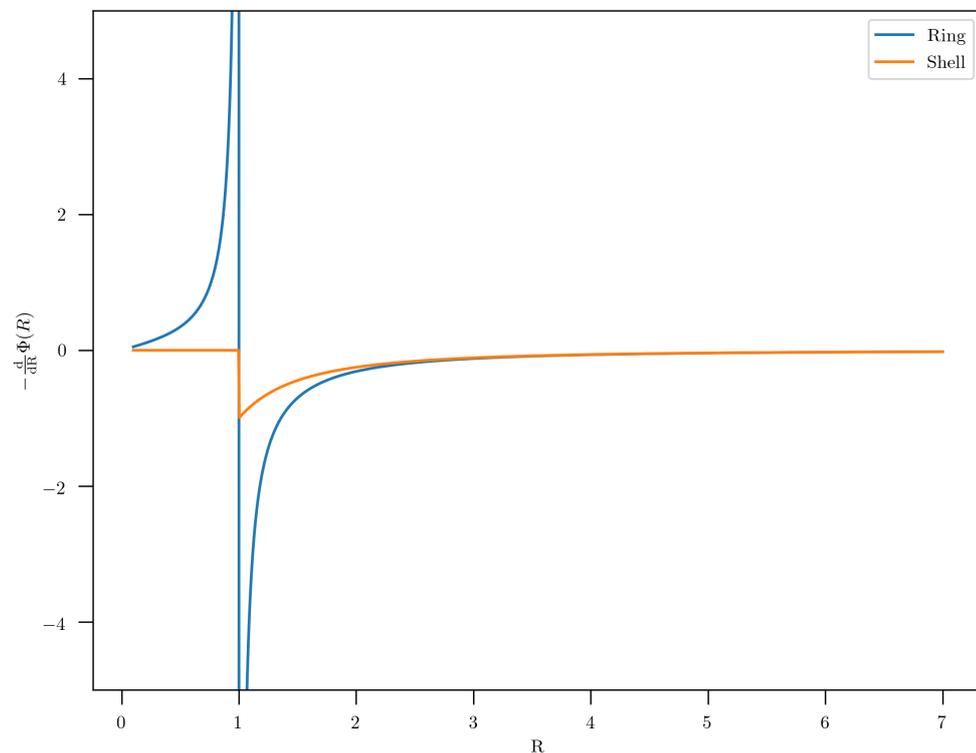
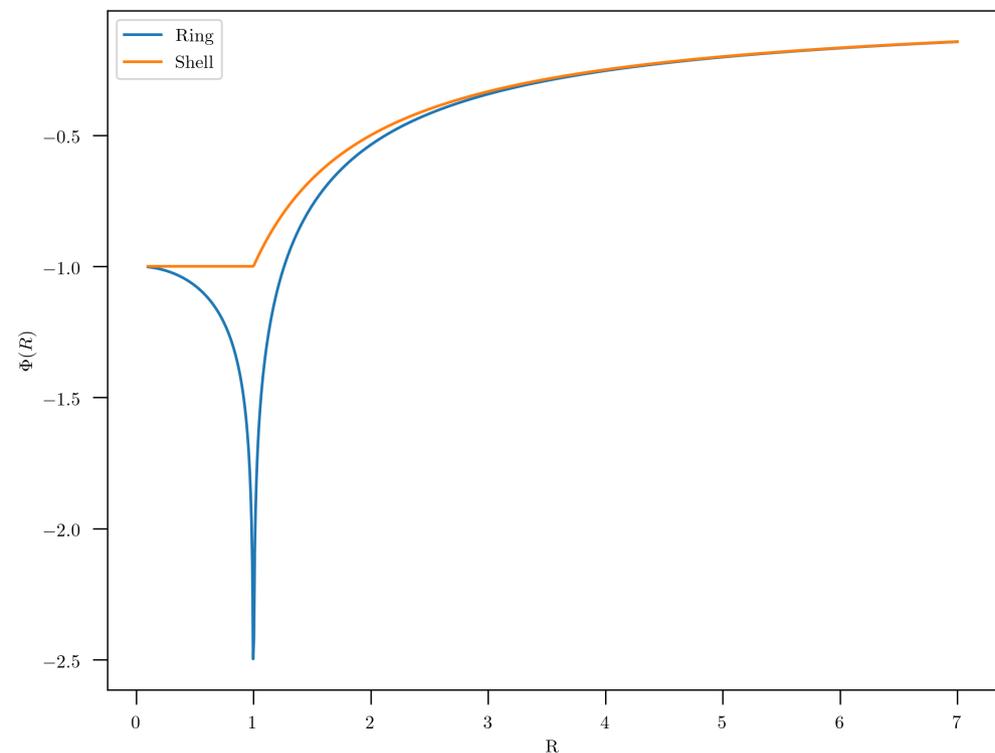
Lass & Blitzer 1982

$= \infty$   
 $k=1$   
 $R_0 = R$

$$\phi(R, z) = - \frac{2GMK(k)}{\pi \sqrt{(R_0 + R)^2 + z^2}}$$

$$\left( k^2 = \frac{4R_0R}{(R + R_0)^2 + z^2} \right)$$

$$g(R, z) = - \frac{GM}{R \pi \sqrt{(R_0 + R)^2 + z^2}} \left[ k \frac{R^2 - R_0^2 - z^2}{4(1-k)R R_0} E(k) + K(k) \right]$$



# Potential of a razor-thin disk of surface density $\Sigma(R)$

---

Sum of rings

$$\phi(R, z) = \int \delta_{R'} \phi(R, z)$$

$$= \int_0^{\infty} - \frac{2 G \delta M' K(k)}{\pi \sqrt{(R' + R)^2 + z^2}}$$

$$\text{with } \delta M' = 2\pi \Sigma(R') R' dR'$$

$$\phi(R, z) = -4 G \int_0^{\infty} dR' \frac{\Sigma(R') R'}{\pi \sqrt{(R' + R)^2 + z^2}} K(k)$$

$$\text{with } k = \sqrt{\frac{4R'R}{(R'+R)^2 + z^2}} \quad z = 0$$

we get

$$\phi(R, z=0) = \frac{-4G}{\sqrt{R}} \int_0^{\infty} dR' \Sigma(R') \sqrt{R'} k K(k)$$

BT 2.265



This integral has a singularity for  $R' \rightarrow R$

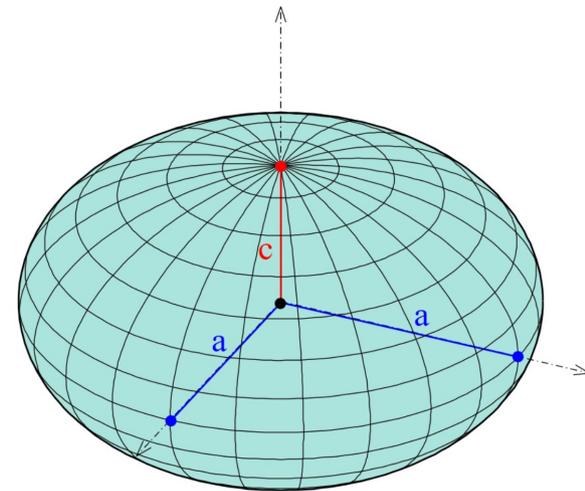
as  $K(1) = \infty$

# Potential Theory

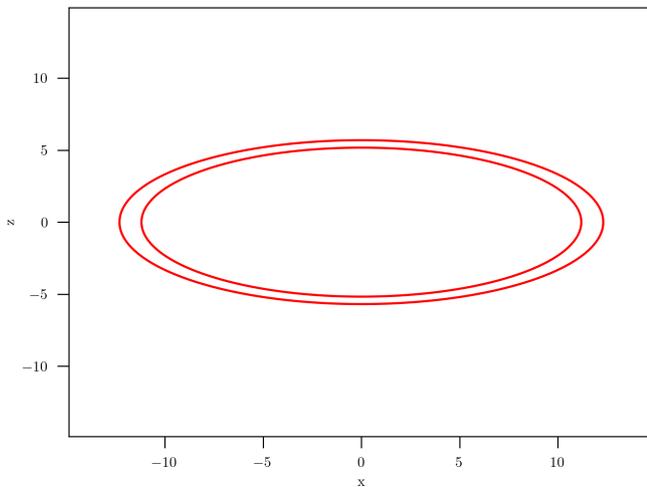
**The potential of infinite thin  
(razor-thin) disks from  
homoeoids**

## Spheroids : ellipse of revolution

$$\frac{R^2}{a^2} + \frac{z^2}{c^2} = 1$$



## Homoeoid : shell of a spheroid of constant density



$$\frac{R^2}{a^2} + \frac{z^2}{c^2} = 1$$

inner shell

$$\frac{R^2}{a^2} + \frac{z^2}{c^2} = (1 + \delta\beta)^2$$

outer shell

The thickness of the homoeoid varies !

