# Potential Theory I

# Outlines

Potential Theory : general results

- Gravitational field force, gravitational potential
- Gauss Law
- Poisson Equation
- Total potential energy

Spherical systems:

- Newton’s Theorems
- Circular speed, circular velocity, circular frequency, escape speed, potential energy

## Potential theory : general results

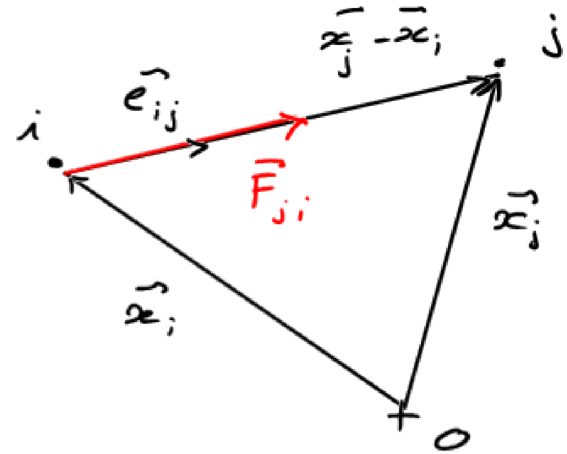
Goal : compute the gravitational potential / forces due to a large number of stars (point masses)

$N \sim 10^{11}$ for a Milky Way like galaxy

As the relaxation time of such system is very large ( $\gg$ the age of the Universe) we can describe the system with a smooth analytical potential / density.

## Newton Law

$$\vec{F}_{ji} = \frac{G m_i m_j}{|\vec{x}_j - \vec{x}_i|^2} \vec{e}_{ij} = \frac{G m_i m_j}{|\vec{x}_{ij}|^3} \vec{x}_{ij}$$

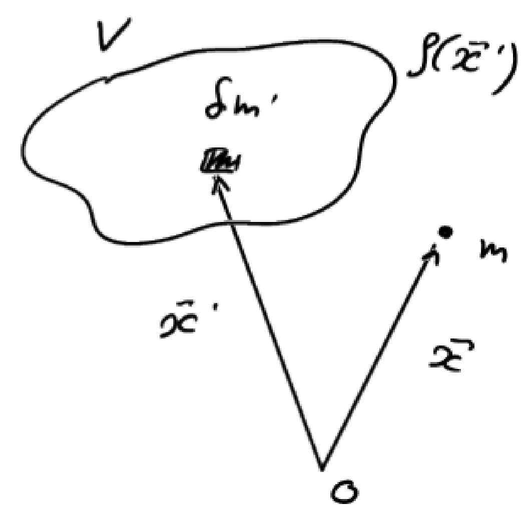



$$\vec{x}_{ij} = \vec{x}_j - \vec{x}_j$$

Force on a particle of mass $m$ in $\vec{x}$ due to a distribution of mass $\rho(\vec{x})$

$$\delta \vec{F}(\vec{x}) = \frac{G m \, \delta m'}{|\vec{x}' - \vec{x}|^3} (\vec{x}' - \vec{x})$$

$$= \frac{G m \, \rho(\vec{x}') \, d^3\vec{x}'}{|\vec{x}' - \vec{x}|^3} (\vec{x}' - \vec{x})$$



So, the total force writes :

$$\boxed{\vec{F}(\vec{x}) = \int_V \frac{G\, m\, \rho(\vec{x}')}{|\vec{x}' - \vec{x}|^3} (\vec{x}' - \vec{x})\, d^3\vec{x}'}$$

$$= m \underbrace{G \int_V \frac{\rho(\vec{x}')}{|\vec{x}' - \vec{x}|^3} (\vec{x}' - \vec{x})\, d^3\vec{x}'}_{}$$

$\vec{g}(\vec{x})$ : gravitational field

$$[\vec{g}] = \frac{cm}{s^2} = \frac{erg}{g} \frac{1}{cm}$$

## Gravitational Potential

It is easy to see that the function

$$\delta V(\bar{x}) = - \frac{G\, m\, \delta m}{|\bar{x}' - \bar{x}|} \qquad \text{is such that}$$

$$\vec{\nabla}\, \delta V(\bar{x}) = - \frac{G\, m\, \delta m}{|\bar{x}' - \bar{x}|^2} \, \frac{(\bar{x}' - \bar{x})}{|\bar{x}' - \bar{x}|} = - \delta \vec{F}(\bar{x})$$

so, by defining

$$\boxed{V(\bar{x}) = - G \int_V \frac{m\, \rho(\bar{x}')}{|\bar{x}' - \bar{x}|} \, d^3\bar{x}'}$$

we ensure that $\quad \vec{\nabla} V(\bar{x}) = - \vec{F}(\bar{x})$

We define the specific potential $\phi(\bar{x}) = \dfrac{V(\bar{x})}{m}$

which writes

$$\boxed{\phi(\bar{x}) = -G \int_V \frac{\rho(\bar{x}')}{|\bar{x}' - \bar{x}|} \, d^3\bar{x}'}$$

$[\phi] = \dfrac{\text{erg}}{\text{g}}$

$\equiv$ specific energy

The gravitational field writes :

$$\boxed{\bar{g}(\bar{x}) = -\bar{\nabla}\phi(\bar{x})}$$

Notes

- The gravity is a conservative force

- $\left.\begin{array}{ll} \phi(\vec{x}) & : \text{scalar field} \\ \vec{g}(\vec{x}) & : \text{vector field} \end{array}\right\}$ contain the same information

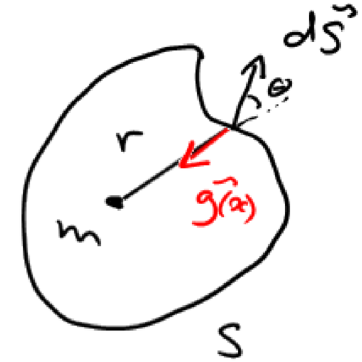
- we will always use "specific" quantities

$$V(\vec{x}) \rightarrow \phi(\vec{x})$$

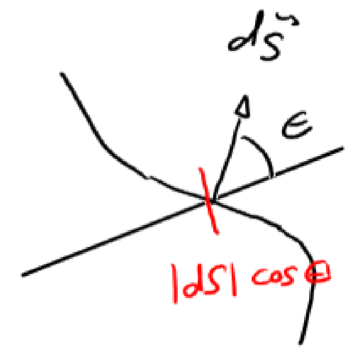
$$K = \frac{1}{2} m \vec{v}^{\,2} \rightarrow \frac{1}{2} \vec{v}^{\,2}$$

$$\frac{1}{2} v^2 + \phi(\vec{x}) = \text{specific energy (conserved quantity)}$$

