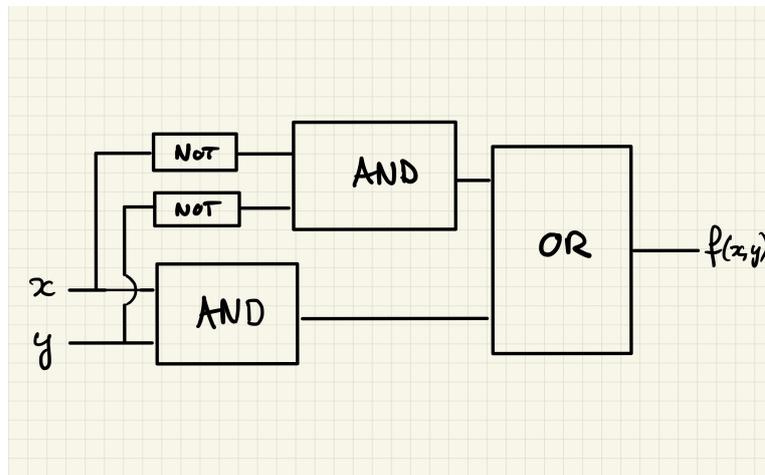**Exercise 1** *Boolean functions and classical circuits*
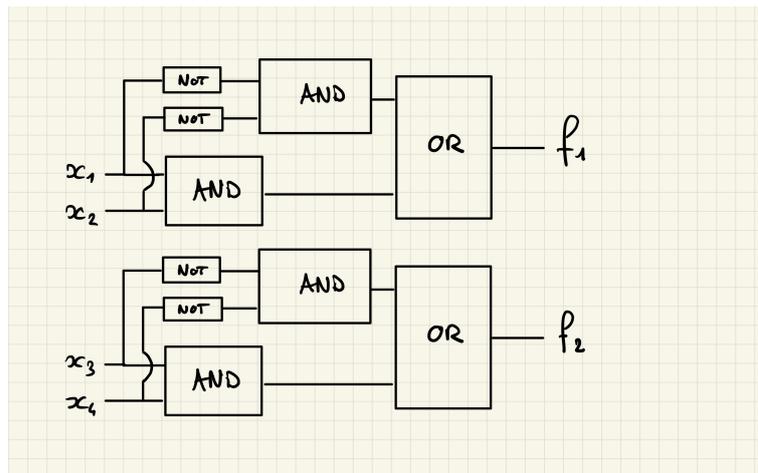
(a) Building a circuit for $f : \{0,1\}^2 \rightarrow \{0,1\}$ such that $f(x,y) = 1$ if and only if $x = y$ can be obtained by noting that

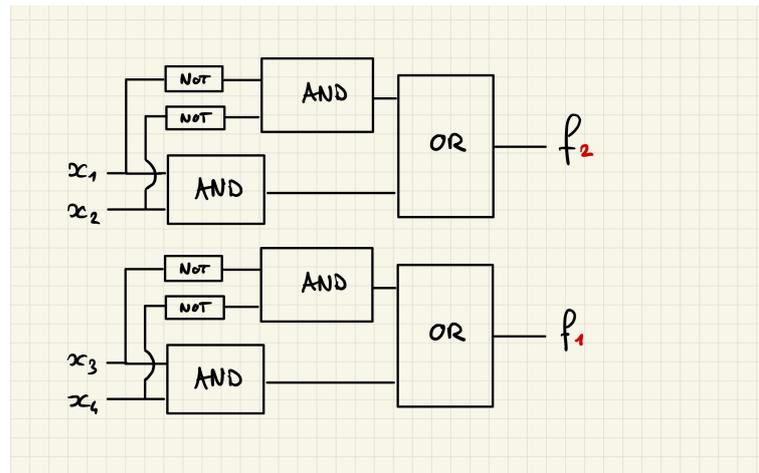$$f(x,y) = 1 \quad \text{if and only if} \quad (x = 1 \text{ and } y = 1) \text{ or } (x = 0 \text{ and } y = 0)$$

so $f(x,y) = (x \text{ AND } y) \text{ OR } (\text{NOT } x \text{ AND NOT } y)$ and the circuit is :