# Newton's Theorems

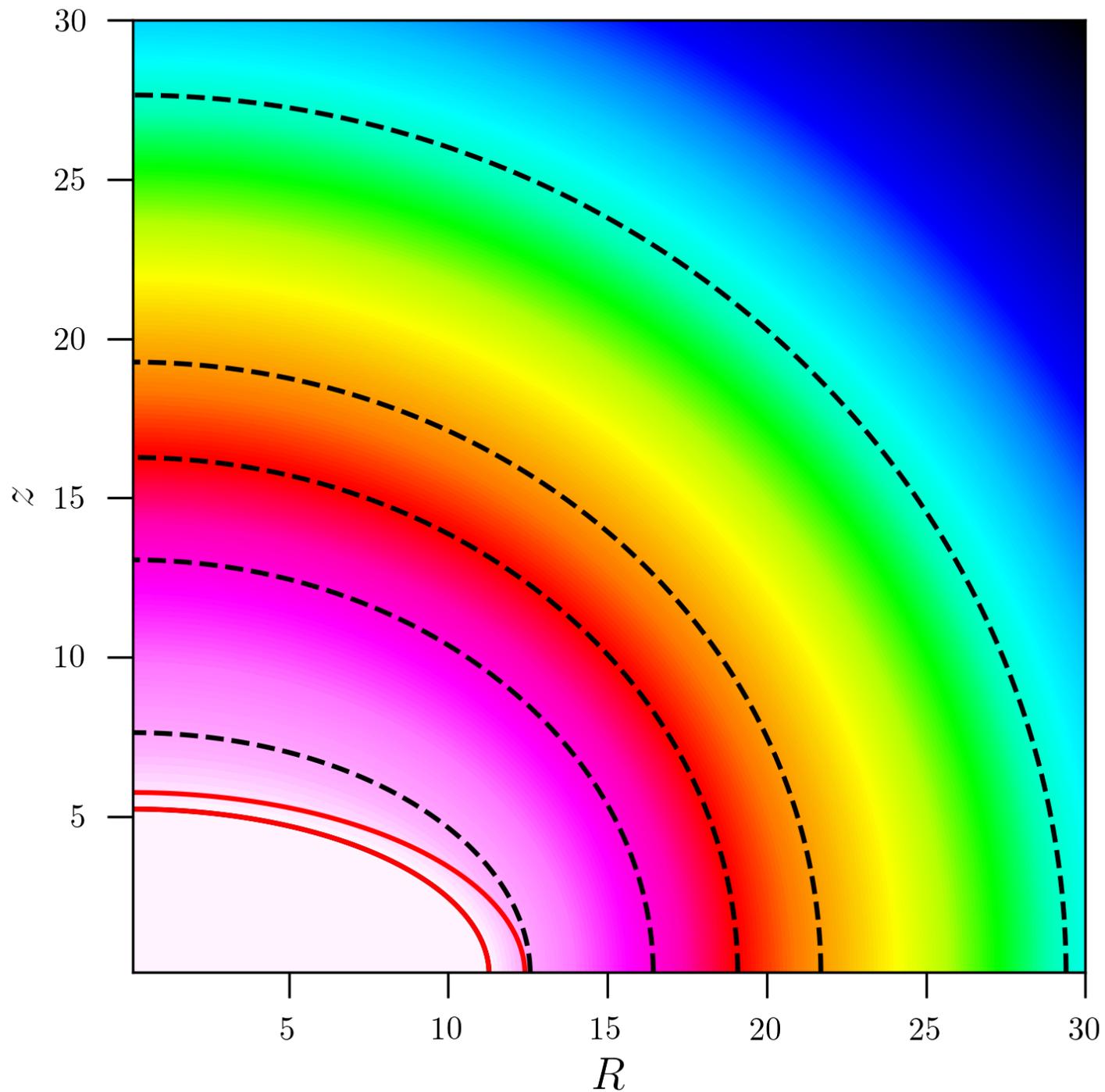
## Homoeoid theorem:

- The exterior iso-potential surfaces of a thin homoeoid are the spheroids that are confocal with the shell itself. Inside the shell, the potential is constant.

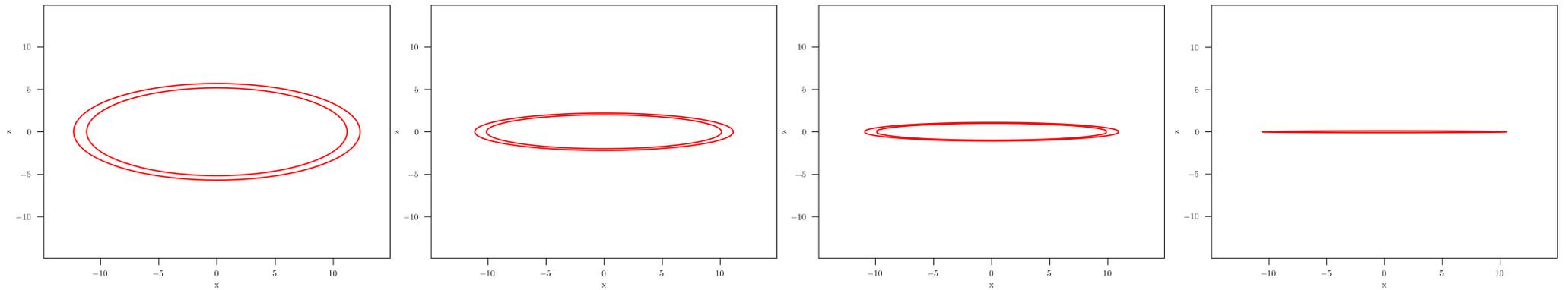
## Newton's third theorem:

- A mass that is inside a homoeoid experiences no net gravitational force from the homoeoid.

# potential of homoeoids

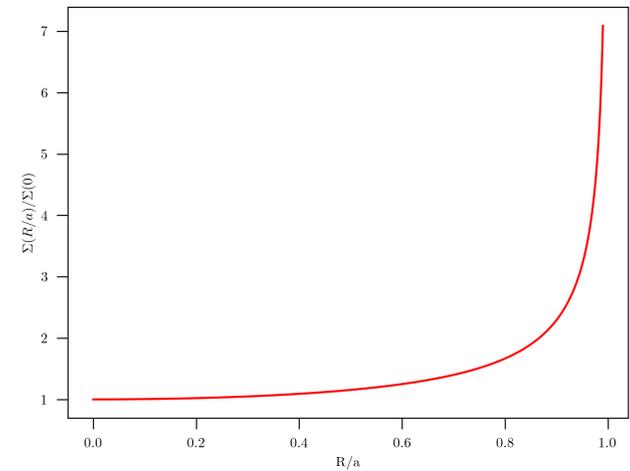


## Zero-thickness Homoeoid



## Surface density :

$$\delta\Sigma(a, R) = \frac{\Sigma(a, 0)}{\sqrt{a^2 - R^2}}$$



## Sum of zero-thickness Homoeoids

$$\Sigma(R) = \sum_{a \leq R} \delta\Sigma(a, R)$$

## Potential

$$\Phi(R, z) = 4G \int_0^R da \arcsin \left( \frac{2a}{\sqrt{+} + \sqrt{-}} \right) \frac{d}{da} \left( \int_a^\infty dR' \frac{R' \Sigma(R')}{\sqrt{R'^2 - a^2}} \right)$$

or

$$\Phi(R, z) = -2\sqrt{2}G \int_0^\infty da \frac{[(a+R)/\sqrt{+}] - [(a-R)/\sqrt{-}]}{\sqrt{R^2 - z^2 - a^2 + \sqrt{+}\sqrt{-}}} \int_a^\infty dR' \frac{R' \Sigma(R')}{\sqrt{R'^2 - a^2}}$$

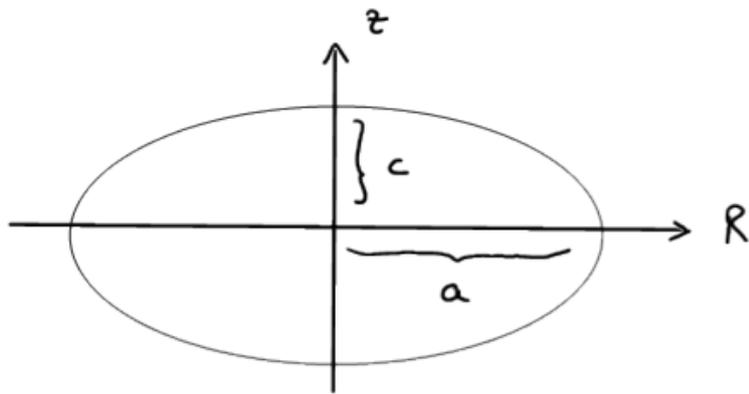
with  $\sqrt{+} = \sqrt{z^2 + (a+R)^2}$        $\sqrt{-} = \sqrt{z^2 - (a+R)^2}$

Circular velocity       $V_c^2(R) = R \frac{\partial}{\partial R} \Phi(R, z) \Big|_{z=0}$

$$V_c^2(R) = -4G \int_0^R da \frac{a}{\sqrt{R^2 - a^2}} \frac{d}{da} \int_a^\infty dR' \frac{R' \Sigma(R')}{\sqrt{R'^2 - a^2}}$$

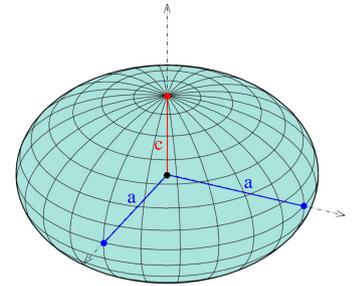
Spheroids  $\equiv$  ellipse of revolution

(axisymmetric system)



Equation of an ellipse

$$\frac{R^2}{a^2} + \frac{z^2}{c^2} = 1$$



we assume a constant density  $\rho$

Eccentricity :  $e^2 = 1 - \frac{c^2}{a^2}$

q tq  $c = q \cdot a$

Mass

$$V = \frac{4}{3} \pi a^2 c$$

$$\Rightarrow M(a) = \frac{4}{3} \pi q a^3 \rho$$

Surface density

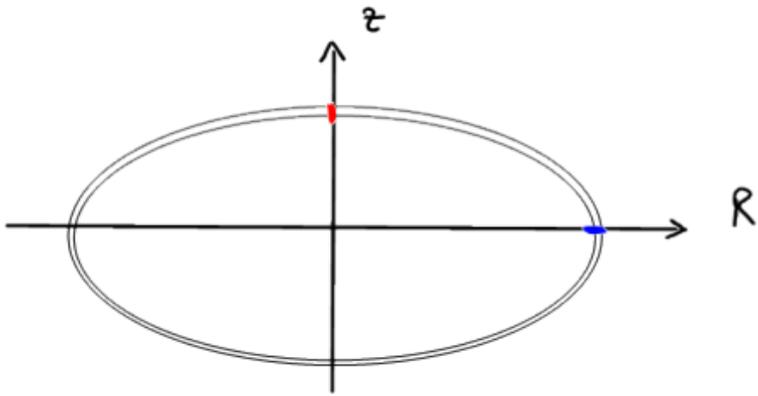
$$\Sigma(R) = \int_{-\infty}^{\infty} dz \rho(R, z) = \rho \int_{-z(R)}^{z(R)} dz = 2\rho z(R)$$

from the ellipse equation  $z^2(R) = c^2 \left(1 - \frac{R^2}{a^2}\right)$   $z(R) = q\sqrt{a^2 - R^2}$

$$\Sigma(R, a) = 2\rho q\sqrt{a^2 - R^2}$$

$$\Sigma_0(a) = \Sigma(0, a) = 2\rho q a$$

Homoeoid : Shell of a spheroid of constant density



$$(i) \text{ inner} \quad \frac{R^2}{a^2} + \frac{z^2}{c^2} = 1$$

$$(o) \text{ outer} \quad \frac{R^2}{a^2} + \frac{z^2}{c^2} = (1 + \delta\beta)^2$$

For  $z = 0$

$$\left. \begin{array}{l} (i) \quad R = a \\ (o) \quad R = a + a\delta\beta \end{array} \right\} \Delta R = a\delta\beta$$

For  $R = 0$

$$\left. \begin{array}{l} (i) \quad z = c \\ (o) \quad z = c + c\delta\beta \end{array} \right\} \Delta z = c\delta\beta$$