# The Gauss's Law



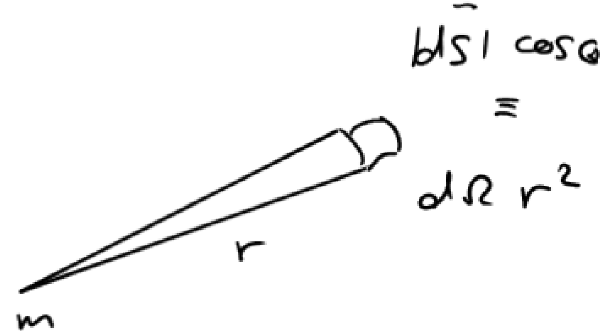

Consider:
- a single point mass $m$
- a surface $S$ around this point
- a point $\vec{x}$ on the surface at a distance $r$
- $\vec{g}(\vec{x})$ the gravitational field
- $d\vec{S}$, the normal at the surface
- $\theta$ the angle between $\vec{g}(\vec{x})$ and $d\vec{S}$




$$\vec{g}(\vec{x}) \cdot d\vec{S} = -|\vec{g}(\vec{x})| \cdot |d\vec{S}| \cos\theta$$

But
$$|d\vec{S}| \cos\theta = r^2 \, d\Omega$$

$$|\vec{g}(x)| = \frac{Gm}{r^2}$$



$$\boxed{\vec{g}(\vec{x}) \cdot d\vec{S} = -Gm \, d\Omega}$$

integrating over any surface

$$\int_S \vec{g}(\vec{x}) \cdot d\vec{S} = \begin{cases} -4\pi G m & \text{if } m \text{ inside } S \\ 0 & \text{instead} \end{cases}$$

For multiple masses $m_i$

$$\int_S \vec{g}(\vec{x}) \cdot d\vec{S} = -4\pi G \sum_{i \text{ in } S} m_i$$

For a continuous mass distribution $\rho(\vec{x})$

$$\boxed{\int_S \vec{g}(\vec{x}) \cdot d\vec{S} = -4\pi G \int_V \rho(\vec{x}) \, d\vec{x} = -4\pi G M}$$

Gauss's Law

## Divergeance of the specific force (A) $\vec{\nabla}_x \cdot \vec{g}(\vec{x})$

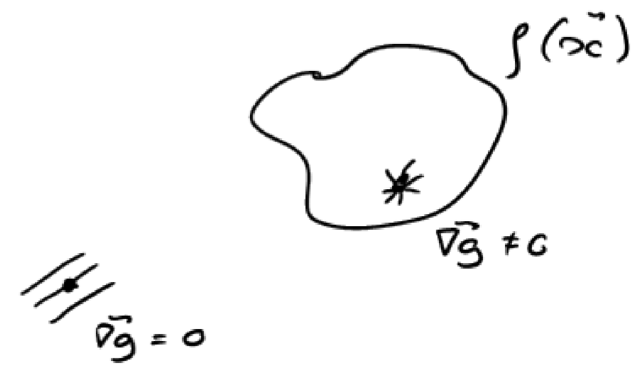
div. theorem

$$\int_V \vec{\nabla} \cdot \vec{g}(\vec{x}) \, d^3\vec{x} = \int_S \vec{g}(\vec{x}) \, d\vec{S}$$

Gauss's Law

$$= -4\pi G \int_V \rho(\vec{x}) \, d\vec{x}$$

$$\boxed{\vec{\nabla}_x \cdot \vec{g}(\vec{x}) = -4\pi G \rho(\vec{x})}$$



# The Poisson Equation

$$\vec{\nabla}_x \cdot \vec{g}(\vec{x}) = -4\pi G \rho(\vec{x})$$

with : $\vec{\nabla}_x \phi(\vec{x}) = -\vec{g}(\vec{x})$ $\qquad \vec{\nabla}_x \cdot (\vec{\nabla}_x) = \vec{\nabla}_x^2$

$$\boxed{\vec{\nabla}_x^2 \phi(\vec{x}) = 4\pi G \rho(\vec{x})}$$

Poisson Equation

<u>Note</u> : To ensure a unique solution, boundary conditions are necessary (2nd order diff. eqn.)

<u>ex</u> : $\phi(\infty) = 0$

$$\vec{\nabla}\phi(\infty) = \vec{g}(x) = 0$$

## Divergence of the specific force (B) $\vec{\nabla}_x \cdot \vec{g}(\vec{x})$

$$\vec{g}(\vec{x}) = G \int_V \frac{\rho(\vec{x}')}{|\vec{x}' - \vec{x}|^3} (\vec{x}' - \vec{x})\, d^3\vec{x}'$$

$$\vec{\nabla}_x \cdot \vec{g}(\vec{x}) = G \int_V \vec{\nabla}_x \cdot \left( \frac{\rho(\vec{x}')}{|\vec{x}' - \vec{x}|^3} (\vec{x}' - \vec{x}) \right) d^3\vec{x}'$$

- $$\vec{\nabla}_x \cdot \left( \frac{\vec{x}' - \vec{x}}{|\vec{x}' - \vec{x}|^3} \right) = \frac{d}{dx_1}\left( \frac{\vec{x}' - \vec{x}}{|\vec{x}' - \vec{x}|^3} \right) + \frac{d}{dx_2}\left( \frac{\vec{x}' - \vec{x}}{|\vec{x}' - \vec{x}|^3} \right) + \frac{d}{dx_3}\left( \frac{\vec{x}' - \vec{x}}{|\vec{x}' - \vec{x}|^3} \right)$$

$$= -\frac{3}{|\vec{x}' - \vec{x}|^3} + \frac{3(\vec{x}' - \vec{x}) \cdot (\vec{x}' - \vec{x})}{|\vec{x}' - \vec{x}|^5}$$

$$= 0 \quad \text{if} \quad \vec{x}' \neq \vec{x}$$

COMPLEMENT

COMPLEMENT

$$\vec{\nabla}_x \cdot \vec{g}(\vec{x}) = G \int_V \vec{\nabla}_x \cdot \left( \frac{\rho(\vec{x}')}{|\vec{x}' - \vec{x}|^3} (\vec{x}' - \vec{x}) \right) d^3\vec{x}'$$


