Astrophysics IV : Stellar and galactic dynamics

Exercises

Problem 1 :

Demonstrate the second Newton theorem using the Gauss Law.

Problem 2 :

By summing the gravitational force generated by infinite shells, show that the specific gravitational force generated by a spherical model for which we know the cummulative mass $M(r)$ can be written as :

$$g(r) \cdot \vec{e}_r = -\frac{GM(r)}{r^2}\vec{e}_r. \tag{1}$$

Problem 3 :

Demonstrate that the Poisson equation can be derived from a variational principle and interpret the meaning of the extremalisation performed.

Problem 4 :

In practice, for spherical systems, it is often useful to derive, for example, the density $\rho(r)$ knowing the gravitational field $g(r) = -\frac{\mathrm{d}\Phi}{\mathrm{d}r}$, or the potential $\Phi(r)$ knowing the cumulative mass $M(r)$. Using the relations presented during the lectures, express successively $\rho(r)$, $\Phi(r)$, $M(r)$ and $\frac{\mathrm{d}\Phi}{\mathrm{d}r}$ as a function of respectively $\rho(r)$, $\Phi(r)$, $M(r)$ and $\frac{\mathrm{d}\Phi}{\mathrm{d}r}$ as given in the following table :

	$\rho(r)$	$\Phi(r)$	$M(r)$	$\frac{\mathrm{d}\Phi}{\mathrm{d}r}$
$\rho(r)$	$\rho(r)$	$\frac{1}{4\,\pi\,G}\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right)$	$\frac{1}{4\pi r^2}\frac{\mathrm{d}M(r)}{\mathrm{d}r}$	$\frac{1}{4\,\pi\,G}\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right)$
$\Phi(r)$	$-\frac{G\,M(r)}{r} - 4\pi G\int_r^\infty \mathrm{d}r'\,r'\,\rho(r')$	$\Phi(r)$	$-G\int_r^\infty \mathrm{d}r'\,\frac{M(r')}{r'^2}$	$-\int_r^\infty \mathrm{d}r'\,\frac{\mathrm{d}\Phi}{\mathrm{d}r}$
$M(r)$	$4\pi\int_0^r \mathrm{d}r'\,r'^2\,\rho(r')$	$\frac{r^2}{G}\frac{\mathrm{d}\Phi}{\mathrm{d}r}$	$M(r)$	$\frac{r^2}{G}\frac{\mathrm{d}\Phi}{\mathrm{d}r}$
$\frac{\mathrm{d}\Phi}{\mathrm{d}r}$	$\frac{4\pi G}{r^2}\int_0^r \mathrm{d}r'\,r'^2\,\rho(r')$	$\frac{\mathrm{d}\Phi}{\mathrm{d}r}$	$\frac{GM(r)}{r^2}$	$\frac{\mathrm{d}\Phi}{\mathrm{d}r}$