$\Rightarrow$

the thickness of the shell varies

$\rho$   $M(a)$  = mass of a spheroid

Mass

$$\delta M(a) = \frac{dM}{da} \delta a = 4\pi \rho a^2 \delta a = 2\pi a \Sigma_0(a) \delta a$$

$$\delta M(a) = 2\pi a \Sigma_0(a) \delta a$$

$\rho$   $\Sigma(a)$  = surface density of a spheroid

Surface density

$$\delta \Sigma(a) = \frac{d\Sigma}{da} \delta a = \frac{2\rho a}{-\sqrt{a^2 - R^2}} \delta a = \frac{\Sigma_0(a)}{-\sqrt{a^2 - R^2}} \delta a$$

$$\delta \Sigma(a) = \frac{\Sigma_0(a)}{-\sqrt{a^2 - R^2}} \delta a$$

# Potential of homocoids

## Spheroidal coordinates

$$(R, \phi, z) \rightarrow (u, \phi, v) \quad \left\{ \begin{array}{l} R = \Delta \cosh u \sin v \\ z = \Delta \sinh u \cos v \end{array} \right. \quad \begin{array}{l} \text{positive constant} \\ \downarrow \end{array}$$

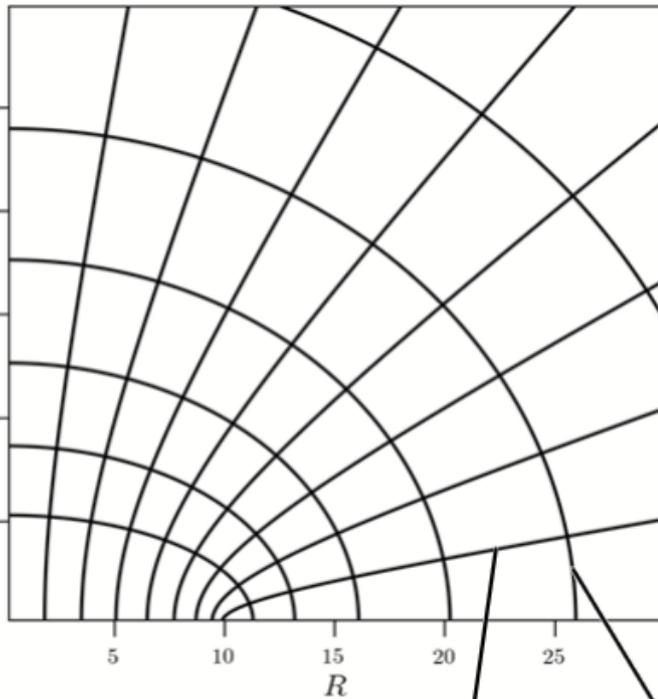
For a constant  $u$  (removing  $v$ )

$$\frac{R^2}{\Delta^2 \cosh^2 u} + \frac{z^2}{\Delta^2 \sinh^2 u} = 1$$

Eccentricity  $e^2 = 1 - \frac{\sinh^2 u}{\cosh^2 u} = 1 - \tanh^2 u$

$$\left\{ \begin{array}{ll} u = 0 & e = 1 \\ u = \infty & e = 0 \end{array} \right.$$

$$\left\{ \begin{array}{l} a = \Delta \cosh u \\ c = \Delta \sinh u \end{array} \right.$$



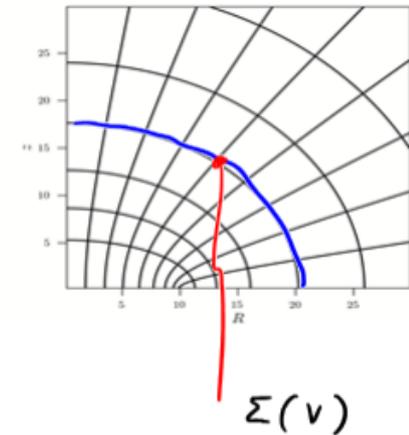
$v = \text{cte}$

$u = \text{cte}$

It is possible to demonstrate that

- 1) the surface density of a thin homoeoid of "radius"  $u_0$  and mass  $\delta M$

$$\Sigma(v) = \frac{\delta M}{4\pi a^2 \sqrt{1 - e^2 \sin^2 v}}$$



- 2) its corresponding potential is

$$\phi(u) = -\frac{G\delta M}{ae} \begin{cases} \arcsin(e) & u \leq u_0 \\ \arcsin\left(\frac{1}{\cosh(u)}\right) & u \geq u_0 \end{cases}$$

## Potential of an homocoid

Assume  $\phi = \phi(u)$  and try to solve  $\nabla^2 \phi = 0$

for  $\phi = \phi(u)$

$$\nabla^2 \phi = \frac{1}{\Delta^2 (\sinh^2 u + \cosh^2 v)} \left[ \frac{1}{\cosh u} \frac{\partial}{\partial u} \left( \cosh u \frac{\partial \phi}{\partial u} \right) \right] = 0$$

$$\frac{\partial}{\partial u} \left( \cosh u \frac{\partial \phi}{\partial u} \right) = 0$$

$$\left( \begin{array}{l} \text{arcsinh} \rightarrow \frac{1}{\sqrt{1-u^2}} \\ \cosh \rightarrow \sinh u \end{array} \right)$$

Solutions

1)  $\phi = \phi_0 = c_1 u$

2)  $\phi = -A \operatorname{arcsinh} \left( \frac{1}{\cosh u} \right) + B$

for  $u \rightarrow \infty$ , using  $R = \Delta \cosh u \cdot \sin v$

$$z = \Delta \sinh u \cos v$$

and  $\cosh^2 u = \sinh^2 u \quad (u \rightarrow \infty)$

we get  $r^2 = R^2 + z^2 = \Delta^2 (\cosh^2 u \sin^2 v + \sinh^2 u \cos^2 v)$   
 $= \Delta^2 \cosh^2 u$

$$\Rightarrow \frac{1}{\cosh(u)} \rightarrow \frac{\Delta}{r}$$

so,  $-A \operatorname{arcsinh} \left( \frac{1}{\cosh(u)} \right) + B \cong -A \operatorname{arcsinh} \left( \frac{\Delta}{r} \right) + B \cong -\frac{\Delta}{r} + B$

$$\Rightarrow \underline{A = \frac{G\delta H}{\Delta}} \quad B = 0$$

So, we get the potential:

$$\phi(u) = -\frac{G\delta H}{\Delta} \begin{cases} \operatorname{arcsinh} \left( \frac{1}{\cosh(u_0)} \right) & u < u_0 \\ \operatorname{arcsinh} \left( \frac{1}{\cosh(u)} \right) & u > u_0 \end{cases}$$

$u_0$  is the surface of an ellipsoid of semi-major/minor axis

$$\begin{cases} a = \Delta \cosh u_0 \\ c = \Delta \sinh u_0 \end{cases} \Rightarrow e = \sqrt{1 - \frac{\cosh^2 u_0}{\sinh^2 u_0}} = \frac{1}{\cosh u_0}$$

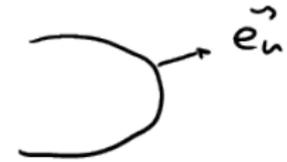
and  $ae = \Delta$

$$\phi(u) = -\frac{GM}{ae} \begin{cases} \operatorname{arcsin}(e) & u < u_0 \\ \operatorname{arcsin}\left(\frac{1}{\cosh(u)}\right) & u > u_0 \end{cases}$$

What is the density at the surface  $u_0$  ?

Gauss theorem :

$$\int \vec{\nabla} \phi \cdot \vec{dS} = 4\pi GM$$



$$\delta M = \frac{\vec{\nabla} \phi \cdot \vec{e}_u \, ds^2}{4\pi G}$$

$$\Sigma(u) = \frac{\delta M}{ds^2} = \frac{\vec{\nabla} \phi \cdot \vec{e}_u}{4\pi G}$$

In elliptical coord :  $\vec{\nabla} \phi(u) = \frac{1}{\Delta \sqrt{\sin^2 u + \cos^2 v}} \frac{\partial \phi}{\partial u} \vec{e}_u$

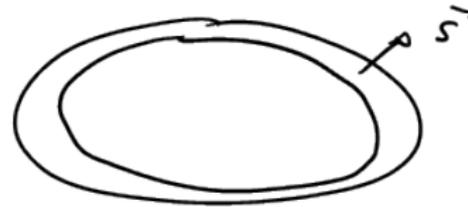
so  $\Sigma(u) = \frac{1}{\Delta \sqrt{\sin^2 u + \cos^2 v}} \left. \frac{\partial \phi}{\partial u} \right|_{u=u_0} \frac{1}{4\pi G}$

$$\Sigma(u) = \frac{\delta M}{4\pi a^2 \sqrt{1 - e^2 \sin^2 v}}$$

Link between  $\Sigma(u)$  and the surf. density of an homocoid

$$\beta^2 = \beta^2(R, z) = \frac{R^2}{a^2} + \frac{z^2}{c^2}$$

$$\vec{s} = s \cdot \vec{e}_n = s \cdot \frac{\vec{\nabla}\beta}{|\vec{\nabla}\beta|}$$



$|\vec{s}| = s =$  thickness of the shell

$$\delta\beta = \vec{s} \cdot \vec{\nabla}\beta$$

$$\beta(R, z) = \beta(R_0, z_0) + \vec{\nabla}\beta \cdot \vec{s}$$

$$\frac{1}{2} (\ )^{-2} \frac{cR}{a^2}$$

$$= s |\vec{\nabla}\beta|$$

$$s = \frac{\delta\beta}{|\vec{\nabla}\beta|}$$

$$\vec{\nabla}\beta = \frac{\partial\beta}{\partial R} \vec{e}_R + \frac{\partial\beta}{\partial z} \vec{e}_z = \frac{1}{R} \frac{R}{a^2} \vec{e}_R + \frac{1}{c} \frac{z}{c^2} \vec{e}_z$$

$$|\vec{\nabla}\beta| = \sqrt{\frac{R^2}{a^4} + \frac{z^2}{c^4}} \beta^{-2}$$

$$s = \left( \frac{R^2}{a^4} + \frac{z^2}{c^4} \right)^{-\frac{1}{2}} \beta \delta\beta$$

$$\Sigma = \rho \cdot s = \rho \left( \frac{R^2}{a^4} + \frac{z^2}{c^4} \right)^{-\frac{1}{2}} \beta \delta\beta$$