$\rho(\vec{x}) = \text{cte}$

$$= G\rho(\vec{x}) \int_{|\vec{x}'-\vec{x}| \le h} \vec{\nabla}_x \cdot \left( \frac{\vec{x}'-\vec{x}}{|\vec{x}'-\vec{x}|^3} \right) d^3\vec{x}'$$

variable exchange

$\vec{\nabla}_x f(\vec{x}-\vec{x}') = -\vec{\nabla}_{x'} f(x-x')$

$$= -G\rho(\vec{x}) \int_{|\vec{x}'-\vec{x}| \le h} \vec{\nabla}_{x'} \cdot \left( \frac{\vec{x}'-\vec{x}}{|\vec{x}'-\vec{x}|^3} \right) d^3\vec{x}'$$

divergence theorem

$$= -G\rho(\vec{x}) \underbrace{\int_{|\vec{x}'-\vec{x}|=h} \frac{\vec{x}'-\vec{x}}{|\vec{x}'-\vec{x}|^3} d^2\vec{S}'}_{4\pi h^2 \cdot \left.\frac{1}{r^2}\right|_{h=r} = 4\pi}$$

$r = |\vec{x}'-\vec{x}| = h$

$$\frac{\vec{x}'-\vec{x}}{|\vec{x}'-\vec{x}|^3} = \frac{1}{r^2}$$

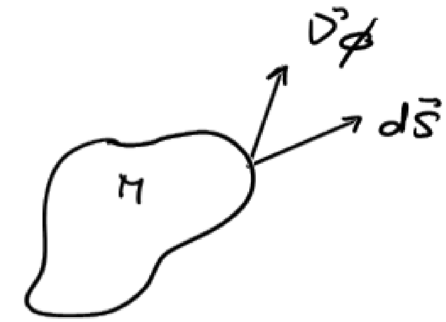
$$\underline{\vec{\nabla}_x \cdot \vec{g}(\vec{x}) = -4\pi G \rho(\vec{x})}$$

COMPLEMENT

## Gauss theorem

(B) integrate the Poisson equation over a volume $V$ that contains a mass $M$

$$\int_V \vec{\nabla}^2 \phi(\vec{x})\, d^3\vec{x} = \underbrace{\int_V 4\pi G \rho(\vec{x})\, d^3\vec{x}}$$




div. theorem

$$\boxed{\int_S d^2\vec{S} \cdot \vec{\nabla}\phi = 4\pi G M}$$

Gauss theorem

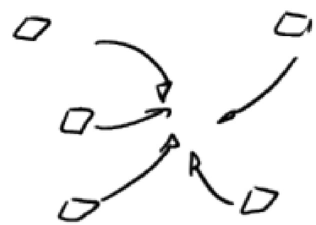
Equivalently :

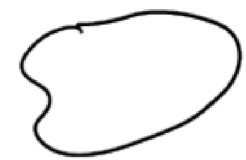
$$\boxed{\int_S d^2\vec{S} \cdot \vec{g}(\vec{x}) = -4\pi G M}$$

Gauss's Law

## Total potential energy (1.0)

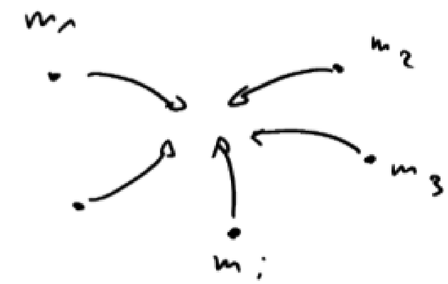
Total work needed to assemble a density distribution $\rho(\vec{x})$





$\rho(\vec{x})$ , $\phi(\vec{x})$

Assume a set of discrete points




- The work to bring the $1^{st}$ point from $\infty$ to $\vec{x}_1$ is $0$
- The work to bring the $2^{nd}$ point from $\infty$ to $\vec{x}_2$ is $-\dfrac{G m_1 m_2}{r_{12}}$
- The work to bring the $3^{dn}$ point from $\infty$ to $\vec{x}_3$ is $-\dfrac{G m_1 m_3}{r_{13}} - \dfrac{G m_2 m_3}{r_{23}}$

The total work is thus

$$W = -\frac{G m_1 m_2}{r_{12}} - \frac{G m_1 m_3}{r_{13}} - \frac{G m_2 m_3}{r_{23}} - \ldots - \sum_{j=1}^{N-1} \frac{G m_{jN}}{r_{jN}}$$

$$= -\sum_{i=1}^{N} \sum_{j=1}^{i-1} \frac{G m_i m_j}{r_{ij}} = -\frac{1}{2} \sum_{i=1}^{N} \sum_{\substack{j=1 \\ i \neq j}}^{N} \frac{G m_i m_j}{r_{ij}}$$

With $\phi_i = -\sum_{\substack{j=1 \\ j \neq i}} \frac{G m_j}{r_{ij}}$ (potential on $i$)

$$W = \frac{1}{2} \sum_{i=1}^{N} m_i \phi_i \equiv \frac{1}{2} \sum_{i=1}^{N} V_i$$

For a continuous mass distribution $\rho(\vec{x})$

$$\boxed{W = \frac{1}{2} \int \rho(\vec{x}) \phi(\vec{x}) \, d^3\vec{x}}$$

## Total potential energy

1.1

From $$W = \frac{1}{2} \int \rho(\vec{x}) \phi(\vec{x}) \, d^3\vec{x}$$

- replace $\rho(\vec{x})$ with the Poisson equation $\rho(\vec{x}) = \frac{1}{4\pi G} \nabla^2 \phi$

$$W = \frac{1}{8\pi G} \int \nabla^2\phi \cdot \phi(\vec{x}) \, d^3\vec{x} = \frac{1}{8\pi G} \int \vec{\nabla} \cdot (\vec{\nabla}\phi) \cdot \phi(\vec{x}) \, d^3\vec{x}$$

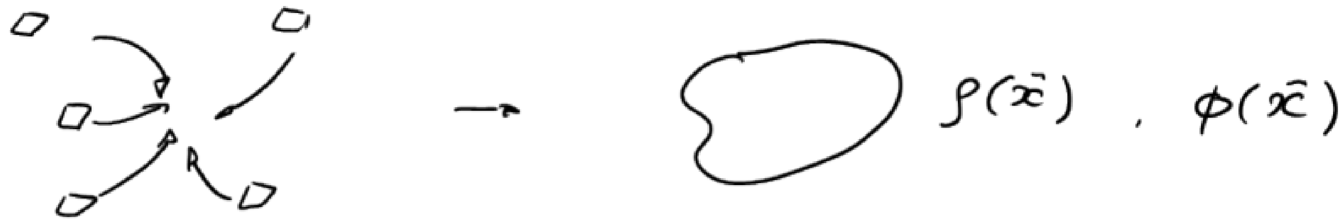
- divergence theorem $\int d^3x \; g \cdot \vec{\nabla} \cdot \vec{F} = \int_S g \cdot \vec{F} \, d\vec{S} - \int d^3x \; \vec{F} \cdot \vec{\nabla} g$

$$W = \frac{1}{8\pi G} \left[ \underbrace{\int \phi \vec{\nabla}\phi \, d\vec{S}}_{=0 \text{ as } \phi(\infty) = \vec{\nabla}\phi(\infty) = 0} - \int d^3\vec{x} \; \vec{\nabla}\phi \cdot \vec{\nabla}\phi \right]$$

