

The gravity force and Newtonian mechanics

Outlines

Newtonian Mechanics:

- refreshing memory

The gravity : a long distance force

- collision-less systems
- the relaxation time

Refreshing memory...

Newtonian mechanics

Newtonian mechanics : a very short remainder

point mass : mass m

position \vec{x}

velocity $\vec{v} = \frac{d\vec{x}}{dt}$

momentum $\vec{p} = m \vec{v} = m \frac{d\vec{x}}{dt}$

Newtonian mechanics : a very short remainder

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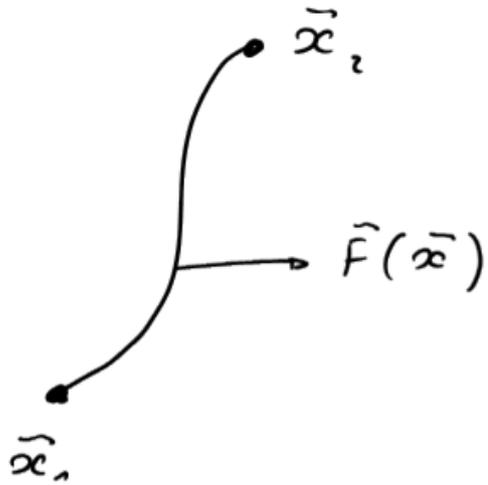
Newton second law

$$\frac{d}{dt}(\vec{p}) = m \frac{d^2\vec{x}}{dt^2} = \vec{F}$$

• \vec{F} : a force

\vec{p} is constant in absence of a force

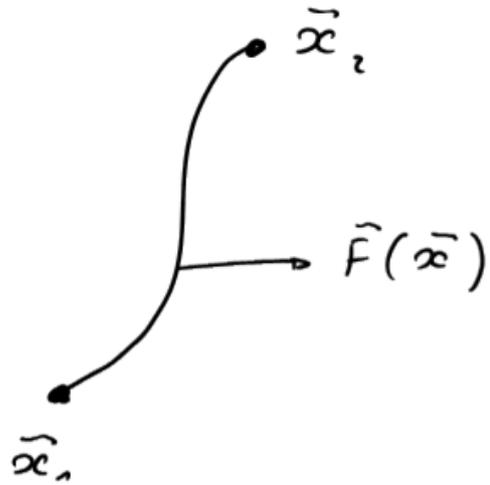
Work : work done by a force in moving the particle
from \vec{x}_1 to \vec{x}_2



$$W_{12} = - \int_{\vec{x}_1}^{\vec{x}_2} \vec{F}(\vec{x}) \cdot d\vec{x}$$

$$[W_{12}] = 9 \frac{\text{cm}^2}{\text{s}^2} \\ = \text{erg}$$

Work : work done against a force that moves the particle from \vec{x}_1 to \vec{x}_2



$$W_{12} = - \int_{\vec{x}_1}^{\vec{x}_2} \vec{F}(\vec{x}) \cdot d\vec{x}$$

$$[W_{12}] = 9 \frac{\text{cm}^2}{\text{s}^2} = \text{erg}$$

Power of a force (energy rate)

$$\frac{\text{erg}}{\text{s}}$$

$$\text{with } \frac{\partial \mathcal{F}(\vec{x})}{\partial x_i} = F_i(\vec{x})$$

$$\frac{d}{dt} W_{12}(x(t)) = - \frac{d}{dt} \int_{\vec{x}_1}^{\vec{x}(t)} \vec{F}(\vec{x}) \cdot d\vec{x} = - \frac{d}{dt} \left(\mathcal{F}(\vec{x}(t)) - \mathcal{F}(\vec{x}_1) \right)$$

$$= - \vec{\nabla}_{\vec{x}} \mathcal{F}(\vec{x}) \cdot \frac{d}{dt} (\vec{x}(t)) = - \vec{F}(\vec{x}) \vec{v}(\vec{x})$$

Kinetic energy

$$W_{12} = - \int_{\vec{x}_1}^{\vec{x}_2} \vec{F}(\vec{x}) \cdot d\vec{x}$$

integration by part gives

$$= -m \left[\vec{v}^2 \Big|_{\vec{x}_1}^{\vec{x}_2} - \int_{\vec{x}_1}^{\vec{x}_2} \vec{v} \frac{d\vec{v}}{dt} dt \right] = -m \vec{v}_2^2 + m \vec{v}_1^2 + m \underbrace{\int_{\vec{x}_1}^{\vec{x}_2} \vec{v} \frac{d\vec{v}}{dt} dt}_{-W_{12}}$$

Thus $W_{12} = \frac{1}{2} m \vec{v}_1^2 - \frac{1}{2} m \vec{v}_2^2$

$$W_{12} = K_1 - K_2$$

Newton
2nd law

$$= -m \int_{\vec{x}_1}^{\vec{x}_2} \frac{d\vec{v}}{dt} \frac{d\vec{x}}{dt} dt$$
$$= -m \int_{\vec{x}_1}^{\vec{x}_2} \frac{d\vec{v}}{dt} \cdot \vec{v} \cdot dt$$

$$\vec{x} = \vec{x}(t)$$
$$d\vec{x} = \frac{d\vec{x}}{dt} dt$$

$$K = \frac{1}{2} m \vec{v}^2 : \text{Kinetic energy}$$

Potential energy and conservative forces

A force $\vec{F}(\vec{x})$ is called conservative if the work done by this force in moving the particle from \vec{x}_1 to \vec{x}_2 is independent of the path.