We introduce  $v$ , such that :

$$R = \beta a \sin(v) \quad z = \beta c \cos(v)$$

$$e = \sqrt{1 - \frac{c^2}{a^2}} \quad e^2 = 1 - \frac{c^2}{a^2}$$

$$e^2 a^2 = a^2 - c^2$$

$$\left( \frac{R^2}{a^4} + \frac{z^2}{c^4} \right)^{-\frac{1}{2}} \beta \int d\beta$$

$$R^2 = \beta^2 a^2 \sin^2 v$$

$$z^2 = \beta^2 c^2 \cos^2 v$$

$$\frac{R^2}{a^4} + \frac{z^2}{c^4}$$

$$\beta^2 \left( \frac{\sin^2 v}{a^2} + \frac{\cos^2 v}{c^2} \right)$$

$$\beta^2 \left( \frac{\sin^2 v}{a^2} + \frac{\cos^2 v}{c^2} \right)$$

$$\beta^2 \left( \frac{R^2 v}{a^2} + \frac{1 - R^2 v}{c^2} \frac{1}{c^2} \right)$$

$$\frac{\beta^2}{a^2 c^2} \left( c^2 \sin^2 v + a^2 - a^2 \cos^2 v \right)$$

$$c^2 = a^2 (1 - e^2)$$

$$c = a \sqrt{1 - e^2}$$

$$\frac{\beta^2}{a^2 c^2} \left( \sin^2 \nu (c^2 - a^2) + a^2 \right)$$

$$a^{2n} \frac{\beta^2}{(1 - e^2)} \left( \cancel{c^2} - e^2 \cancel{c^2} \sin^2 \nu \right)$$

$$\Sigma = \int \left( \frac{R^2}{a^4} + \frac{r^2}{c^4} \right)^{-\frac{1}{2}} \beta \, d\beta$$

$$= \frac{a \sqrt{1 - e^2}}{\beta \sqrt{1 - e^2 \sin^2 \nu}} \int \cancel{\beta} \, d\beta = \frac{a \sqrt{1 - e^2}}{\sqrt{1 - e^2 \sin^2 \nu}} \int d\beta$$

Volume of the ellipsoid  $\therefore V = \frac{4}{3} \pi a^2 c \beta^3 = \frac{4}{3} \pi a^3 \beta^2 \sqrt{1 - e^2}$

$$\delta M = \delta(\rho M) = 4\pi a^3 \beta^2 \rho \sqrt{1 - e^2} \, d\beta$$

introduce  $\delta\beta$  in  $\Sigma = \frac{a \sqrt{1 - e^2}}{\sqrt{1 - e^2 \sin^2 \nu}} \int d\beta$

and set  $\beta = 1$

$$\Sigma(\nu) = \frac{a \sqrt{1-e^2} \rho \delta M}{4\pi a^3 \beta^2 \rho \sqrt{1-e^2} \sqrt{1-e^2} \Omega^2 \nu}$$

$$\Sigma(\nu) = \frac{\delta M}{4\pi a^2 \sqrt{1-e^2} \Omega^2 \nu}$$

#

# Potential Theory

**The potential of infinite thin  
(razor-thin) disks from  
homoeoids**

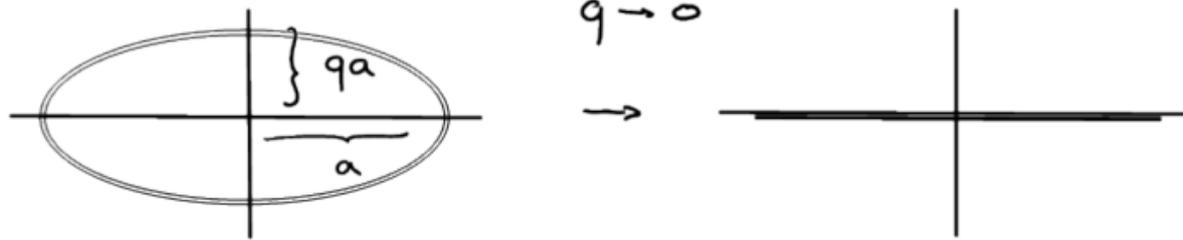
The potential of zero thickness (razor-thin) disks from homocoids

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Idea Reproduce any surface density  $\Sigma(R)$  by summing a set of infinitely flattened homocoids

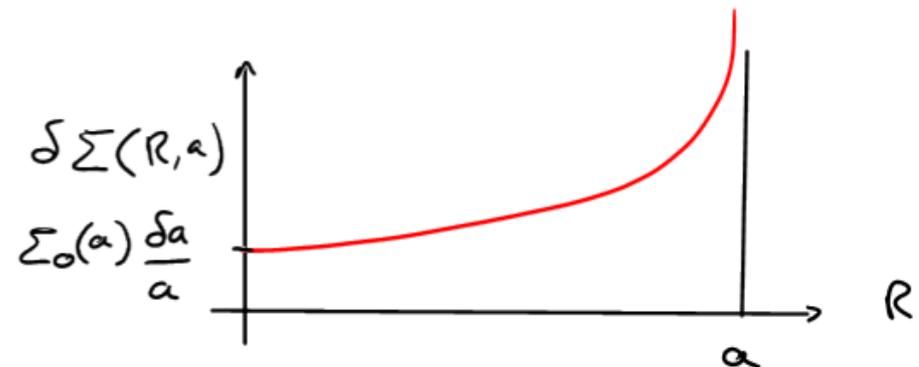
Infinitely flattened homocoids

---



The surface density remains the same (indep. of  $q$ )

$$\delta\Sigma(a, R) = \frac{\Sigma_0(a)}{\sqrt{a^2 - R^2}} \delta a$$



## Summing infinitely flattened homocoids

$$\begin{aligned}\Sigma(R) &= \sum_{a \geq R} \delta \Sigma(a, R) = \sum_{a \geq R} \frac{d}{da} \Sigma(a, R) \delta a = \sum_{a \geq R} \frac{\Sigma_0(a)}{\sqrt{a^2 - R^2}} \delta a \\ &= \int_R^{\infty} \frac{\Sigma_0(a)}{\sqrt{a^2 - R^2}} da \quad \text{Abel integral}\end{aligned}$$

Solution:

$$\Sigma_0(a) = -\frac{2}{\pi} \frac{d}{da} \left( \int_a^{\infty} dR \frac{R \Sigma(R)}{\sqrt{R^2 - a^2}} \right)$$

For a given  $\Sigma(R)$  we can compute  $\Sigma_0(a)$  (the weights)

such that 
$$\Sigma(R) = \int_R^{\infty} \delta \Sigma(a, R)$$

# Potential of infinitely flattened homocoids

$$\phi(u) = -\frac{GM}{ae} \operatorname{arcsin}\left(\frac{1}{\cosh(u)}\right) \quad u \geq u_0$$

The potential is continuous across the plane  $z=0$

we can compute it just above the plane i.e., outside the shell

with  $GM = 2\pi a \Sigma_0(a) \Delta a$  and for  $u \geq u_0$   
and noting that for  $q \rightarrow 0$   $e \rightarrow 1$

$$\begin{aligned} \Delta\phi_a(R, z) &= -\frac{G 2\pi a \Sigma_0(a) \Delta a}{ae} \operatorname{arcsin}\left(\frac{1}{\cosh(u)}\right) \\ &= -2\pi G \Sigma_0(a) \Delta a \operatorname{arcsin}\left(\frac{1}{\cosh(u)}\right) \end{aligned}$$

Expression for  $u$

$$\text{from } \begin{cases} R = \Delta \cosh u \sin v \\ z = \Delta \sinh u \cos v \end{cases} \quad \text{and}$$

$$\cos^2 v + \sin^2 v = 1$$

$$\cosh^2 u = \frac{1}{4a^2} \left[ \underbrace{\sqrt{z^2 + (a+R)^2}}_{\sqrt{+}} + \underbrace{\sqrt{z^2 - (a-R)^2}}_{\sqrt{-}} \right]^2$$

Thus

$$\delta\phi_a(R, z) = -2\pi G \Sigma_0(a) \arcsin\left(\frac{2a}{\sqrt{+} + \sqrt{-}}\right) \delta a$$

Summing the contribution of all homocoids

$$\phi(R, z) = \int_0^{\infty} \delta\phi_a(R, z) = -2\pi G \int_0^{\infty} \Sigma_0(a) \arcsin\left(\frac{2a}{\sqrt{+} + \sqrt{-}}\right) da$$

$$\text{but } \Sigma_0(a) = -\frac{2}{\pi} \frac{d}{da} \left( \int_a^{\infty} dR' \frac{R' \Sigma(R')}{\sqrt{R'^2 - a^2}} \right)$$

$$\phi(R, z) = 4G \int_0^{\infty} da \arcsin\left(\frac{2a}{\sqrt{+} + \sqrt{-}}\right) \frac{d}{da} \left( \int_a^{\infty} dR' \frac{R' \Sigma(R')}{\sqrt{R'^2 - a^2}} \right)$$

dep. only on "a" : can be tabulated

Integrating by part gives

$$\phi(R, z) = -2\sqrt{2}G \int_0^{\infty} da \frac{\left[ \frac{(a+R)/\sqrt{+}}{\sqrt{R^2-z^2-a^2 + \sqrt{+} \cdot \sqrt{-}}} \right] - \left[ \frac{(a-R)/\sqrt{-}}{\sqrt{R^2-z^2-a^2 + \sqrt{+} \cdot \sqrt{-}}} \right]}{\sqrt{R^2-z^2-a^2 + \sqrt{+} \cdot \sqrt{-}}} \int_a^{\infty} dR' \frac{R' \Sigma(R')}{\sqrt{R'^2-a^2}}$$

Circular velocity

$$V_c^2(R) = R \left. \frac{\partial \phi}{\partial R} \right|_{z=0}$$

$$\frac{d}{dR} \arcsin \left( \frac{2a}{|a+R| + |a-R|} \right)$$

•  $R < a \rightarrow a+R + a-R = 2a \Rightarrow \arcsin(1) \Rightarrow \frac{d}{dR} = 0$

•  $R > a \rightarrow a+R - a+R = 2R \Rightarrow \arcsin\left(\frac{a}{R}\right) \Rightarrow \frac{d}{dR} = -\frac{a/R^2}{\sqrt{1-(a/R)^2}}$

$$V_c^2(R) = -4G \int_0^R da \frac{a}{\sqrt{R^2-a^2}} \frac{d}{da} \left( \int_a^{\infty} dR' \frac{R' \Sigma(R')}{\sqrt{R'^2-a^2}} \right)$$

# Potential Theory

**The potential of infinite thin  
(razor-thin) disks from  
homoeoids :  
applications**

# Exponential disk

$$\Sigma(R) = \Sigma_0 e^{-R/R_d}$$

The integral in the razor-thin potential equation is then:

$$\int_a^\infty R' \frac{R' \Sigma_0 e^{-R'/R_d}}{\sqrt{R'^2 - a^2}} = \Sigma_0 a K_1(a/R_d)$$

The potential:

$$\Phi(R, z) = -2\sqrt{2} G \int_0^\infty a \frac{\frac{a+R}{\sqrt{z^2+(a+R)^2}} - \frac{a-R}{\sqrt{z^2+(a-R)^2}}}{\sqrt{R^2 - z^2 - a^2 + \sqrt{z^2 + (a+R)^2}} \sqrt{z^2 + (a-R)^2}} \times \Sigma_0 a K_1(a/R_d)$$