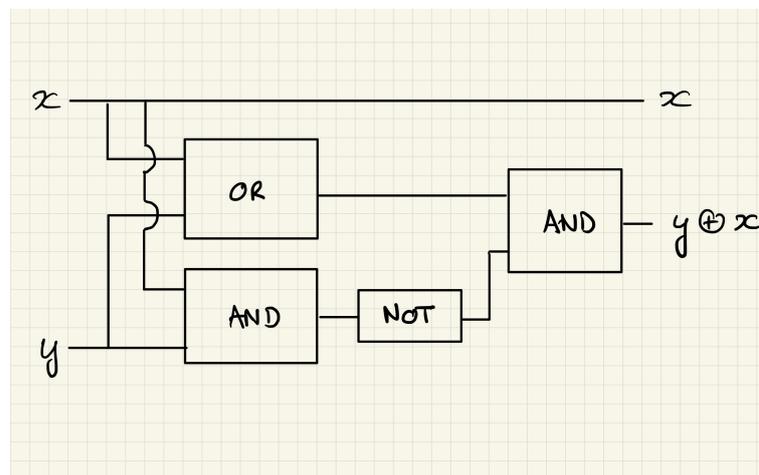The final circuit is given by :

(b) In this case, it suffices to inverts the outputs of f_1 and f_2 in order to obtain what we want :
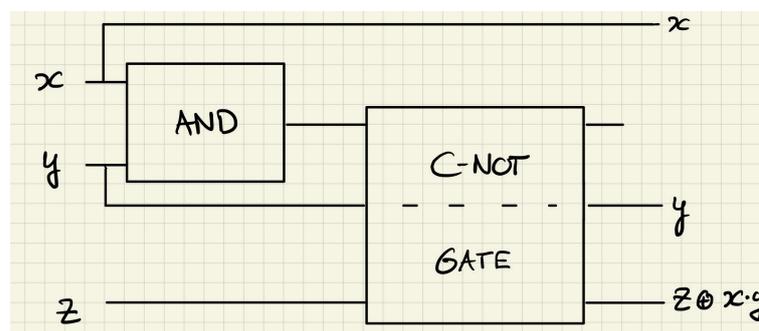

Exercise 2 *NOT, C-NOT, CC-NOT gates*

The C-NOT gate is obtained by noting that $x \oplus y = (x \text{ OR } y) \text{ AND NOT } (x \text{ AND } y)$:

And here is the CC-NOT or Toffoli gate, obtained via the previous gate :

Exercise 3 *Production of Bell states*

(a) 1. Let $|\psi\rangle = \alpha_{00}\,|00\rangle + \alpha_{01}\,|01\rangle + \alpha_{10}\,|10\rangle + \alpha_{11}\,|11\rangle$. Then

$$\det \begin{pmatrix} \alpha_{00} & \alpha_{01} \\ \alpha_{10} & \alpha_{11} \end{pmatrix} = 0 \tag{1}$$

if and only if the rows are colinear, which means we can re-parametrize :

$$\begin{pmatrix} \alpha_{00} & \alpha_{01} \\ \alpha_{10} & \alpha_{11} \end{pmatrix} = \begin{pmatrix} x & y \\ ax & ay \end{pmatrix} \tag{2}$$

with some $x, y, a \in \mathbb{C}$. This happens if and only if

$$|\psi\rangle = \frac{1}{\sqrt{1 + |a|^2}}(|0\rangle + a\,|1\rangle) \otimes \frac{1}{\sqrt{|x|^2 + |y|^2}}(x\,|0\rangle + y\,|1\rangle)\,. \tag{3}$$

2. State (i) is a Bell entangled state (see below).
State (ii) is a product state $= |0\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$.
State (iii) is an entangled state.
State (iv) is a product state $= \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$.
State (v) is also a product state $= \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$.

(b) A direct computation gives

$$\begin{aligned}
(CNOT)\,(H \otimes I)\,|x\rangle \otimes |y\rangle &= (CNOT)\,(H\,|x\rangle \otimes |y\rangle) \\
&= (CNOT)\left(\frac{1}{\sqrt{2}}(|0\rangle + (-1)^x\,|1\rangle) \otimes |y\rangle \right) \\
&= \frac{1}{\sqrt{2}}\,CNOT\,|0, y\rangle + \frac{(-1)^x}{\sqrt{2}}\,CNOT\,|1, y\rangle \\
&= \frac{1}{\sqrt{2}}\,|0, y\rangle + \frac{(-1)^x}{\sqrt{2}}\,|1, \bar{y}\rangle
\end{aligned}$$