Then, for any given point \vec{x}_0

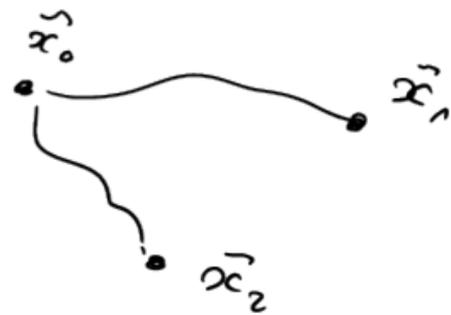
we can define the function (potential) $V_0(\vec{x})$

$$V_0(\vec{x}) := V_{0\vec{x}} = - \int_{\vec{x}_0}^{\vec{x}} \vec{F}(\vec{x}') d\vec{x}'$$

Then $W_{12} = W_{10} + W_{02}$

$W_{12} = V(\vec{x}_2) - V(\vec{x}_1)$

Useful convention : $\vec{x}_0 \rightarrow \infty$ (far away from all interacting bodies)



Gradient of the potential

$$\vec{\nabla}_{\vec{x}} \cdot V(\vec{x}) = -\vec{\nabla}_{\vec{x}} \cdot \left[\int_{\vec{x}_0}^{\vec{x}} \vec{F}(\vec{x}') d\vec{x}' \right] = -\vec{\nabla}_{\vec{x}} \left(\underbrace{F(\vec{x}) - F(\vec{x}_0)}_{\text{ck} \Rightarrow 0} \right) = -\vec{F}(\vec{x})$$

with $\frac{\partial F(\vec{x})}{\partial x_i} = F_i(\vec{x})$

$$\vec{\nabla}_{\vec{x}} \cdot V(\vec{x}) = -\vec{F}(\vec{x})$$

We can represent a conservative force field by its potential

Total Energy E

$$E := K + V = \frac{1}{2} m \vec{v}^2 + V(\vec{x})$$

Theorem

The energy E of a system evolving under conservative forces $\vec{F}(\vec{x})$ (associated to a potential $V(\vec{x})$) is constant.



$$E_1 = E(\vec{x}_1) = \frac{1}{2} m v_1^2 + V(\vec{x}_1)$$

$$E_2 = E(\vec{x}_2) = \frac{1}{2} m v_2^2 + V(\vec{x}_2)$$

$$E_1 - E_2 = \underbrace{K_1 - K_2}_{W_{12}} + \underbrace{V(\vec{x}_1) - V(\vec{x}_2)}_{-W_{12}}$$

$$= 0$$

#

Angular momentum and Torque

Angular momentum $\vec{L} = \vec{x} \times \vec{p}$

Torque $\vec{N} = \vec{x} \times \vec{F}$

$$\begin{aligned} \frac{d\vec{L}}{dt} &= \frac{d\vec{x}}{dt} \times \vec{p} + \vec{x} \times \frac{d\vec{p}}{dt} \\ &= \underbrace{\vec{v} \times \vec{p}}_{=0} + \underbrace{\vec{x} \times \vec{F}}_{\vec{N}} \end{aligned}$$

$$\frac{d\vec{L}}{dt} = \vec{N}$$

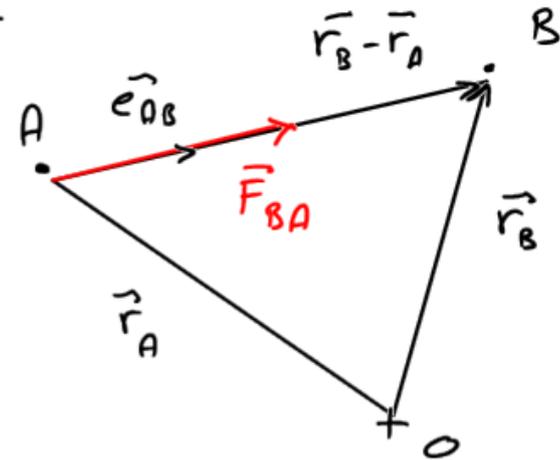
**The Gravity :
a long distance force**

The gravity : a long range force

$$\vec{F}_{BA} = \frac{G m_A m_B}{|\vec{r}_B - \vec{r}_A|^2} \vec{e}_{AB}$$

$$\vec{F}_{BA} = \frac{G m_A m_B}{|\vec{r}_B - \vec{r}_A|^3} \vec{r}_B - \vec{r}_A$$

$$|\vec{F}_{BA}| = \frac{G m_A m_B}{r_{AB}^2}$$



$$\vec{e}_{AB} = \frac{\vec{r}_B - \vec{r}_A}{|\vec{r}_B - \vec{r}_A|}$$

$$r_{AB} = |\vec{r}_B - \vec{r}_A|$$

$$[G] = \frac{\text{cm}^3}{\text{g s}^2} \equiv \frac{\text{erg cm}}{\text{g}^2}$$

$$[\text{erg}] = \frac{\text{cm}^2}{\text{s}^2} \text{ g}$$

Contrary to, for example, molecular forces, gravity is a long range force, i.e.: **we cannot neglect distant regions**

Illustration: an homogeneous medium ($\rho(\vec{r}) = \rho_0$)

Force on point O due to a thin shell of mass Δm at distance r

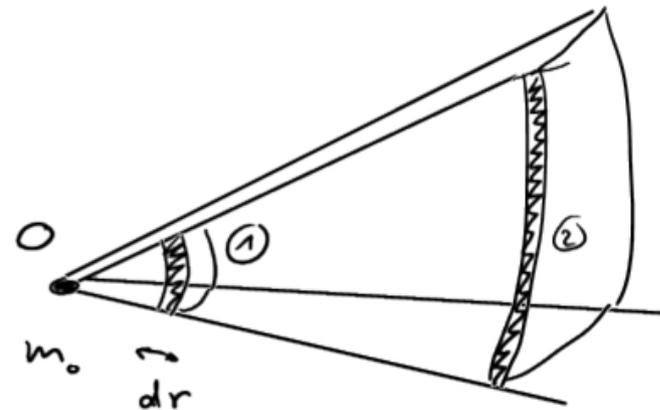
$$\Delta F = \frac{G m_0 \Delta m}{r^2}$$

but $\Delta m = \rho r^2 \Delta \Omega dr$

thus

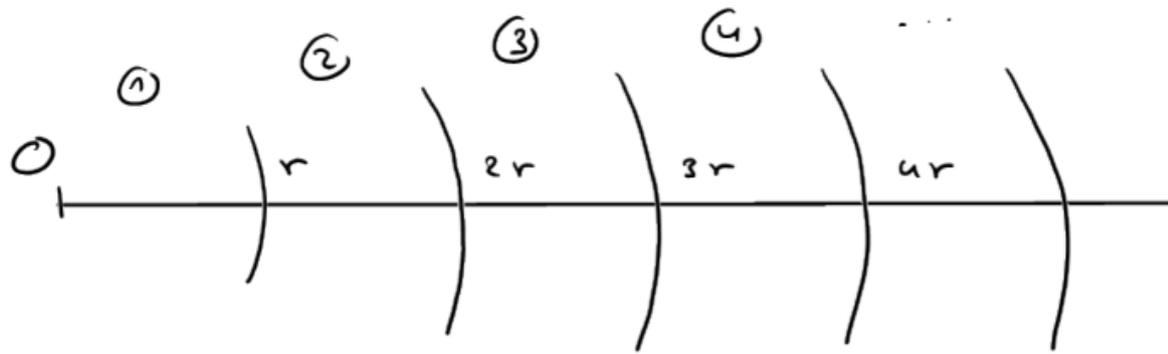
$$\Delta F = G m_0 \rho \Delta \Omega dr$$