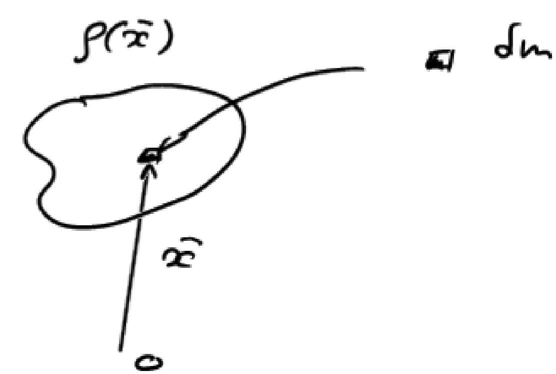
$$\boxed{W = -\frac{1}{8\pi G} \int d^3\vec{x} \, |\vec{\nabla}\phi|^2}$$

## Total potential energy (2.0)

Total work needed to assemble a density distribution $\rho(\vec{x})$




(1) Work done to assemble a piece of mass $\delta m = \delta\rho \, d\vec{x}^3$ from $\infty$ to $\vec{x}$ assuming an existing mass distribution $\rho(\vec{x})$, $\phi(\vec{x})$



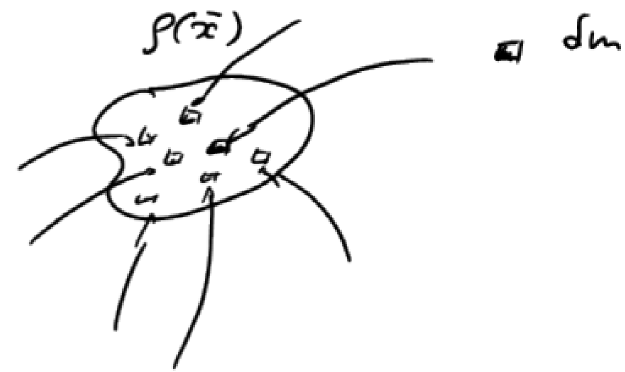

$$\delta W_{\vec{x}} = V(\vec{x}) - \underbrace{V(\infty)}_{=0}$$

$$= \delta m \, \phi(\vec{x}) = \delta\rho(\vec{x}) \, d^3\vec{x} \; \phi(\vec{x})$$

COMPLEMENT

To increase everywhere the mass distribution by $\delta\rho$

$$\rho(\vec{x}) \to \rho(\vec{x}) + \delta\rho(\vec{x})$$




$$\delta W = \int \delta\rho(\vec{x})\, d^3\vec{x}\ \phi(\vec{x})$$

Poisson: $\delta\rho(\vec{x}) = \frac{1}{4\pi G} \vec{\nabla}^2 \delta\phi(x)$

$$\delta W = \frac{1}{4\pi G} \int \vec{\nabla}^2 \delta\phi(x)\ \phi(\vec{x})\ d^3\vec{x}$$

divergence theorem

$$\int_V d^3x\, g\, \vec{\nabla}\cdot\vec{F} = \int_S g\, \vec{F}\cdot d^2\vec{s} - \int_V d^3x\, \vec{F}\cdot\vec{\nabla} g$$

$$= \underbrace{\frac{1}{4\pi G} \int_{S \text{ at } \infty} \phi(\vec{x})\, \vec{\nabla}\delta\phi(\vec{x})}_{=0} - \frac{1}{4\pi G} \int \vec{\nabla}\phi(\vec{x}) \cdot \vec{\nabla}(\delta\phi(\vec{x}))\ d^3\vec{x}$$

as $\phi(\infty) = 0$

$\vec{\nabla}\delta\phi(\infty) = \delta g(\infty) = 0$

$$\delta W = -\frac{1}{4\pi G}\int \vec{\nabla}\phi(\vec{x})\cdot\vec{\nabla}(\delta\phi(\vec{x}))\, d^3\vec{x}$$

with

$$\frac{1}{2}\,\delta|\vec{\nabla}\phi(\vec{x})|^2 = \delta\vec{\nabla}\phi(\vec{x})\cdot\vec{\nabla}\phi(\vec{x}) = \vec{\nabla}(\delta\phi(\vec{x}))\cdot\vec{\nabla}\phi(\vec{x})$$

$$\delta W = -\frac{1}{8\pi G}\int \delta|\vec{\nabla}\phi|^2\, d^3x \equiv -\frac{1}{8\pi G}\,\delta\int |\vec{\nabla}\phi|^2\, d^3x$$

## 2 Contribution of all $\delta W$ to $W$

$$\boxed{W = -\frac{1}{8\pi G}\int |\vec{\nabla}\phi|^2\, d^3x}$$

COMPLEMENT

## Total potential energy (2.2)

From $W = -\frac{1}{8\pi G}\int |\vec{\nabla}\phi|^2 d^3x = -\frac{1}{8\pi G}\int \vec{\nabla}\phi \cdot \vec{\nabla}\phi \, d^3\vec{x}$

- divergence theorem $\int d^3x \, \vec{F}\cdot\vec{\nabla} g = \int_S g \cdot \vec{F} \, d\vec{S} - \int d^3x \, g \, \vec{\nabla}\vec{F}$

$$W = -\frac{1}{8\pi G}\left[\underbrace{\int_S \phi \vec{\nabla}\phi \, d\vec{S}}_{=0 \text{ as } \phi(\infty) = \vec{\nabla}\phi(\infty) = 0} - \int d^3\vec{x} \, \phi \, \underbrace{\vec{\nabla}(\vec{\nabla}\phi)}_{4\pi G \rho \text{ (Poisson)}}\right]$$

$$= \frac{1}{8\pi G} 4\pi G \int d^3\vec{x} \, \phi(\vec{x}) \, \rho(\vec{x})$$

$$\boxed{W = \frac{1}{2}\int \rho(\vec{x}) \, \phi(\vec{x}) \, d^3\vec{x}}$$

COMPLEMENT

## Total potential energy : Summary

$$\boxed{W = \frac{1}{2} \int \rho(\vec{x})\, \phi(\vec{x})\, d^3\vec{x}}$$

$$\boxed{W = -\frac{1}{8\pi G} \int |\vec{\nabla}\phi|^2\, d^3x}$$

Other useful expression

$$\boxed{W = -\int \rho(\vec{x})\, \vec{x} \cdot \vec{\nabla}\phi(\vec{x})\, d^3\vec{x}}$$

# Relation between the potential energy and the Poisson equation

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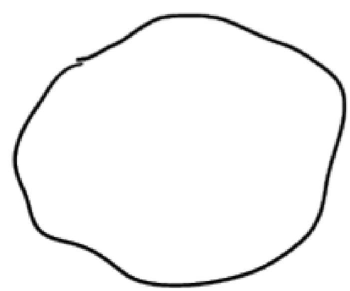
What is the relation that must hold between the density $\rho(\bar{x})$ and potential $\phi(\bar{x})$ in order to minimize the potential energy of a system?