More explicitly, we enumerate all the cases :

$$(CNOT)\,(H \otimes I)\,|00\rangle = (CNOT)\frac{1}{\sqrt{2}}\big(|00\rangle + |10\rangle\big) = \frac{1}{\sqrt{2}}\big(|00\rangle + |11\rangle\big) = |B_{00}\rangle$$

$$(CNOT)\,(H \otimes I)\,|01\rangle = (CNOT)\frac{1}{\sqrt{2}}\big(|01\rangle + |11\rangle\big) = \frac{1}{\sqrt{2}}\big(|01\rangle + |10\rangle\big) = |B_{01}\rangle$$

$$(CNOT)\,(H \otimes I)\,|10\rangle = (CNOT)\frac{1}{\sqrt{2}}\big(|00\rangle - |10\rangle\big) = \frac{1}{\sqrt{2}}\big(|00\rangle - |11\rangle\big) = |B_{10}\rangle$$

$$(CNOT)\,(H \otimes I)\,|11\rangle = (CNOT)\frac{1}{\sqrt{2}}\big(|00\rangle - |11\rangle\big) = \frac{1}{\sqrt{2}}\big(|01\rangle - |10\rangle\big) = |B_{11}\rangle$$

(c) The operation $(CNOT)(H \otimes I)$ examined in the previous section is a unitary operation (as it is a composition of unitaries). Therefore, the image of an orthonormal basis under

3

it is an orthonormal basis itself. To be explicit, we define the set $\mathcal{B}$ as the image of the standard basis under $(CNOT)(H \otimes I)$:

$$\mathcal{B} := \{(CNOT)(H \otimes I) \ket{xy}\}_{x,y \in \{0,1\}} \ . \tag{4}$$

We recognize now that $\mathcal{B}$ is the set of the four Bell-states $\ket{B_{xy}}$ and therefore the set forms an orthonormal basis.

(d) The reverse circuit is given by $U = (H \otimes I)(CNOT)$, as

$$U\,(CNOT)\,(H \otimes I) = (H \otimes I)\,(CNOT)^2\,(H \otimes I) = (H \otimes I)\,(I \otimes I)\,(H \otimes I) = I \otimes I$$

(e) See the Jupyter notebook "Worksheet 2 Solutions".

Exercise 4 *Matrix representation of a few gates / circuits*

(a) The Hilbert space here is $\mathbb{C}^8$ and its matrix representation in the computational basis $\left\{ |000\rangle, |001\rangle, |010\rangle, |011\rangle, |010\rangle, |101\rangle, |110\rangle, |111\rangle \right\}$ is given by

$$CCNOT = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

(b) The matrix representation of this circuit in the computational basis $\left\{ |00\rangle, |01\rangle, |10\rangle, |11\rangle \right\}$ of $\mathbb{C}^4$ is given by

$$NOT_y = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

(c) The matrix representation of this circuit (in the same basis) is given by

$$NOT_x = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

(d) The matrix representation of this circuit (in the same basis) is given by

$$NOT_x \cdot CNOT \cdot NOT_x = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

All the above matrices are permutation matrices, and are also equal to their own inverse.

Exercise 5 *Hadamard Transform of NOT-Gate*

(a) A matrix U is unitary if $UU^\dagger = U^\dagger U = I$. Note that for Hadamard and NOT(X) gates, we have $HH^\dagger = H^\dagger H = I$, $XX^\dagger = X^\dagger X = I$. For HXH, we have

$$HXH(HXH)^\dagger = HXHH^\dagger X^\dagger H^\dagger = HXX^\dagger H^\dagger = HH^\dagger = I$$

With similar computations, $(HXH)^\dagger HXH = I$. Thus HXH is unitary.

(b) We obtain successively for $|\varphi\rangle = \alpha_0 |0\rangle + \alpha_1 |1\rangle$:

$$H |\varphi\rangle = \frac{\alpha_0 + \alpha_1}{\sqrt{2}} |0\rangle + \frac{\alpha_0 - \alpha_1}{\sqrt{2}} |1\rangle$$

$$XH |\varphi\rangle = \frac{\alpha_0 + \alpha_1}{\sqrt{2}} |1\rangle + \frac{\alpha_0 - \alpha_1}{\sqrt{2}} |0\rangle$$

$$HXH |\varphi\rangle = \frac{\alpha_0 + \alpha_1}{\sqrt{2}} \frac{|0\rangle - |1\rangle}{\sqrt{2}} + \frac{\alpha_0 - \alpha_1}{\sqrt{2}} \frac{|0\rangle + |1\rangle}{\sqrt{2}} = \alpha_0 |0\rangle - \alpha_1 |1\rangle$$

(c) The above gives

$$HXH|0\rangle = |0\rangle \qquad HXH|1\rangle = -|1\rangle \qquad HXH|+\rangle = |-\rangle \qquad HXH|-\rangle = |+\rangle$$