etc indep. of r



solid angle $\Delta \Omega$

Split the space in shells of thickness r



$$\Delta F_1 = \int_0^r \Delta F = \int_0^r dr G m_o \rho \Delta R = G m_o \rho \Delta R r$$

$$\Delta F_2 = \int_r^{2r} \Delta F = \dots = G m_o \rho \Delta R r$$

As the contribution of all shell is the same, the contribution of the stars with $r' > r$ will dominate over the ones with $r' < r$

We cannot neglect regions at large distances !

Corollary : As the force is dominated by the mass at large scale, the force varies smoothly along the trajectory of a particle (star).

Stellar systems can be modeled by smooth mass distributions



Notes

① 2D case $\delta F = \frac{Gm_0}{r^2} \delta m = \frac{Gm_0}{r^2} \Sigma \delta \theta r dr = Gm_0 \Sigma \delta \theta dr \frac{1}{r}$
 Σ : surface density

$$\delta F_0 = \int_0^r Gm_0 \rho \delta \theta \frac{1}{r} dr = Gm_0 \rho \delta \theta [\ln(r) - \ln(0)] = \infty$$

(iii) r

$$\delta F_i = \int_{ir} Gm_0 \rho \delta \theta \frac{1}{r} dr = Gm_0 \rho \delta \theta [\ln(i+1) - \ln(i)] < \infty$$

= 0 (i \rightarrow ∞)

1D case $\delta F = \frac{Gm_0}{r^2} \delta m = \frac{Gm_0}{r^2} \lambda dr$ λ : linear density

$$\delta F_0 = \int_0^r Gm_0 \lambda \frac{1}{r^2} dr = Gm_0 \lambda \left[-\frac{1}{r} + \frac{1}{\infty} \right] = \infty$$

(iii) r

$$\delta F_i = \int_{ir} Gm_0 \lambda \frac{1}{r^2} dr = Gm_0 \lambda \left[\frac{1}{r} \left(\frac{1}{i} - \frac{1}{i+1} \right) \right] < \infty$$

= 0 (i \rightarrow ∞)

② Molecular dynamics

"long distance" attraction force
between two molecules,

Van der Waals Force $\sim r^{-7}$

\Rightarrow local molecules dominates

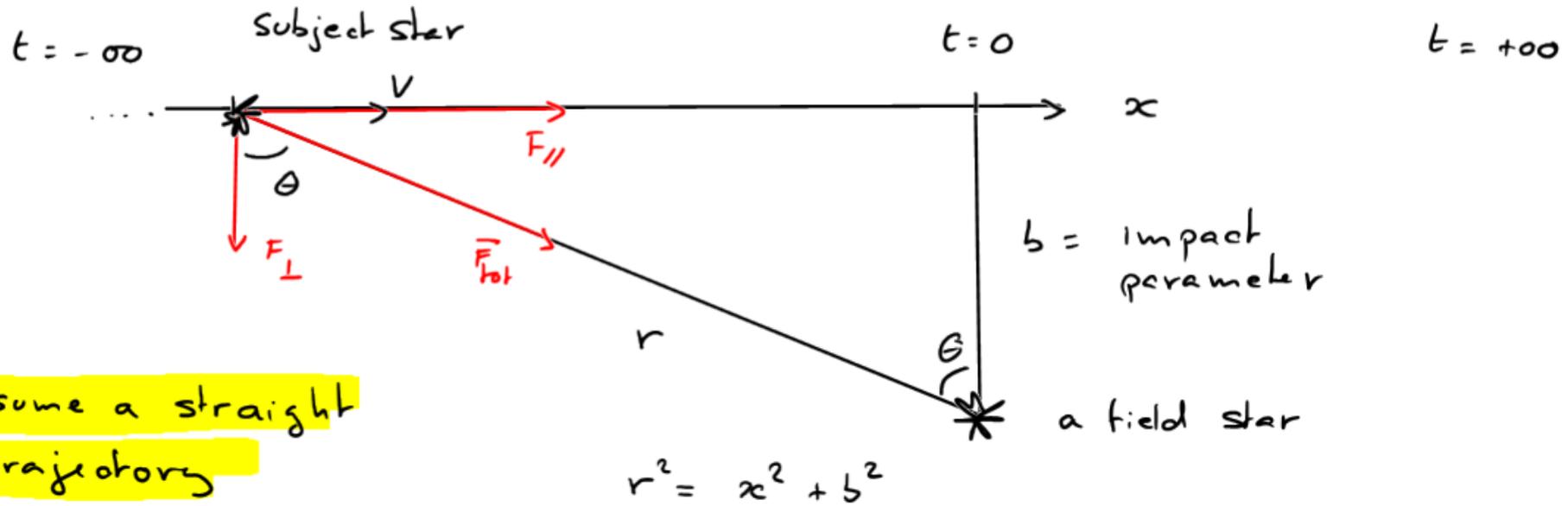
Relaxation Time

Question :

How accurate is the assumption that a galaxy may be modelled as a smooth distribution ?

- 1) Effect of one star on the orbit of a peculiar star.
- 2) Effect of all stars of a stellar system on a peculiar star.
- 3) Under which conditions the orbit of a peculiar star is strongly influenced by the discrete nature of the stellar system (importance of “collision” with other stars).