$\rho(\bar{x})$

$\phi(\bar{x})$



$W$ : potential energy

<u>Answer</u> : the Poisson equation $\nabla^2 \phi = 4 \pi G \rho$

# Spherical Systems

$$\rho(\vec{x}) = \rho(r)$$

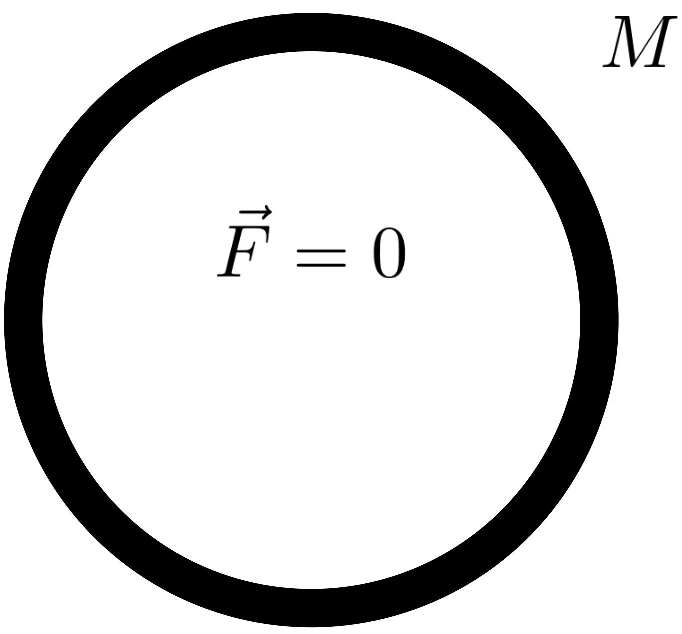
$$r = \sqrt{x^2 + y^2 + z^2}$$

# Newton’s Theorems

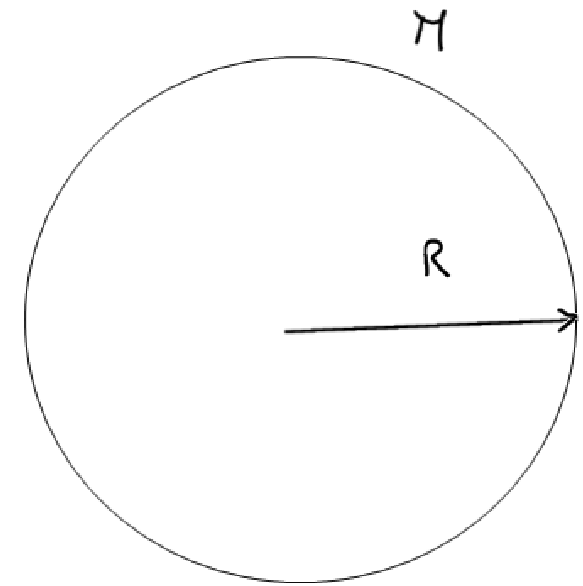
Newton (1642-1727)

First theorem:

A body that is inside a spherical shell of matter experiences no net gravitational force from the shell.



# Spherical infinitly thin shell




Radius : $R$

Mass : $M$

Density : $\rho(r) = \frac{M}{4\pi r^2}\,\delta(R-r)$

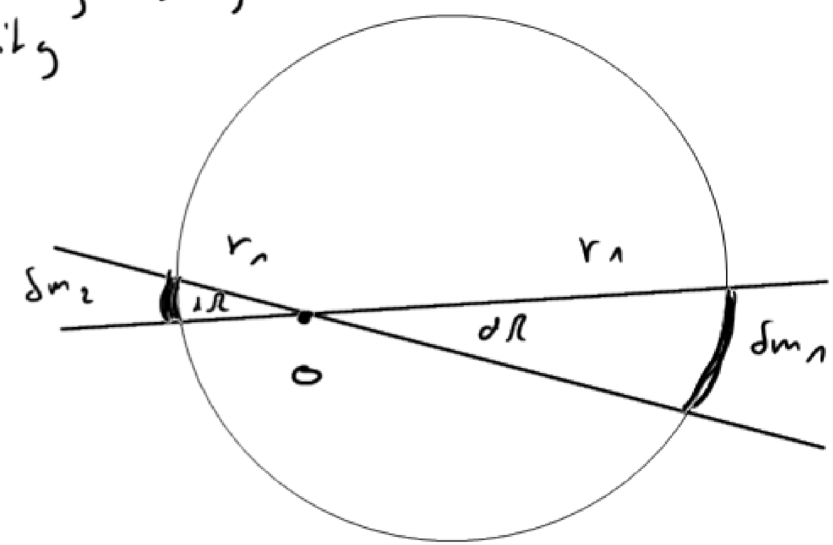
indeed : $M := 4\pi \int_0^\infty dr\, r^2\, \rho(r)$

$$= 4\pi \int_0^\infty dr\, r^2\, \frac{M}{4\pi r^2}\,\delta(R-r) = M$$

# First Newton theorem

A body that is inside a spherical shell of matter experiences no net gravitational force from that shell

a shell of constant density $\rho(\vec{x}) = \rho$




$$\begin{cases} \delta m_1 = \rho(r_1) \cdot r_1^2 \, d\Omega \, dr \\ \delta m_2 = \rho(r_2) \cdot r_2^2 \, d\Omega \, dr \end{cases}$$

thus : $\dfrac{\delta m_1}{\delta m_2} = \dfrac{r_1^2}{r_2^2}$

and $\dfrac{\delta m_1}{r_1^2} = \dfrac{\delta m_2}{r_2^2}$

<u>consequently</u> : $\delta \vec{F_1} = - \delta \vec{F_2}$

by integrating over the entire shell ($d\Omega$)

all forces cancel out !