The circular velocity:

$$v_c^2 = 4\pi G \Sigma_0 R_d y^2 [I_0(y)K_0(y) - I_1(y)K_1(y)]$$

$$y = \frac{R}{2 R_d}$$

Bessel functions

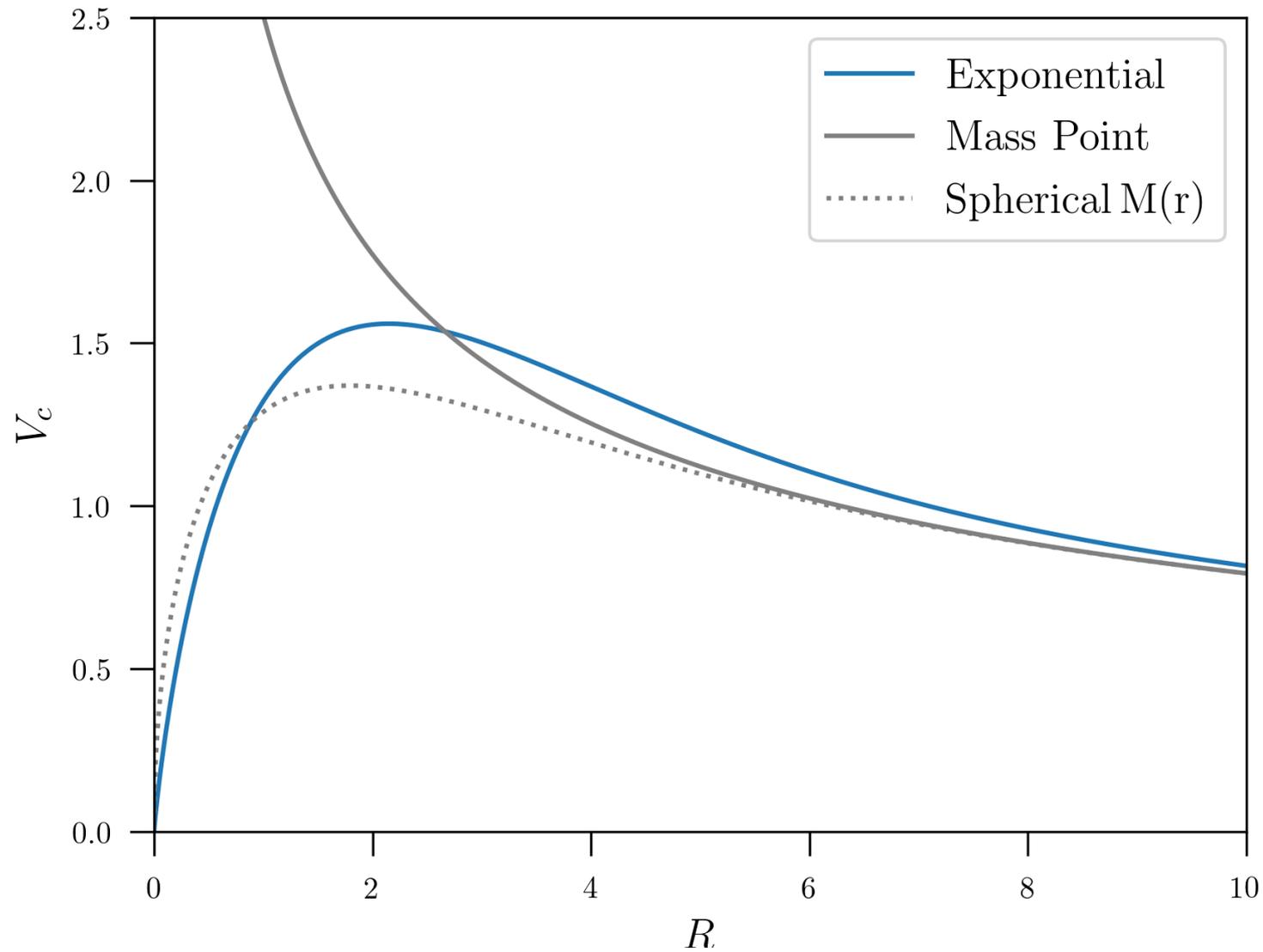
$$I_\nu(z) = i^{-\nu} J_\nu(iz)$$

$$K_\nu(z) = \frac{\pi}{2} \frac{I_{-\nu}(z) - I_\nu(z)}{\sin(\nu\pi)}$$

$$J_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(\nu+k)!} \left(\frac{1}{2}z\right)^{\nu+2k}$$

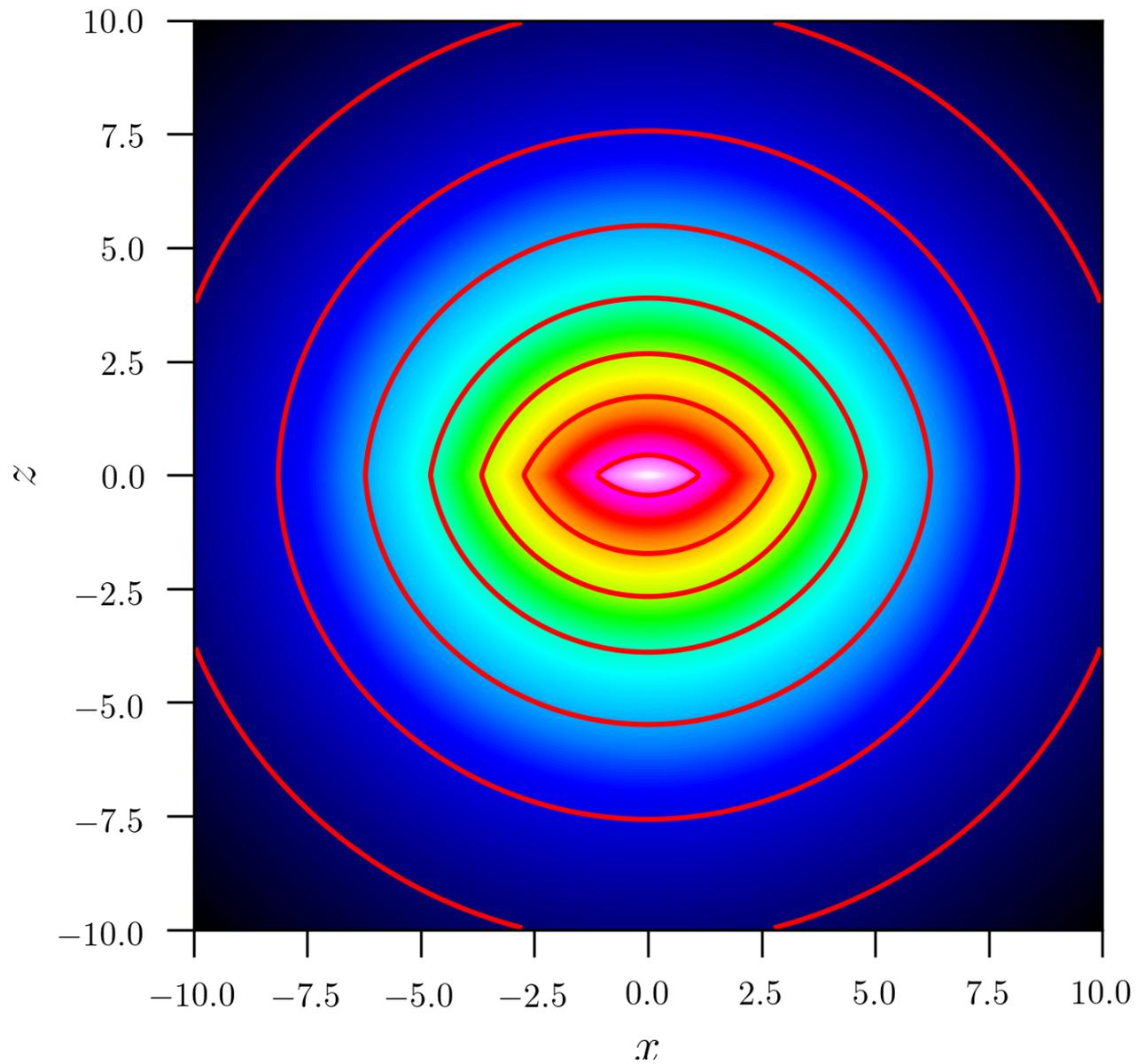
# Exponential disk

## Circular velocity rotation curve



# Exponential disk

## Potential



# Mestel disk

$$\Sigma(R) = \begin{cases} \frac{v_0^2}{2\pi GR} & (R < R_{\max}) \\ 0 & (R \geq R_{\max}) \end{cases}$$

“2D” version of the Isothermal sphere

for  $R_{\max} \rightarrow \infty$

$$v_c^2 = \frac{2v_0^2}{\pi} \int_0^R \frac{a}{\sqrt{R^2 - a^2}} = v_0^2 = \text{cte}$$

Computing the cumulative mass:

**EXERCICE**

$$M(R) = 2\pi \int_0^R R' \Sigma(R') = \frac{v_0^2 R}{G}$$

we get:

$$v_0^2 = v_c^2(R) = \frac{GM(R)}{R}$$

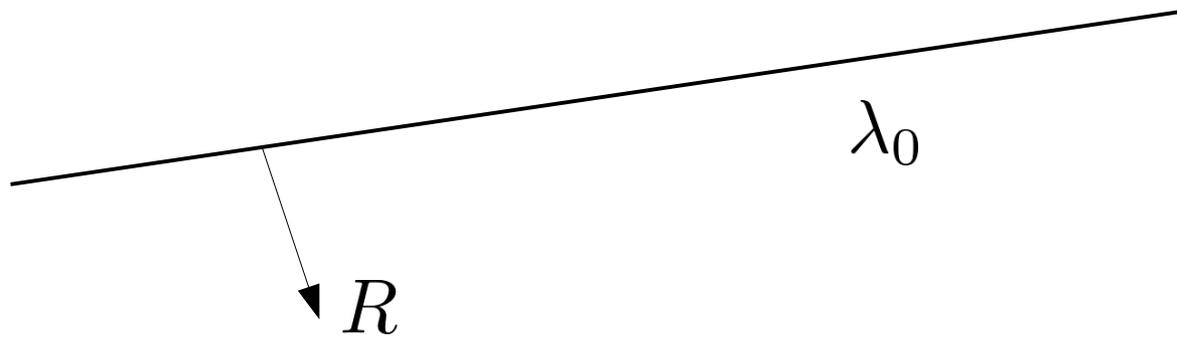


This is very specific to the Mestel disk...  
In general the external mass matter.

