Astrophysics IV, Dr. Yves Revaz — EPFL

4th year physics — Exercises week 3

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Astrophysics IV : Stellar and galactic dynamics

Solutions

Problem 1 :

For a shell of mass M and radius R centered on 0, consider the surface s of a sphere of radius r ($r > R$). The Gauss Law states that :

$$\int_S \vec{g}(\vec{x}) \cdot d\vec{S} = -4\pi GM. \tag{1}$$

Benefiting from the symmetry of the problem, we can write :

$$\vec{g}(\vec{x}) = g(r) \cdot \vec{e}_r \tag{2}$$

and

$$d\vec{S} = r^2 d\Omega \cdot \vec{e}_r. \tag{3}$$

So, we obtain :

$$\int_S \vec{g}(\vec{x}) \cdot d\vec{S} = \int_S g(r) \cdot \vec{e}_r \cdot r^2 d\Omega \cdot \vec{e}_r = r^2 g(r) \int_S d\Omega = 4\pi r^2 g(r) = -4\pi GM, \tag{4}$$

so,

$$g(r) = -\frac{GM}{r^2}. \tag{5}$$

The corresponding potential is thus :

$$\Phi(r) = -\frac{GM}{r}. \tag{6}$$

Problem 2 :

The norm of the specific force of the spherical model can be written as an integral over the norm of forces $\delta g_{r'}(r)$ generated by individual shells of radius r' :

$$g(r) = \int_0^\infty \delta g_{r'}(r) \tag{7}$$

Lets split the integral into two parts, one including the contribution of shells with a radius smaller than r' and one with radius larger :

$$g(r) = \int_0^r \delta g_{r'}(r) + \int_r^\infty \delta g_{r'}(r). \tag{8}$$

Using the Newton theorem, we know that the norm of the specific gravitational field of a shell of mass $\delta M_{r'}$ is :

$$\delta g_{r'}(r) = -\frac{G\delta M_{r'}}{r^2}, \tag{9}$$

and is null for any point inside the shell. For a shell of density $\rho(r')$, $\delta M_{r'}$ writes :

$$\delta M_{r'} = 4\pi\rho(r')r^{'2}dr'. \tag{10}$$

$\delta g_{r'}(r)$ is thus :

$$\delta g_{r'}(r) = -\frac{4\pi G\rho(r')r^{'2}}{r^2}dr'. \tag{11}$$

Inserting the latter in Eq. 8, and recognizing that the second integral is zero (the contribution of shells with a radius larger that r), we get :

$$g(r) = \int_0^r \delta g_{r'}(r) = -\frac{1}{r^2}4\pi G\int_0^r \rho(r')r^{'2}dr', \tag{12}$$

As :

$$4\pi\int_0^r \rho(r')r^{'2}dr' = M(r), \tag{13}$$

we obtain the final result :

$$g(r) = -\frac{GM(r)}{r^2}, \tag{14}$$

and

$$g(r)\cdot\vec{e}_r = -\frac{GM(r)}{r^2}\vec{e}_r. \tag{15}$$

Problem 3 :

Lets define the following Lagrangian, a function of the potential ϕ and its gradient $\vec{\nabla}\phi$:

$$\mathcal{L}(\phi,\vec{\nabla}\phi,\vec{x}) = \frac{1}{8\,\pi\,G}(\vec{\nabla}\phi)^2 + \rho\,\phi, \tag{16}$$

We associate to this Lagrangian an action :

$$\mathcal{S}[\phi] = \int \mathrm{d}^3\vec{x}\,\mathcal{L}\left(\phi,\vec{\nabla}\phi,\vec{x}\right). \tag{17}$$

Extremalizing this action amounts to solving the Euler-Lagrange equation :

$$\frac{\partial\mathcal{L}}{\partial\Phi} - \vec{\nabla}\cdot\frac{\partial\vec{\mathcal{L}}}{\partial\vec{\nabla}\phi} = 0, \tag{18}$$

Plugging the Lagrangian (Eq. 16) to this equation, we obtain :

$$\vec{\nabla}^2\phi = 4\,\pi\,G\,\rho. \tag{19}$$

which is nothing but the Poisson equation.
Interpretation : What is the physical meaning of the Lagrangian ?

From the potential theory, the total potential energy of a system is :

$$W = \frac{1}{2}\int \mathrm{d}^3\vec{x}\,\rho(\vec{x})\,\phi(\vec{x}). \tag{20}$$

or

$$W = -\frac{1}{8\,\pi\,G}\int \mathrm{d}^3\vec{x}\,(\vec{\nabla}\phi)^2. \tag{21}$$

The physical meaning of $\mathcal{L}(\phi, \vec{\nabla}\phi, \vec{x})$ is now obvious and is nothing else than the total potential energy written as $W = -W + 2\,W$. Thus, the variational principle answers the following question : *For a given density field, what is the relationship between the density and the potential that render the total potential energy extremum ?* The answer is : *The Poisson equation.*

Problem 4 :

Using the following relations for spherical systems, derived during the lectures :
the Poisson equation in Spherical coordinates :

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right) = 4\,\pi\,G\,\rho(r) \tag{22}$$

the mass inside a radius r due to a spherical distribution of matter $\rho(r')$:

$$M(r) = 4\pi\int_0^r \mathrm{d}r'\,r'^2\,\rho(r'), \tag{23}$$

the gravitational field due to a spherical distribution of matter $\rho(r')$

$$\vec{g}(r) = -\frac{G\,M(r)}{r^2}\cdot\vec{e}_r, \tag{24}$$

the potential due to a spherical distribution of matter $\rho(r')$

$$\Phi(r) = -\frac{G\,M(r)}{r} - 4\pi G\int_r^\infty \rho(r')r'\mathrm{d}r', \tag{25}$$

the gradient of the potential due to a spherical distribution of matter $\rho(r')$

$$\frac{\mathrm{d}\Phi}{\mathrm{d}r} = \frac{G\,M(r)}{r^2}, \tag{26}$$

we can express $\rho(r)$, $\Phi(r)$, $M(r)$ and $\frac{\mathrm{d}\Phi}{\mathrm{d}r}$ as a function of respectively $\rho(r)$, $\Phi(r)$, $M(r)$ and $\frac{\mathrm{d}\Phi}{\mathrm{d}r}$:

$\rho(r)$

- as a function of $\rho(r)$: -
- as a function of $\Phi(r)$: use the Poisson equation Eq. (22)
- as a function of $M(r)$: use Eq. (23)
- as a function of $\frac{\mathrm{d}\Phi}{\mathrm{d}r}$: compute the first derivative of $M(r)$ from Eq. (23)

$\Phi(r)$

— as a function of $\rho(r)$: use Eq. (25)
— as a function of $\Phi(r)$: -
— as a function of $M(r)$: integrate Eq. (26)
— as a function of $\frac{\mathrm{d}\Phi}{\mathrm{d}r}$: integrate $\Phi(r)$

$M(r)$

— as a function of $\rho(r)$: use Eq. (23)
— as a function of $\Phi(r)$: use Eq. (26)
— as a function of $M(r)$: -
— as a function of $\frac{\mathrm{d}\Phi}{\mathrm{d}r}$: use Eq. (26)

$\frac{\mathrm{d}\Phi}{\mathrm{d}r}$

— as a function of $\rho(r)$: use Eq. (26) and express $M(r)$ with Eq. (23)
— as a function of $\Phi(r)$: compute the first derivative of $\Phi(r)$
— as a function of $M(r)$: use Eq. (26)
— as a function of $\frac{\mathrm{d}\Phi}{\mathrm{d}r}$: -