#

**Corollary** The gravitational potential $\phi(\bar{x})$ is constant inside the sphere.

As $\vec{\nabla}_x \phi(\bar{x}) = \bar{g} = 0$ $\quad \phi(\bar{x}) = \text{ct}$ #

**What is the value of $\phi(\bar{x})$ ?**

$$\phi(\bar{x}) = -\int_V \frac{G\,\rho(\bar{x}')}{|\bar{x}' - \bar{x}|}\, d^3\bar{x}'$$

**Spherical coordinates** $\quad d^3\bar{x} = r^2\, dr\, d\Omega = 4\pi\, r^2 dr$

**At the center** $\quad \bar{x} = 0$

$$\phi(0) = -4\pi G \int_0^\infty \frac{\rho(r')}{r'}\, r'^2 dr' = -4\pi G \int_0^\infty \rho(r')\, r' dr'$$

with : $\rho(r') = \dfrac{M}{4\pi r'^2} \delta(R - r')$

$$\phi(r) = -GM \int_0^{\infty} \frac{\delta(R-r)}{r^2} r \, dr = -\frac{GM}{R}$$

As the potential is constant for $r < R$

$$\boxed{\phi(\bar{x}) = -\frac{GM}{R} \qquad \bar{x} \in \text{sphere}}$$

# Newton's Theorems

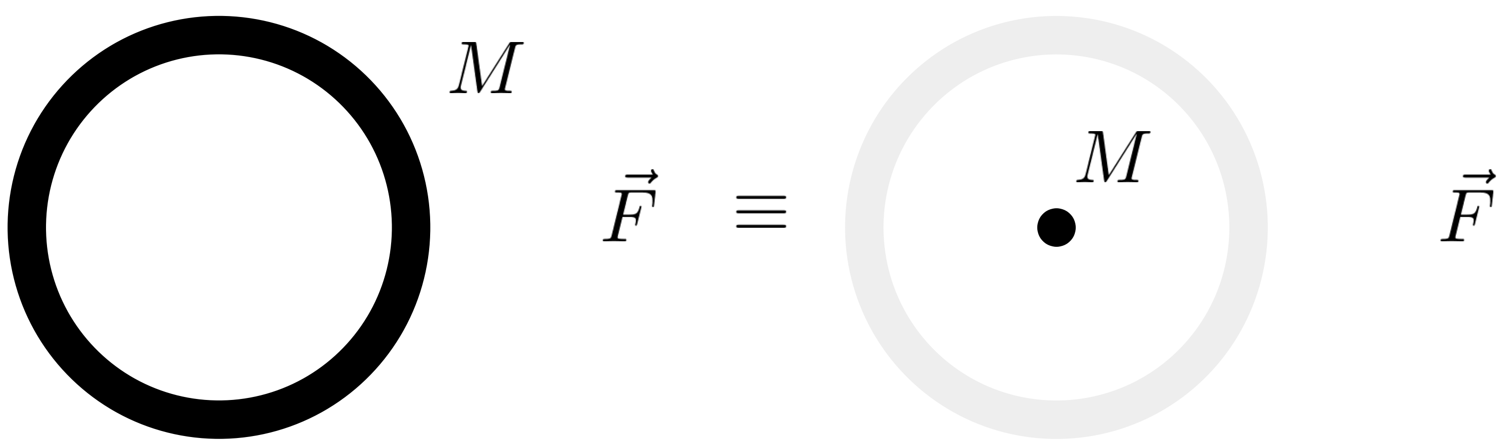
Newton (1642-1727)

First theorem:

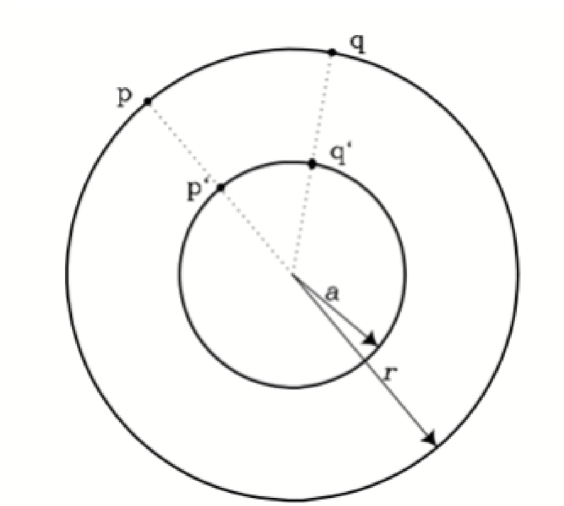
A body that is inside a spherical shell of matter experiences no net gravitational force from the shell.

Second theorem:

The gravitational force on a body that lies outside a spherical shell of matter is the same as it would be if all the shell's matter where concentrated into a point at its centre.



## Second Newton theorem



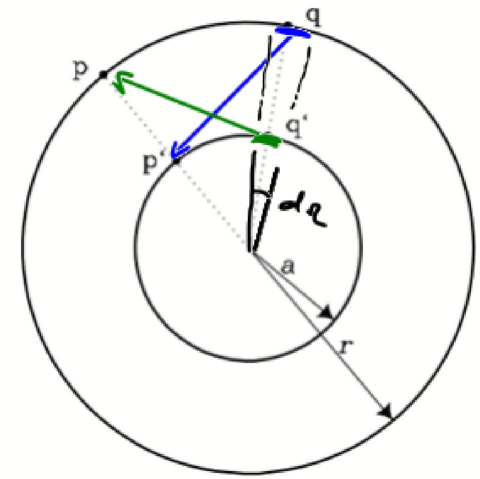

The gravitational force on a body that lies outside a spherical shell of matter is the same as it would be if all the shell's matter were concentrated into a point at its center

## Consider two shells

1. inner, with radius a and mass M
2. outer, with radius r and mass M

Compute  1. $\phi_p = \phi_i(r)$

2. $\phi_{p'} = \phi_o(a) = -\dfrac{GM}{r}$




1. contribution of shell$_i$, in $q'$ with solid angle $\delta\Omega$

- $$\delta\phi_i(p) = -\frac{G\,\delta m_{q'}}{|p-q'|} = -\frac{GM}{|p-q'|}\frac{\delta\Omega}{4\pi}$$

mass inside the solid angle

$$\delta m = \frac{M\,\delta\Omega}{4\pi}$$

2. contribution of shell$_o$, in $q$ with solid angle $\delta\Omega$

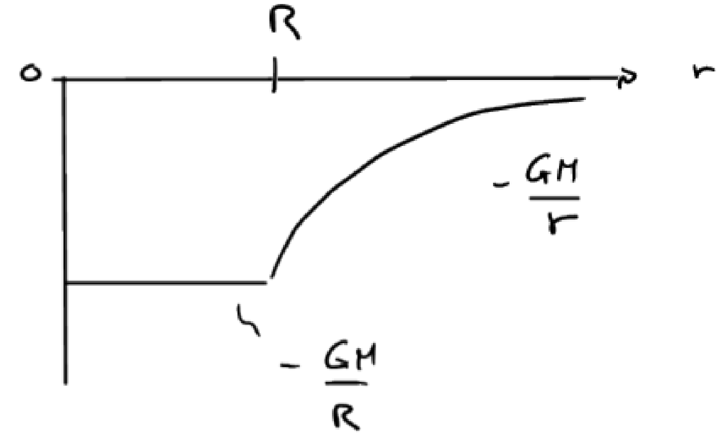
- $$\delta\phi_o(p') = -\frac{G\,\delta m_q}{|p'-q|} = -\frac{GM}{|p'-q|}\frac{\delta\Omega}{4\pi} = \delta\phi_i(p)$$