- ① Estimate the effect of one star on the trajectory of a peculiar star
-



We assume a straight line trajectory

- 1) acceleration along x (F_{\parallel}) does not matter, as it is symmetric (the star decelerate after passing the field star)
- 2) acceleration perpendicular to x (F_{\perp})

$$\begin{aligned}
 |F_{\perp}| &= |F_{\text{tot}}| \cos \Theta = \frac{Gmm}{r^2} \cos \Theta && \text{but } \cos \Theta = \frac{b}{r} \\
 &= \frac{Gmm}{x^2 + b^2} \frac{b}{\sqrt{x^2 + b^2}} = \frac{Gmm}{(x^2 + b^2)^{3/2}} b \\
 & && r^2 = x^2 + b^2 \\
 &= \frac{Gmm}{\left(1 + \frac{x^2}{b^2}\right)^{3/2}} \frac{1}{b^2} = \frac{Gmm}{b^2} \left(1 + \frac{x^2}{b^2}\right)^{-3/2}
 \end{aligned}$$

with $x = vt$

$$F_{\perp} = |F_{\perp}| = \frac{Gmm}{b^2} \left(1 + \left(\frac{vt}{b}\right)^2\right)^{-3/2}$$

Newton 2nd law

$$\begin{aligned}
 F_{\perp} &= m a_{\perp} = m \frac{dV_{\perp}}{dt} \\
 dV_{\perp} &= \frac{F_{\perp}}{m} dt
 \end{aligned}$$

Integrating over time from $t = -\infty$ to $t = \infty$

$$\Delta V_{\perp} = \frac{1}{m} \int_{-\infty}^{\infty} F_{\perp} dt \quad \text{net velocity increase}$$

$$= \frac{1}{m} \int_{-\infty}^{\infty} \frac{GmM}{b^2} \left(1 + \left(\frac{vt}{b}\right)^2\right)^{-\frac{3}{2}} dt$$

with : $s = \frac{vt}{b}$ $ds = \frac{v}{b} dt$

$$\Delta V_{\perp} = \frac{Gm}{b^2} \int_{-\infty}^{\infty} \left(1 + s^2\right)^{-\frac{3}{2}} dt = 2 \frac{Gm}{bv}$$

This can be written as :

$$\delta V_{\perp} = \frac{Gm}{b^2} \cdot \frac{2b}{v}$$

acceleration
at the closest
approach

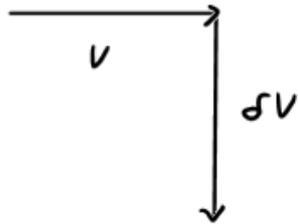
"duration"
of the closest
approach

$$= 2 \frac{Gm}{bv}$$

Note : our hypothesis of a straight line is ok if $\frac{\delta V}{V} \ll 1$

$$\Rightarrow b \gg \frac{2Gm}{v^2} := b_{90}$$

b_{90} define
as $V = \delta V$



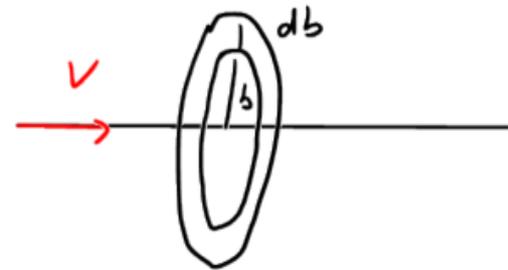
② Effect of all stars of a stellar system on the trajectory of a peculiar star

N : total number of stars

R typical size of the system

- number density of stars per unit of surface $n = \frac{N}{\pi R^2}$
- number of stars met by the star with $[b, b + db]$

$$\begin{aligned}\delta N &= 2\pi b db \cdot n \\ &= 2\pi b db \frac{N}{\pi R^2} = \frac{2N b db}{R^2}\end{aligned}$$