# Potential Theory

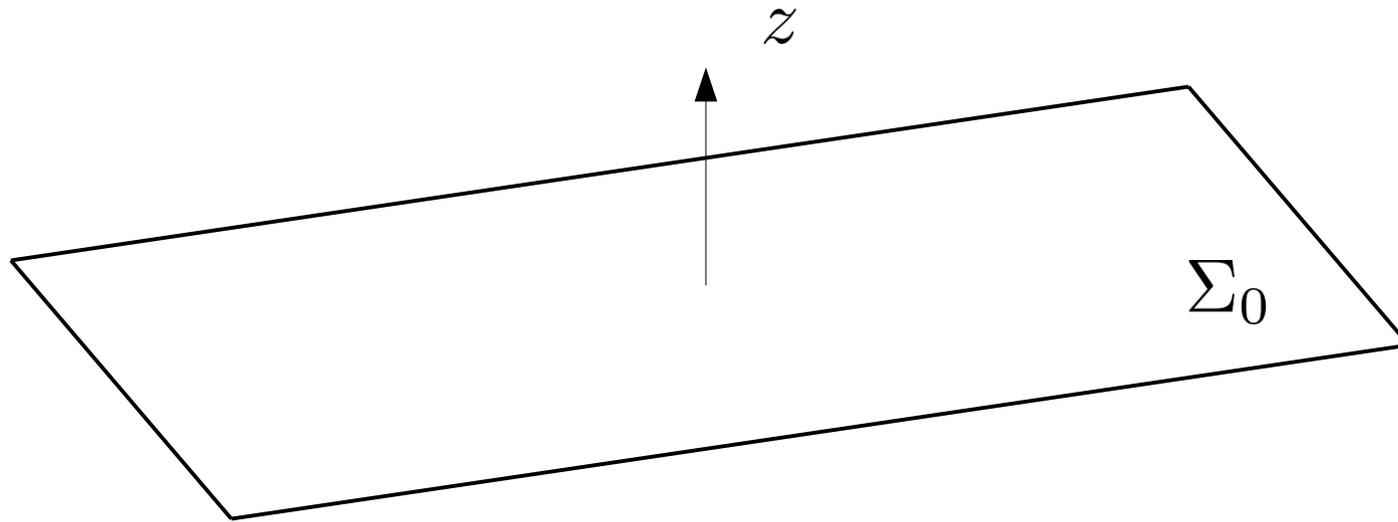
**Idealized but useful models**

# Potential of an infinite wire of constant linear density



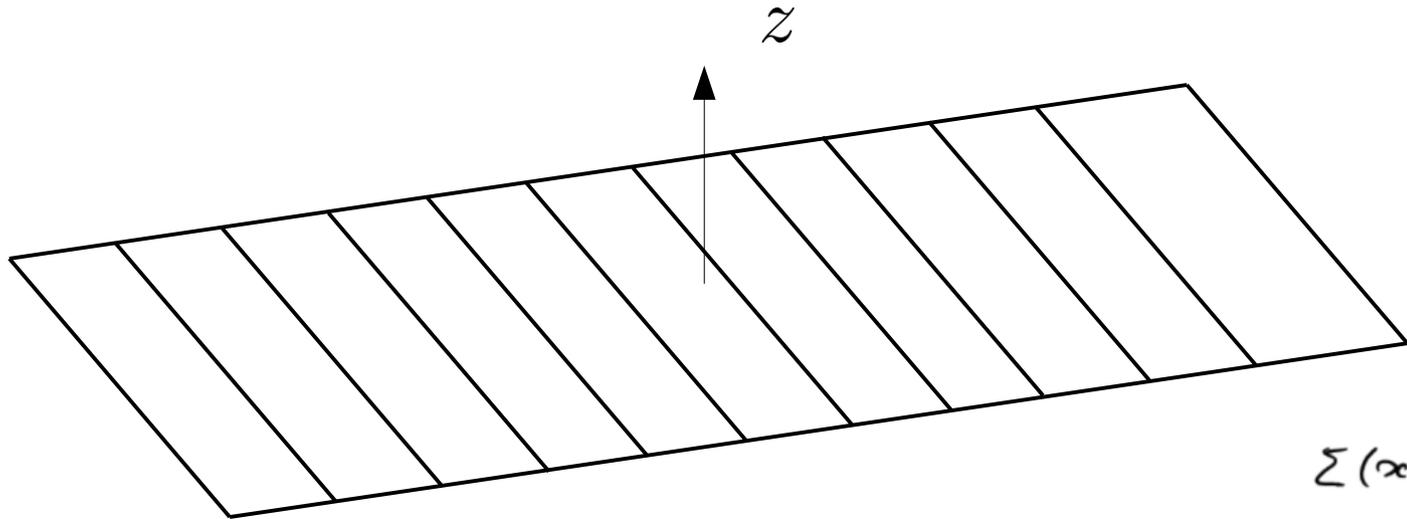
$$\Phi(R) = 2 G \lambda_0 \ln(R) + C$$

# Potential of an infinite slab of constant surface density



$$\Phi(z) = 2\pi G \Sigma_0 |z| + C$$

# Potential of an infinite slab with an oscillatory surface density



$$k = |\vec{k}| = \frac{2\pi}{\lambda}$$

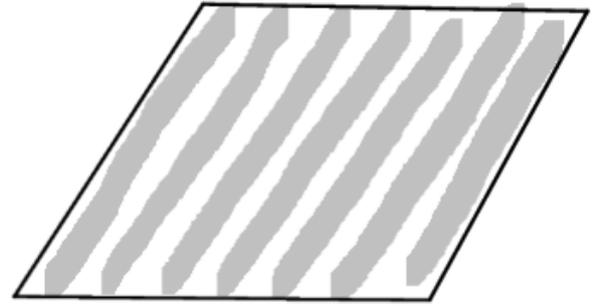
$$\Sigma(x, y) = \Sigma_1 \operatorname{Re} \left( e^{i(\vec{k} \cdot \vec{x})} \right)$$

**! will be negative !**

$$\Phi(x, y, z) = -\frac{2\pi G \Sigma_1}{|\vec{k}|} \operatorname{Re} \left( e^{i(\vec{k} \cdot \vec{x})} \right) e^{-|\vec{k}| z}$$

# Potential of an infinite slab with an oscillatory surface density

$$\Sigma(x, y) = \operatorname{Re} \left( \Sigma_0 e^{i(kx^2)} \right)$$



Without loss of generality we can restrict to:

$$\Sigma(x) = \Sigma_0 e^{ikx}$$



Poisson equation

$$\nabla^2 \phi(x, z) = 4\pi G \Sigma(x) \delta(z)$$

Assume a corresponding potential of the type

$$\phi(x, z) = \phi_0 e^{ikx - |kz|}$$

Method: Integrate the Poisson equation over  $\tau$

---

$$\nabla^2 \phi = 4\pi G \rho$$

$$\int_{-\xi}^{\xi} d\tau \nabla^2 \phi = \int_{-\xi}^{\xi} d\tau 4\pi G \rho$$

and take the limit  $\xi \rightarrow 0$

$$\underbrace{\lim_{\xi \rightarrow 0} \int_{-\xi}^{\xi} d\tau \nabla^2 \phi}_{(1)} = \underbrace{\lim_{\xi \rightarrow 0} \int_{-\xi}^{\xi} d\tau 4\pi G \rho}_{(2)}$$

$$\textcircled{1} \quad \nabla^2 = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$$

$\frac{\partial^2 \phi}{\partial x^2}(x, y, 0)$  is continuous across  $z=0$

$$\lim_{\xi \rightarrow 0} \int_{-\xi}^{\xi} dz \frac{\partial^2 \phi}{\partial x^2}(x, y, z) = 0$$

$$\lim_{\xi \rightarrow 0^+} = \lim_{\xi \rightarrow 0^-}$$

$$\lim_{\xi \rightarrow 0} \int_{-\xi}^{\xi} dz \frac{\partial^2 \phi}{\partial y^2}(x, y, z) = 0$$

$$\lim_{\xi \rightarrow 0} \int_{-\xi}^{\xi} dz \frac{\partial^2 \phi}{\partial z^2}(x, y, z) = \lim_{\xi \rightarrow 0} \left. \frac{\partial \phi}{\partial z} \right|_{-\xi}^{\xi}$$

$$= \lim_{\xi \rightarrow 0} \phi_0 |k| \operatorname{sgn}(z) e^{ikx - |kz|} \Big|_{-\xi}^{\xi} = -2|k| \phi_0 e^{ikx}$$

$$\begin{aligned}
 \textcircled{2} \quad \lim_{\xi \rightarrow 0} \int_{-\xi}^{\xi} dz \quad 4\pi G \rho &= \lim_{\xi \rightarrow 0} \int_{-\xi}^{\xi} dz \quad 4\pi G \Sigma_0 e^{ikx} \delta(z) \\
 &= 4\pi G \Sigma_0 e^{ikx}
 \end{aligned}$$

Combining  $\textcircled{1}$  and  $\textcircled{2}$

$$-2|\kappa| \phi_0 e^{ikx} = 4\pi G \Sigma_0 e^{ikx}$$

$$\phi_0 = -\frac{2\pi G \Sigma_0}{|\kappa|}$$

$$\phi(x, y, z) = -\frac{2\pi G \Sigma_0}{|\kappa|} e^{ikx - |kz|}$$

Thus for  $\Sigma(x, y) = \Sigma_0 e^{i\vec{k}\vec{x}}$

$$\phi(x, y, z) = -\frac{2\pi G \Sigma_0}{|\vec{k}|} e^{i\vec{k}\vec{x} - |\vec{k}|z}$$