Summing over all $q'$ = Summing over all $q$

$$\phi_i(p) = \phi_o(p') = -\frac{GM}{r} \qquad \#$$

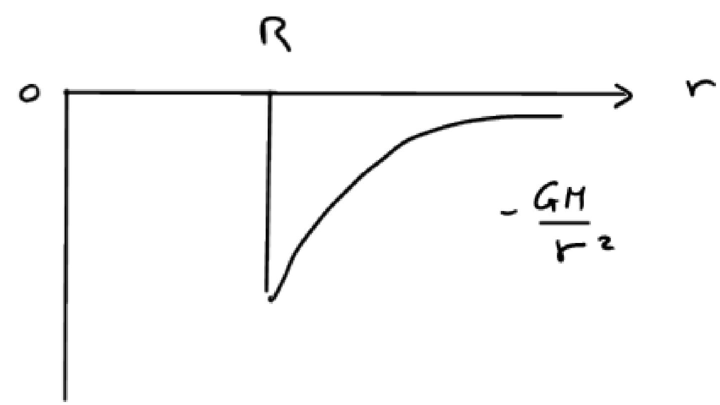
## Total potential of a shell of mass M, radius R

$$\phi(r) = \begin{cases} -\dfrac{GM}{R} & r < R \\[2ex] -\dfrac{GM}{r} & r \geq R \end{cases}$$




## Total gravitational field of a shell of mass M, radius R

$$\vec{g}(r) = \begin{cases} 0 & r < R \\[2ex] -\dfrac{GM}{r^2}\,\vec{e}_r & r \geq R \end{cases}$$



# Spherical Systems
# general distribution of mass

$$\rho(\vec{x}) = \rho(r)$$

$$r = \sqrt{x^2 + y^2 + z^2}$$

# Potential and gravitational field of any density $\rho(r)$

Build any density by summing shells of
of size $R$, mass $M_R$ and density $\rho_R(r)$

$$\rho(r) = \sum_R \delta\rho_R(r) = \int dR \, \frac{\partial \rho_R(r)}{\partial R}$$

But : $\rho_R = \frac{M_R}{4\pi r^2} \delta(R-r)$ $\qquad \frac{\partial \rho_R}{\partial R} = \frac{\partial M_R}{\partial R} \frac{1}{4\pi r^2} \delta(R-r)$

$$\rho(r) = \int_0^\infty dR \, \underbrace{\frac{\partial M_R}{\partial R}}_{\text{mass per unit length}} \frac{1}{4\pi r^2} \delta(R-r) = \frac{\partial M_r}{\partial r} \frac{1}{4\pi r^2}$$

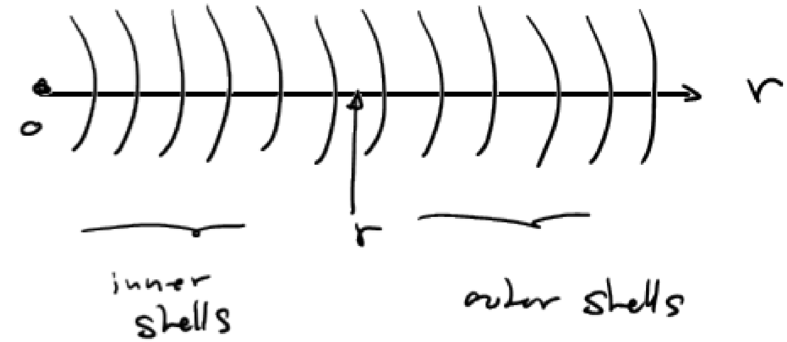
$$\boxed{dM_R = 4\pi R^2 \rho(R) \, dR}$$

Each shell contributing to the total density has thus a potential

$$\delta\phi_R(r) = \begin{cases} -\dfrac{G\,\overbrace{4\pi R^2\,\rho(R)\,dR}^{\delta M_R}}{R} & r < R \\ -\dfrac{G\,4\pi R^2\,\rho(R)\,dR}{r} & r \geq R \end{cases}$$

$$\boxed{\delta\phi_R(r) = \begin{cases} -4\pi G R\,\rho(R)\,dR & r < R \\ \dfrac{-4\pi G\,R^2\,\rho(R)\,dR}{r} & r \geq R \end{cases}}$$

# Total Potential




$$\phi(r) = \int_0^\infty \delta\phi_R(r)$$

$$= \int_0^r \underbrace{\delta\phi_R(r)}_{\substack{\text{inner shells} \\ r \geq R}} + \int_r^\infty \underbrace{\delta\phi_R(r)}_{\substack{\text{outer shells} \\ r < R}}$$

$$= \underbrace{-4\pi G \int_0^r \frac{R^2 \rho(R)}{r}\, dR}_{-\frac{GM(r)}{r}} - 4\pi G \int_r^\infty R\, \rho(R)\, dR$$

$$\boxed{\phi(r) = \underbrace{-\frac{GM(r)}{r}}_{\substack{\text{contribution}\\ \text{of the mass}\\ \text{inside } r}} \underbrace{- 4\pi G \int_r^{\infty} dR\, R\, \rho(R)}_{\substack{\text{contribution}\\ \text{of the mass}\\ \text{outside } r}}}$$