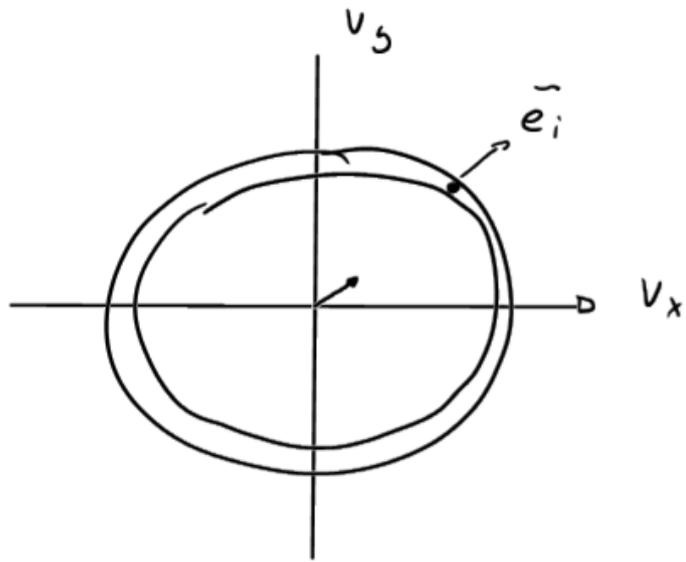


Note: each of those stars (alone) induces a change of velocity

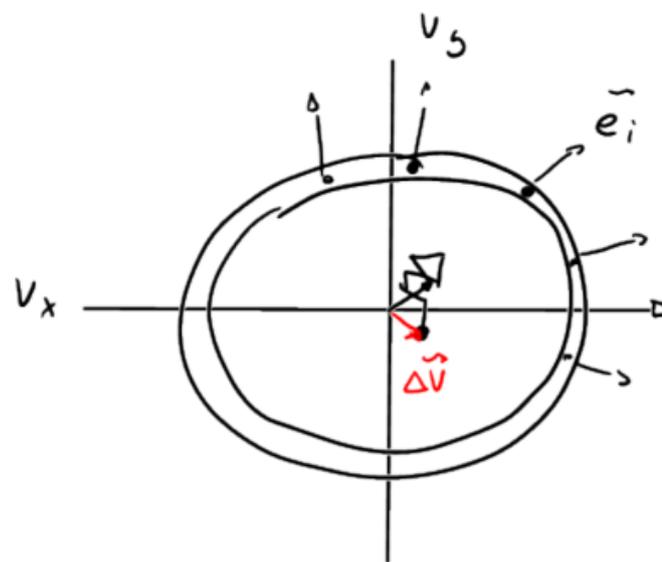
$$\delta V_{\perp} = \frac{2Gm}{bv}$$

Velocity change due to δN stars in a ring

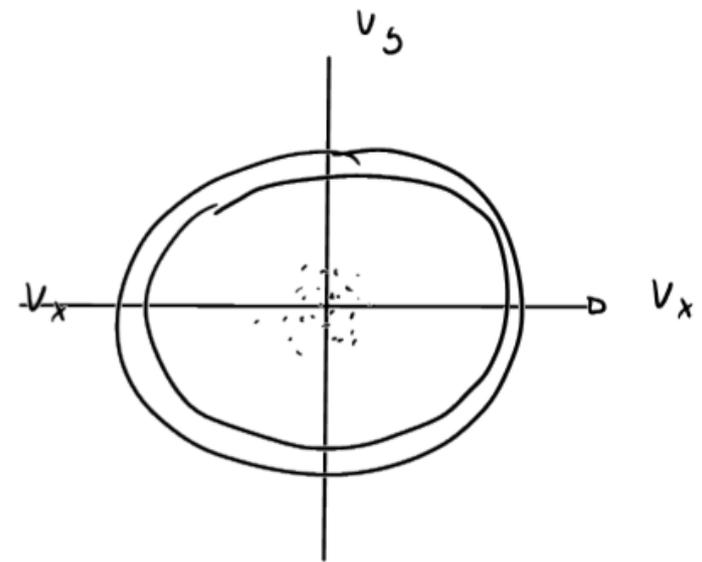
$$\Delta \vec{v} = \sum_i \delta v_{\perp,i} \vec{e}_i$$



velocity change due to one star in the ring



velocity change due to δN stars in the ring



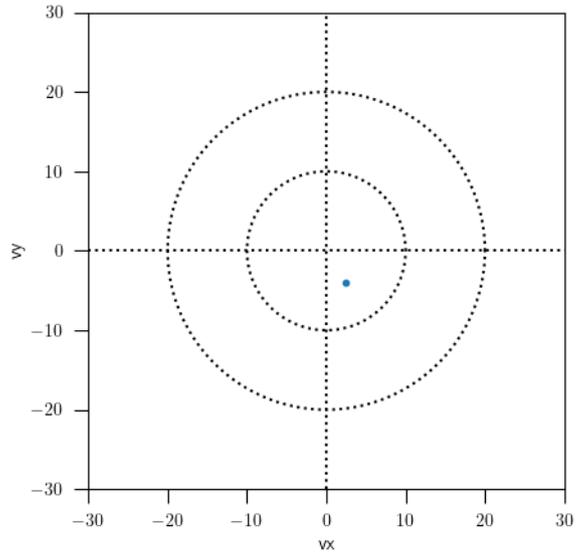
Statistical distribution of velocity changes

$$\Delta \vec{v} = \sum_i \delta v_{\perp,i} \vec{e}_i = 0$$

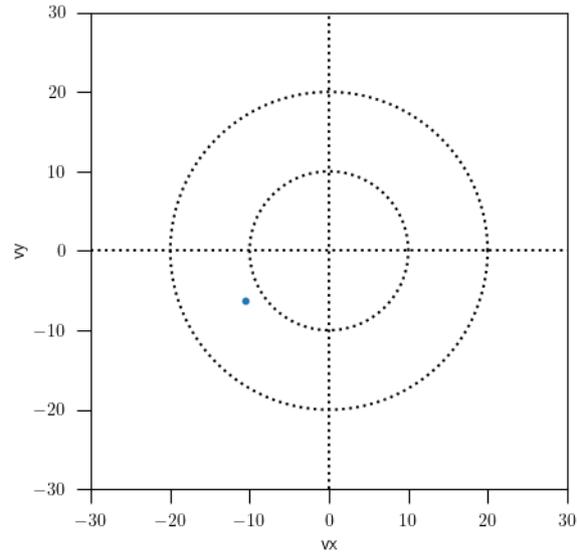
$$\begin{aligned} \Delta v^2 &= \sum_i \delta v_{\perp,i}^2 \neq 0 \\ &= \delta N \delta v^2 = \frac{2N b db}{R^2} \left(\frac{2Gm}{bv} \right)^2 = \frac{8NG^2 m^2}{v^2 R^2} \frac{db}{b} \end{aligned}$$