Note if the surface density evolves as a plane wave

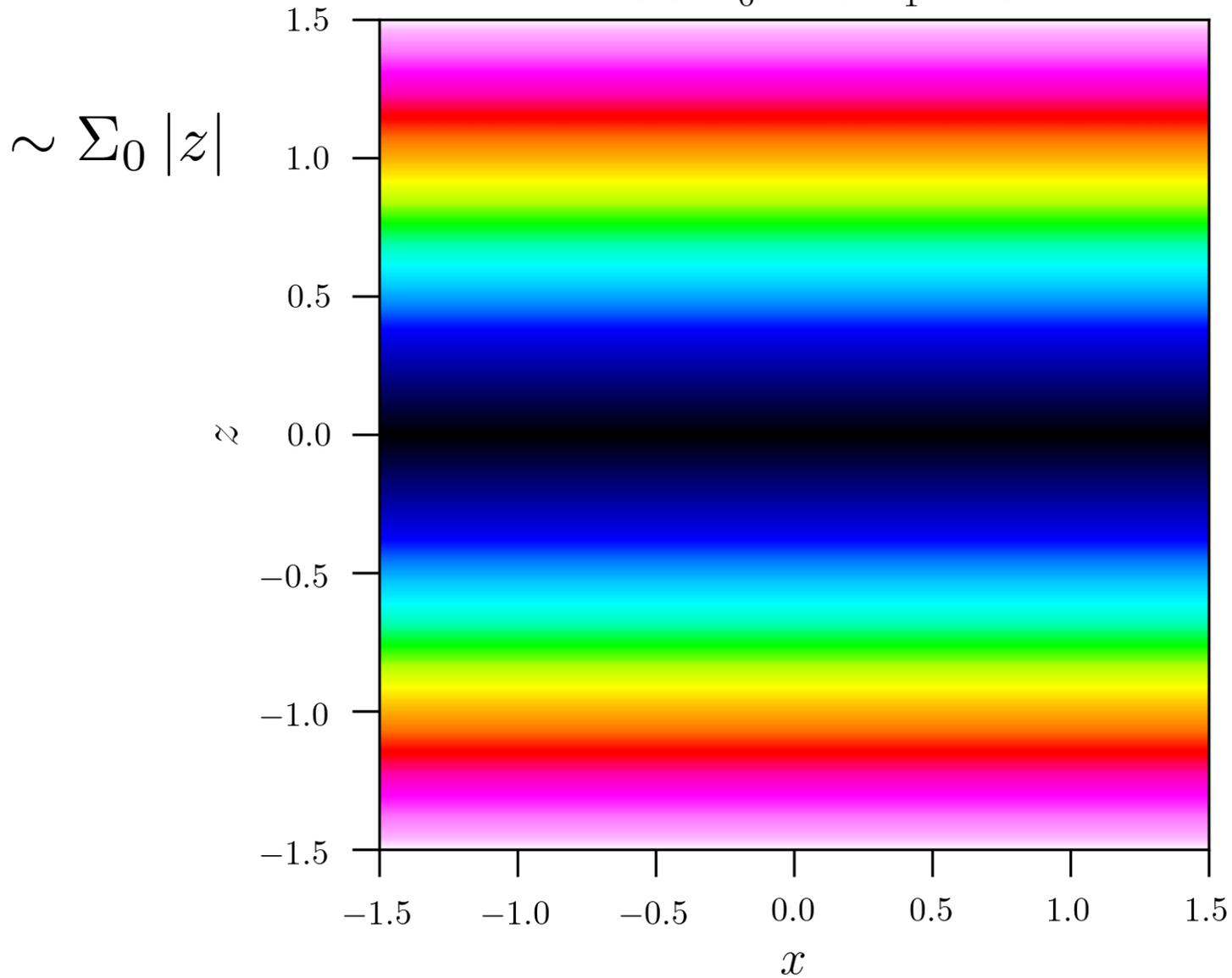
$$\Sigma(x, y, t) = \Sigma_0 e^{i(\vec{k}\vec{x} - \omega t)}$$

$$\phi(x, y, z, t) = -\frac{2\pi G \Sigma_0}{|\vec{k}|} e^{i(\vec{k}\vec{x} - \omega t) - |\vec{k}|z}$$

# Potential of an Infinite slab

$$\Sigma(x) = \Sigma_0 + \Sigma_1 \operatorname{Re}(e^{ikx})$$

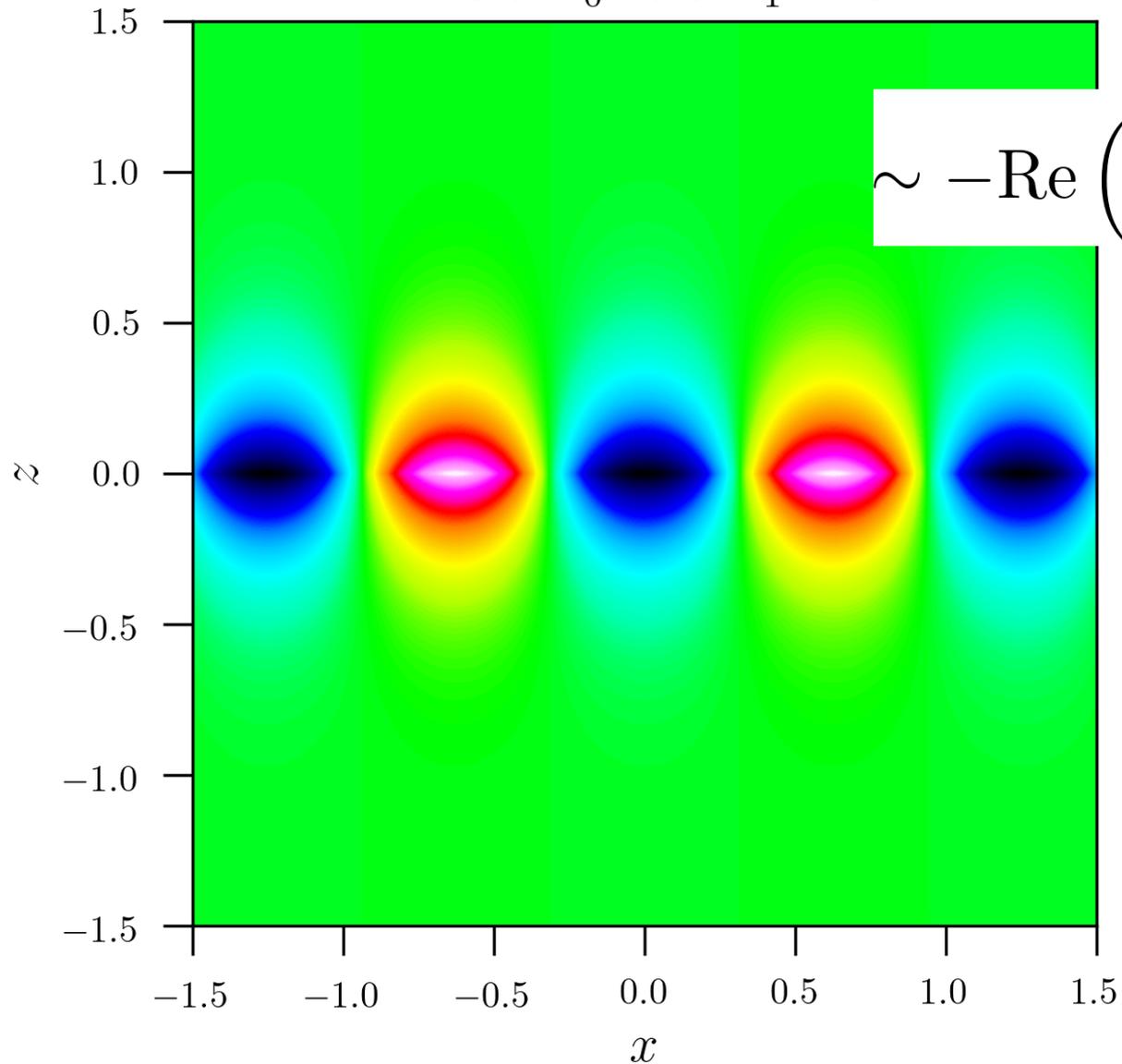
$$k=5.0 \quad \Sigma_0=1.0 \quad \Sigma_1=0.0$$



# Potential of an Infinite slab

$$\Sigma(x) = \Sigma_0 + \Sigma_1 \operatorname{Re} (e^{ikx})$$

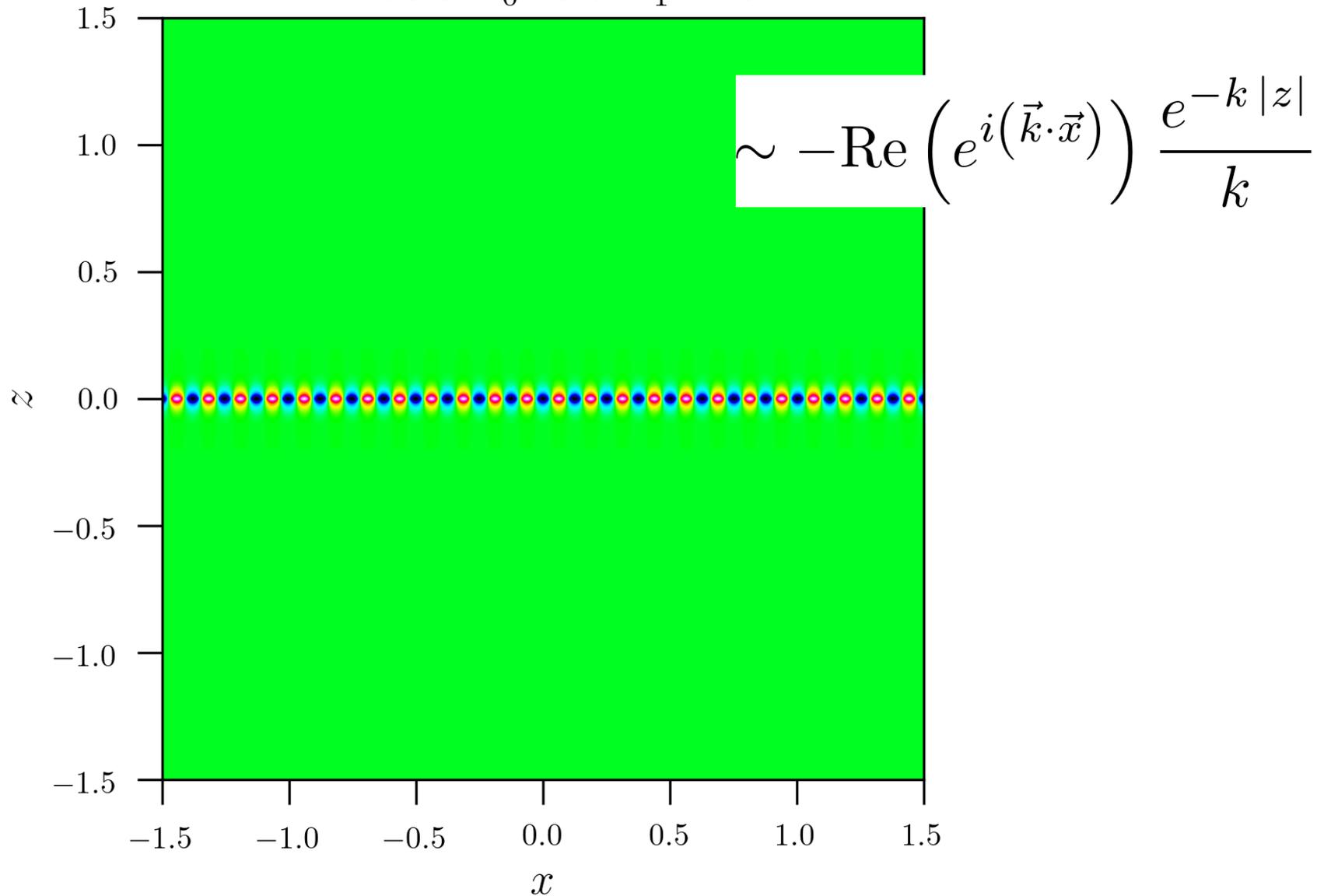
$$k=5.0 \quad \Sigma_0=0.0 \quad \Sigma_1=1.0$$