## Gravitational field of a spherical model $\rho(r)$

From the potential $\phi(r)$ $\quad \vec{g}(\vec{x}) = -\vec{\nabla}\phi(\vec{x})$

$$g(r) = -\frac{\partial \phi}{\partial r} = -\frac{\partial}{\partial r}\left[ -\frac{G M(r)}{r} - 4\pi G \int_r^{\infty} \rho(r') r' dr' \right]$$

$$= -\frac{G M(r)}{r^2} + \frac{G}{r} 4\pi \frac{\partial}{\partial r}\int_0^r dr' r'^2 \rho(r') + 4\pi G \frac{\partial}{\partial r}\int_r^{\infty} dr' r' \rho(r')$$

$$= -\frac{G M(r)}{r^2} + \underbrace{\frac{G}{r} 4\pi r^2 \rho(r) - 4\pi G r \rho(r)}_{=0}$$

$$\boxed{g(r) = -\frac{G M(r)}{r^2}}$$

contribution
of the mass
inside $r$

COMPLEMENT

# Gravitational field of a spherical model $\rho(r)$

## Sum of shells

$$g(r) = \int_0^\infty \delta g_{r'}(r)$$

$\delta g_{r'}(r)$ = force due to the shell of radius $r'$

EXERCICE

## Summary: for any spherical mass distribution $\rho(r)$

$$\boxed{g(r) = -\frac{GM(r)}{r^2}}$$

$$M(r) = 4\pi \int_0^{\infty} \rho(r')\, r'^2\, dr'$$

$$\boxed{\phi(r) = -\frac{GM(r)}{r} - 4\pi G \int_r^{\infty} \rho(r')\, r'\, dr'}$$

<u>Note</u> $\quad g(r) = -\dfrac{\partial \phi}{\partial r}$ $\quad$ as expected from

$$\vec{g}(\vec{x}) = \vec{\nabla} \phi(\vec{x})$$

# Spherical systems : circular speed, circular velocity

Speed of a test particle in a cirular orbit in the potential $\phi(r)$ at a radius $r$ :

$$\vec{g}_s = \vec{a}_c$$

$\vec{a}_c$ : centripetal acceleration $\qquad \frac{V_c^2}{r}$

$\vec{g}_s$ : gravity acceleration ( spec force ) $\qquad -\frac{GM(r)}{r^2} = -\frac{\partial\phi}{\partial r}$

$$\boxed{V_c^2 = \frac{GM(r)}{r}} \qquad \boxed{V_c^2 = r\,\frac{\partial\phi}{\partial r}}$$

$$\boxed{GM(r) = r^2\,\frac{\partial\phi}{\partial r}}$$

$[V_c^2]$ : $\frac{erg}{g}$

as $\phi$

$\equiv$ specific energy

# Velocity composition

**Note**: $V_o^2$ scale with the mass ($M(r)$) : it is thus the "important" quantity (spec. energy)

Multi-components system : ex: bulge + stellar halo + DM halo

$$\begin{cases} \rho_B(r) \quad , \quad M_B(r) \quad , \quad \phi_B(r) \quad \longrightarrow \quad V_{c,B}(r) \\ \rho_H(r) \quad , \quad M_H(r) \quad , \quad \phi_H(r) \quad \longrightarrow \quad V_{c,H}(r) \\ \rho_{DM}(r) \quad , \quad M_{DM}(r) \quad , \quad \phi_{DM}(r) \quad \longrightarrow \quad V_{c,DM}(r) \end{cases}$$

$$V_{c,tot}^2 = \frac{G M_{tot}(r)}{r} = \frac{G}{r} \sum_i M(r)$$

$$\boxed{V_{c,tot}^2 = \sum_i V_{c,i}^2}$$

$V_c^2 \sim$ energy : extensive quantity

## Period of the circular orbit

$$\boxed{T(r) = \frac{2\pi r}{v_c(r)} = 2\pi\sqrt{\frac{r^3}{GM(r)}} = 2\pi\sqrt{\frac{r}{\frac{\partial\phi}{\partial r}}}}$$

## Circular frequency (angular frequency)

$$\boxed{\Omega(r) = \frac{2\pi}{T(r)} = \sqrt{\frac{GM(r)}{r^3}} = \sqrt{\frac{1}{r}\frac{\partial\phi}{\partial r}}}$$

## Escape speed $v_e$

if $\frac{1}{2}v_e^2 > \phi(r) = E > 0$

the partcle may escape the system

$$\boxed{v_e(r) = \sqrt{2|\phi(r)|}}$$

## Potential energy

from $W = - \int \rho(\vec{x}) \, \vec{x} \cdot \vec{\nabla} \phi(\vec{x}) \, d^3\vec{x}$

$$\boxed{W = -4\pi G \int_0^\infty \rho(r) \, M(r) \, r \, dr}$$

## Gravitational radius

radius at which $\frac{GM^2}{r} = W$

(estimation of the system size)

$$\boxed{r_g = \frac{GM^2}{|W|}}$$

# Spherical systems : useful relations

| | $\rho(r)$ | $\Phi(r)$ | $M(r)$ | $\frac{\mathrm{d}\Phi}{\mathrm{d}r}$ |
|---|---|---|---|---|
| $\rho(r)$ | $\rho(r)$ | $\frac{1}{4\,\pi\,G}\,\frac{1}{r^2}\,\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\,\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right)$ | $\frac{1}{4\pi r^2}\,\frac{\mathrm{d}M(r)}{\mathrm{d}r}$ | $\frac{1}{4\,\pi\,G}\,\frac{1}{r^2}\,\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\,\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right)$ |
| $\Phi(r)$ | $-\frac{G\,M(r)}{r} - 4\pi G\,\int_r^\infty \mathrm{d}r'\,r'\,\rho(r')$ | $\Phi(r)$ | $-G\,\int_r^\infty \mathrm{d}r'\,\frac{M(r')}{r'^2}$ | $-\int_r^\infty \mathrm{d}r'\,\frac{\mathrm{d}\Phi}{\mathrm{d}r}$ |
| $M(r)$ | $4\pi\,\int_0^r \mathrm{d}r'\,r'^2\,\rho(r')$ | $\frac{r^2}{G}\,\frac{\mathrm{d}\Phi}{\mathrm{d}r}$ | $M(r)$ | $\frac{r^2}{G}\,\frac{\mathrm{d}\Phi}{\mathrm{d}r}$ |
| $\frac{\mathrm{d}\Phi}{\mathrm{d}r}$ | $\frac{4\pi G}{r^2}\,\int_0^r \mathrm{d}r'\,r'^2\,\rho(r')$ | $\frac{\mathrm{d}\Phi}{\mathrm{d}r}$ | $\frac{G\,M(r)}{r^2}$ | $\frac{\mathrm{d}\Phi}{\mathrm{d}r}$ |

Poisson in spherical coordinates

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right) = 4\,\pi\,G\,\rho(r)$$

Potential in spherical coordinates

$$\Phi(r) = -\frac{G\,M(r)}{r} - 4\pi G\int_r^\infty \rho(r')r'\mathrm{d}r'$$

Mass inside a radius r

$$M(r) = 4\pi\int_0^r dr'\,r'^2\,\rho(r')$$

Gradient of the potential in spherical coordinates

$$\frac{\mathrm{d}\Phi(r)}{\mathrm{d}r} = \frac{G\,M(r)}{r^2}$$

# The End