Spread in the velocity space due to two-body encounters

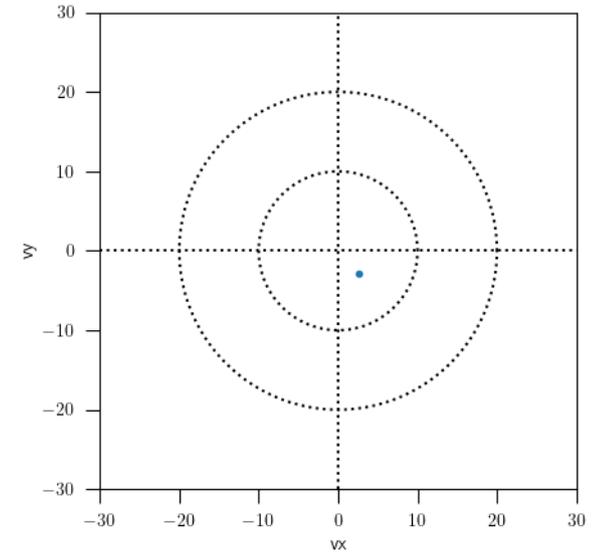
N=1



N=1

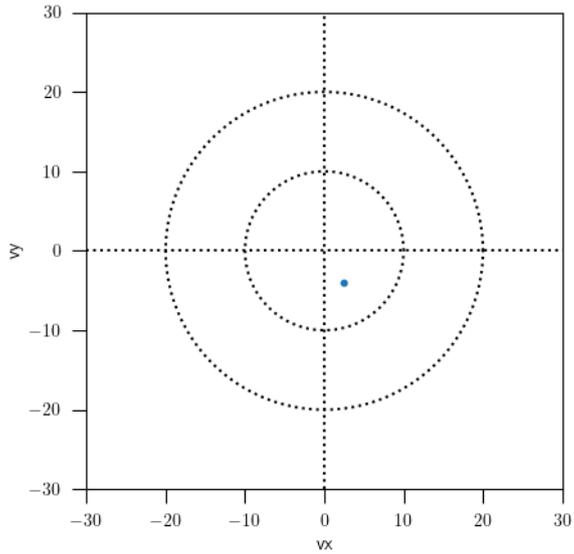


N=1

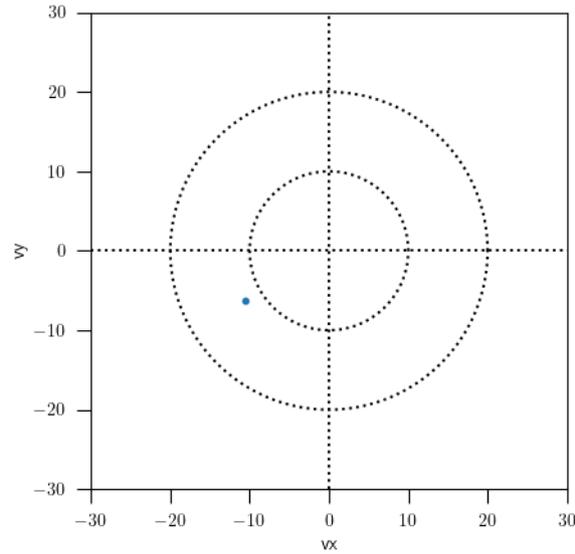


Spread in the velocity space due to two-body encounters

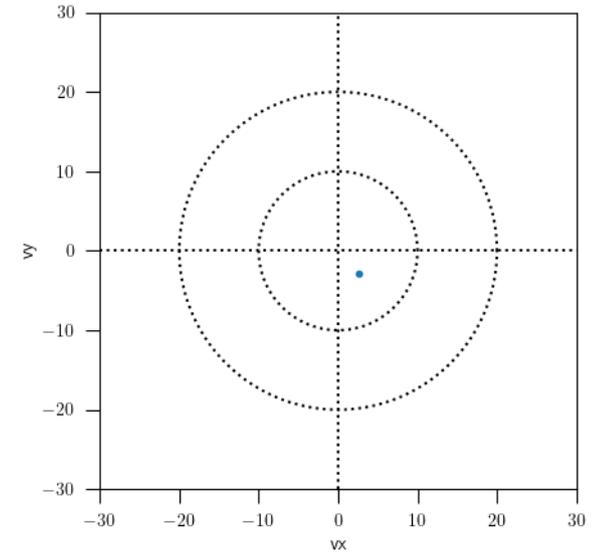
N=1



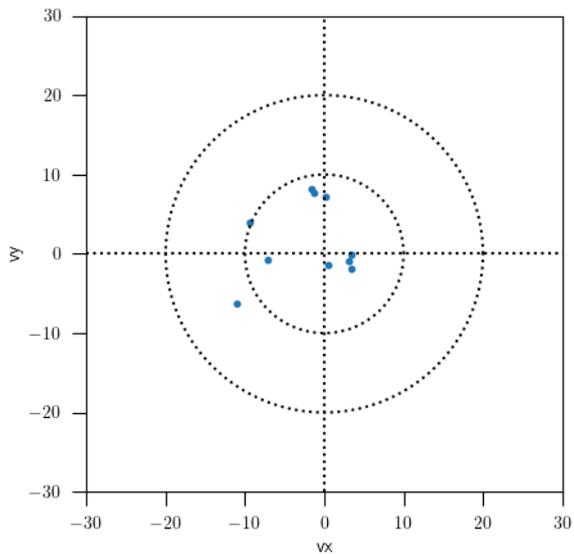
N=1



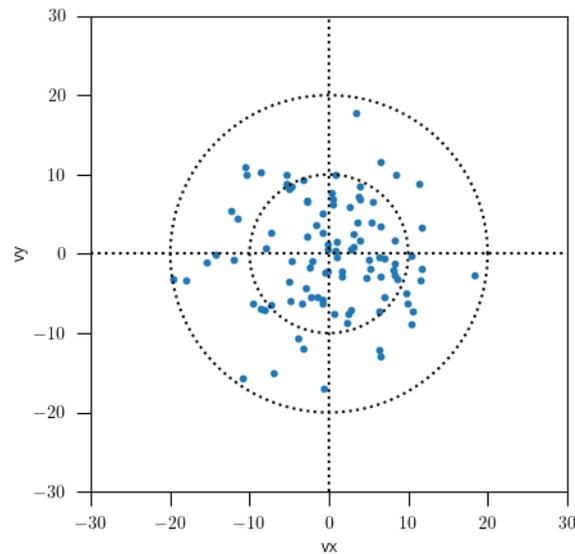
N=1



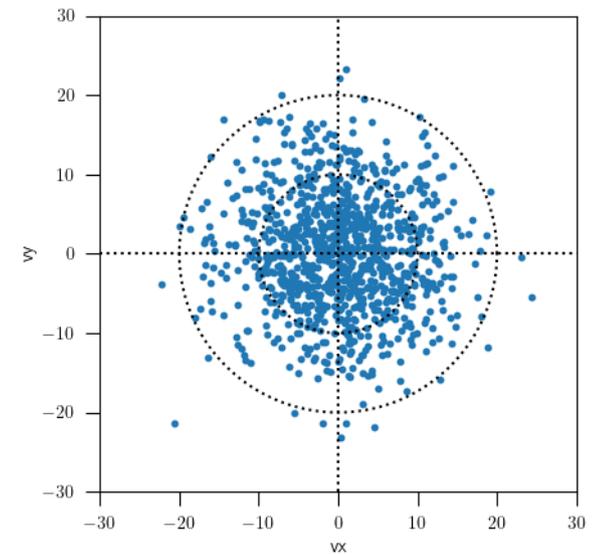
N=10



N=100



N=1000



For all encounters, we integrate over b from b_{\min} to b_{\max}

$$b_{\min} := \beta_1 b_{s0} \quad \text{if } b < b_{s0} \quad \Delta V \sim v \quad \beta_1 \approx 1$$

$$b_{\max} := \beta_2 R \quad \text{if } b > R \quad \text{the density is no longer constant} \quad \beta_2 \approx 1$$

we get

ln : Coulomb logarithm

$$\Delta V^2 = 8N \left(\frac{Gm}{vR} \right)^2 \int_{b_{\min}}^{b_{\max}} \frac{db}{b} = 8N \left(\frac{Gm}{vR} \right)^2 \ln \left(\frac{b_{\max}}{b_{\min}} \right)$$

$$\Delta V^2 = 8N \left(\frac{Gm}{vR} \right)^2 \left[\ln \left(\frac{R}{b_{s0}} \right) + \ln \left(\frac{\beta_2}{\beta_1} \right) \right]$$

- variation due to one crossing.