# Potential of an Infinite slab

$$\Sigma(x) = \Sigma_0 + \Sigma_1 \operatorname{Re} (e^{ikx})$$

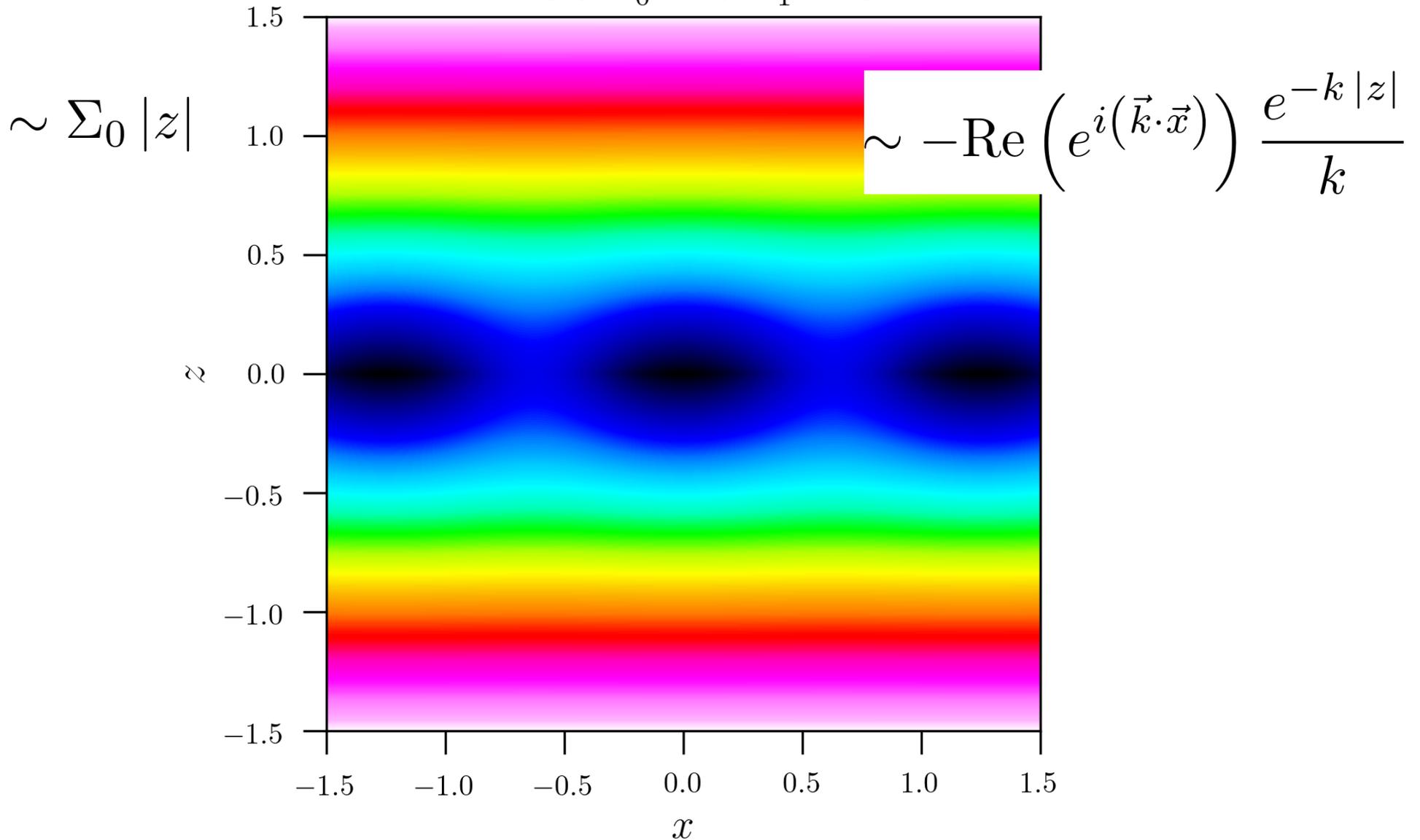
$$k=50.0 \quad \Sigma_0=0.0 \quad \Sigma_1=1.0$$



# Potential of an Infinite slab

$$\Sigma(x) = \Sigma_0 + \Sigma_1 \operatorname{Re} (e^{ikx})$$

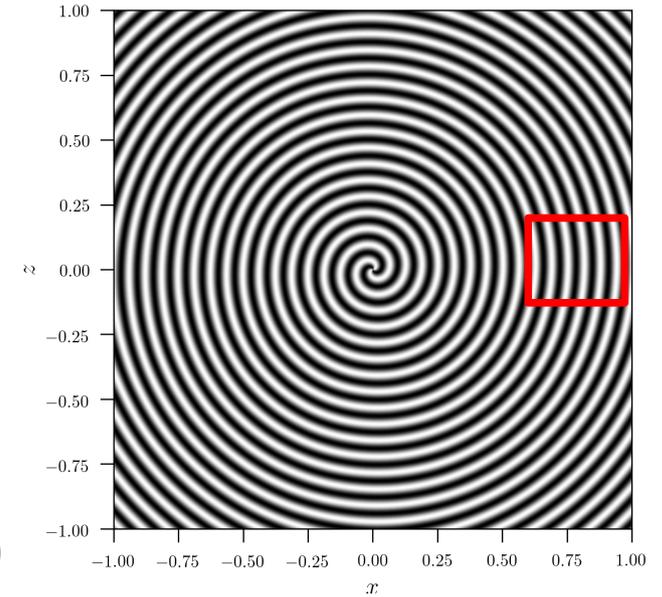
$$k=5.0 \quad \Sigma_0=1.0 \quad \Sigma_1=1.0$$



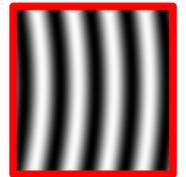
# Potential of an infinite slab with a tightly wound spiral pattern

$$\Sigma(R, \phi) = H(R) \operatorname{Re} \left( e^{i[m\phi + f(R)]} \right)$$

$m=2$



if  $\left| \frac{\partial f}{\partial R} \cdot R \right| \ll 1$       WKB approximation  
(Wentzel, Kramers, Brillouin)



$$\Phi(R, \phi) = -\frac{2\pi G \Sigma_0}{\left| \frac{\partial f}{\partial R} \right|} H(R) \operatorname{Re} \left( e^{if(R)} \right) e^{-\left| \frac{\partial f}{\partial R} \cdot z \right|}$$

# Potential of an infinite slab with a tightly wound spiral pattern

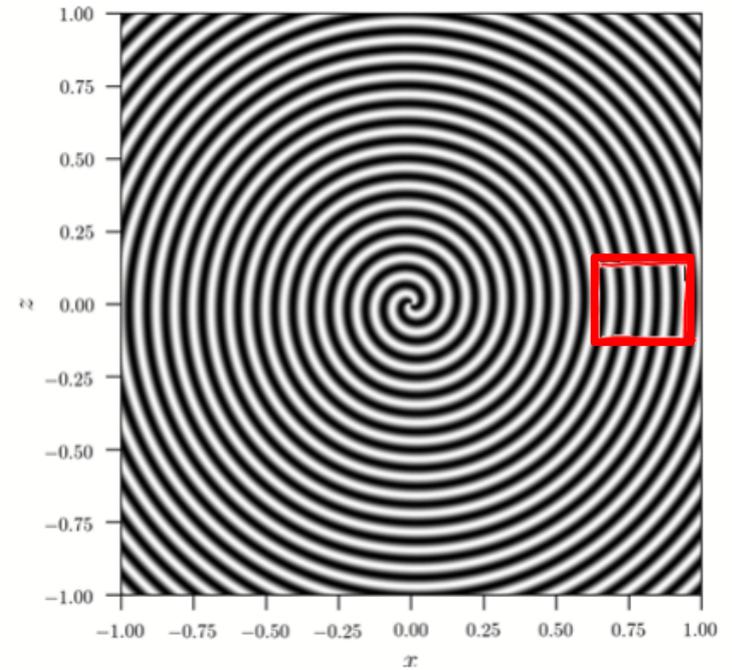
$m=2$

$$\Sigma(R, \phi) = \text{Re} \left( \underbrace{U(R)}_{\text{slow variation}} \underbrace{e^{i(m\theta + f(R))}}_{\text{rapid variation}} \right)$$

Note

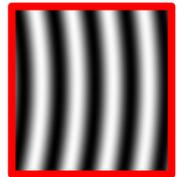
$$m\theta + f(R) = \text{cte}$$

describe a spiral  $f(R) = \text{shape function}$



Idea: WKB approximation

far from the center,  $\Sigma$  is nearly  $\sim e^{i(kx)}$



Indeed

Developping  $f(R)$  around  $R_0$  gives

$$f(R) \approx f(R_0) + \left. \frac{\partial f}{\partial R} \right|_{R_0} (R - R_0)$$

For  $\theta = 0$

$$\Sigma(R, \theta) = \underbrace{U(R_0)}_{\text{no radial dependency}} e^{i\psi(R_0)} e^{i \left. \frac{\partial \psi}{\partial R} \right|_{R_0} (R - R_0)}$$

no radial  
dependency

$$e^{ikx}$$

$$\left\{ \begin{array}{l} k = \left. \frac{\partial \psi}{\partial R} \right|_{R_0} \\ x = R - R_0 \end{array} \right.$$

We directly have the solution from the infinite slab

---

$$\phi(R, \theta) = - \frac{2\pi G}{\left| \frac{\partial \psi}{\partial R} \right|} U(R_0) e^{i\psi(R_0)} e^{i \left. \frac{\partial \psi}{\partial R} \right|_{R_0} (R - R_0)} e^{-\left| \frac{\partial \psi}{\partial R} \right| z}$$

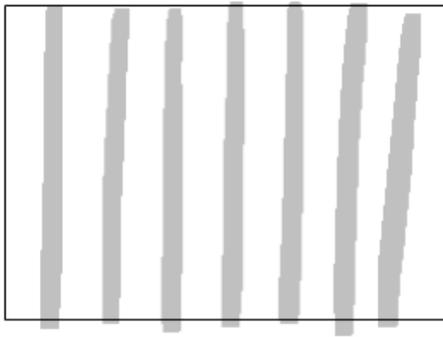
Choosing  $R_0 = R$

$$\phi(R, \theta) = - \frac{2\pi G}{\left| \frac{\partial \psi}{\partial R} \right|} U(R) e^{i\psi(R)} e^{-\left| \frac{\partial \psi}{\partial R} \right| z}$$

## Validity of the approximation

---

- we want a large number of "oscillations" over a small radius compared to  $R$



$\sim R$

$$\left| \frac{\partial \mathcal{L}}{\partial R} \right| \cdot R \gg 1$$

**The End**