≈ 0

Replacing R with v

Typical velocity of one star (circular orbit)

$$v^2 \sim \frac{GNm}{R} \Rightarrow R = \frac{GNm}{v^2}$$

$$\Delta v^2 = 8N \left(\frac{Gm}{vR} \right)^2 \ln \Lambda = 8N \left(\frac{v}{N} \right)^2 \ln \Lambda = 8 \frac{v^2}{N} \ln \Lambda$$

$$\Delta v^2 = 8 \frac{v^2}{N} \ln \Lambda$$

Relaxation

How many crossing of the system (n_{relax}) will lead to a substantial change of the orbit? $\Delta V_{\text{tot}}^2 \approx V^2$

$$n_{\text{relax}} \Delta V^2 \approx V^2$$

$$n_{\text{relax}} \approx \frac{V^2}{8 \ln \mathcal{L}} \approx V^2$$

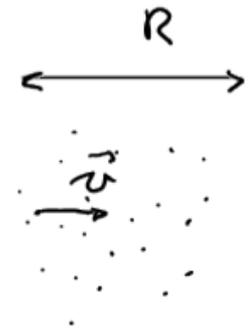
$$n_{\text{relax}} = \frac{N}{8 \ln \mathcal{L}}$$

Crossing time

Time needed for a star to cross the system

$$t_{\text{cross}} \approx \frac{R}{v}$$

$$\left(v^2 \sim \frac{GM_{\text{in}}}{R} \right)$$



Relaxation time

Time after which the orbit substantially changes

$$\Delta V_{\text{tot}}^2 \approx V^2$$

≡

Time to cross the system n_{relax} times

$$t_{\text{relax}} = n_{\text{relax}} \cdot t_{\text{cross}} = \frac{N}{8 \ln \Lambda} t_{\text{cross}}$$

if $t \ll t_{\text{relax}}$: the perturbation of nearby stars is weak
the system is **collision-less**

Estimations for stellar systems

N, m, R

using

$$b_{50} = \frac{2Gm}{v^2} \quad \text{circular orbit: } a = \frac{v^2}{r} = \frac{GNm}{r^2} \Rightarrow v^2 = \frac{GNm}{r}$$

we get

$$b_{50} = \frac{2R}{N}$$

$$\frac{R}{b_{50}} = \frac{N}{2}$$

$$\ln \mathcal{L} \approx \ln N$$

$$t_{\text{rdax}} = \frac{N}{8 \ln \mathcal{L}} \cdot t_{\text{cross}} \Rightarrow$$

$$t_{\text{rdax}} \approx \frac{0.1 N}{\ln N} t_{\text{cross}}$$

$$\begin{aligned} \ln \mathcal{L} &\approx \ln N \\ \frac{1}{8} &\approx 0.1 \end{aligned}$$

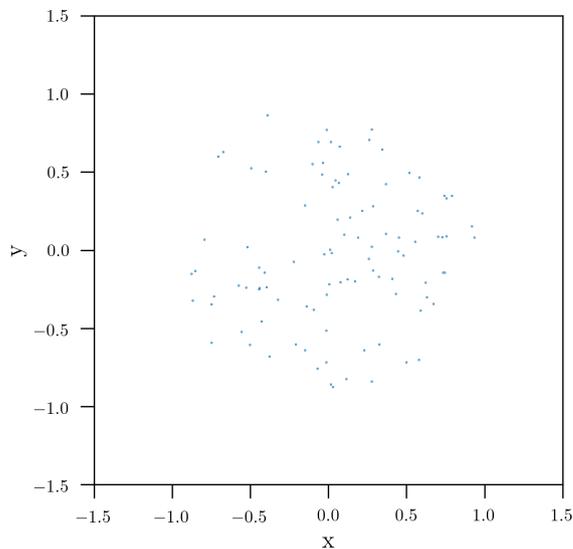
Numerical application / Relaxation Time

	N	R	b_{90}	$\ln(R/b_{90})$	t_{relax}
Globular Cluster	10^5	10 pc	2×10^{-4} pc	~ 10	~ 1 Gyr
Dwarf Galaxy	10^6	1 kpc	2×10^{-3} pc	~ 13	$\gg t_{\text{Hubble}}$
Spiral Galaxy	10^{10}	15 kpc	3×10^{-6} pc	~ 20	$\gg t_{\text{Hubble}}$
Galaxy Cluster	10^{13}	1 Mpc	2×10^{-7} pc	~ 30	$\gg t_{\text{Hubble}}$

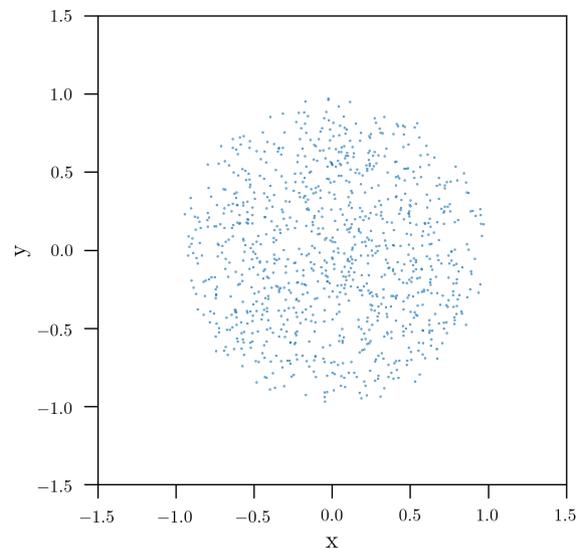
Numerical illustration

Orbit of a point mass in an homogeneous sphere
sampled with a discrete number of stars.

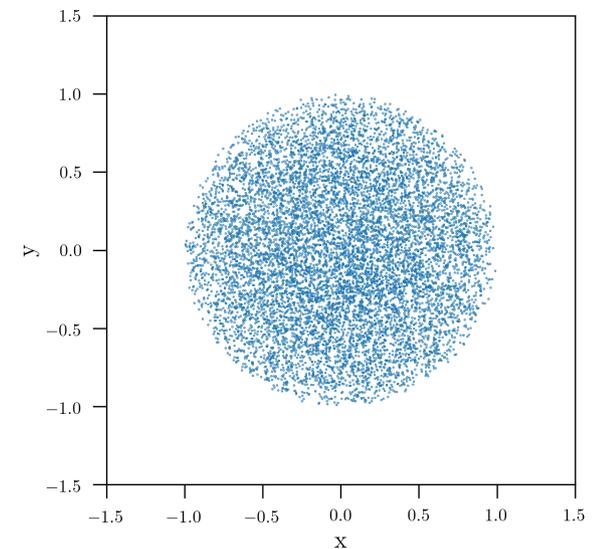
$N = 100$



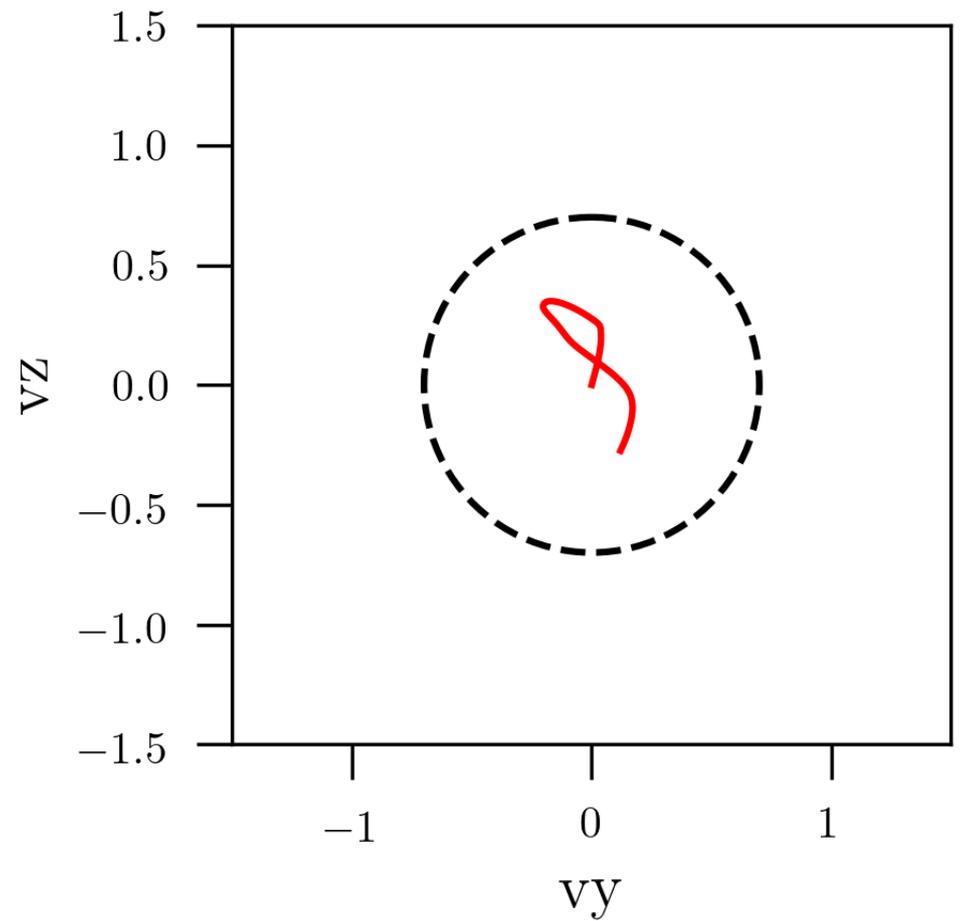
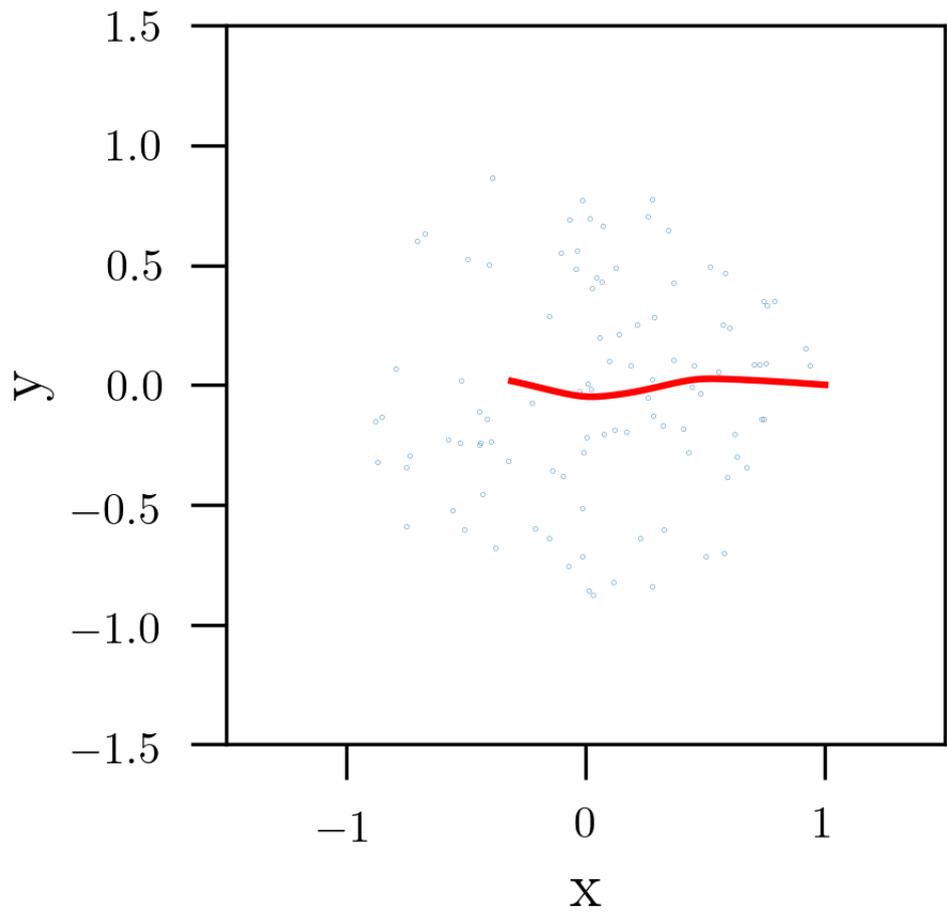
$N = 1000$



$N = 10000$

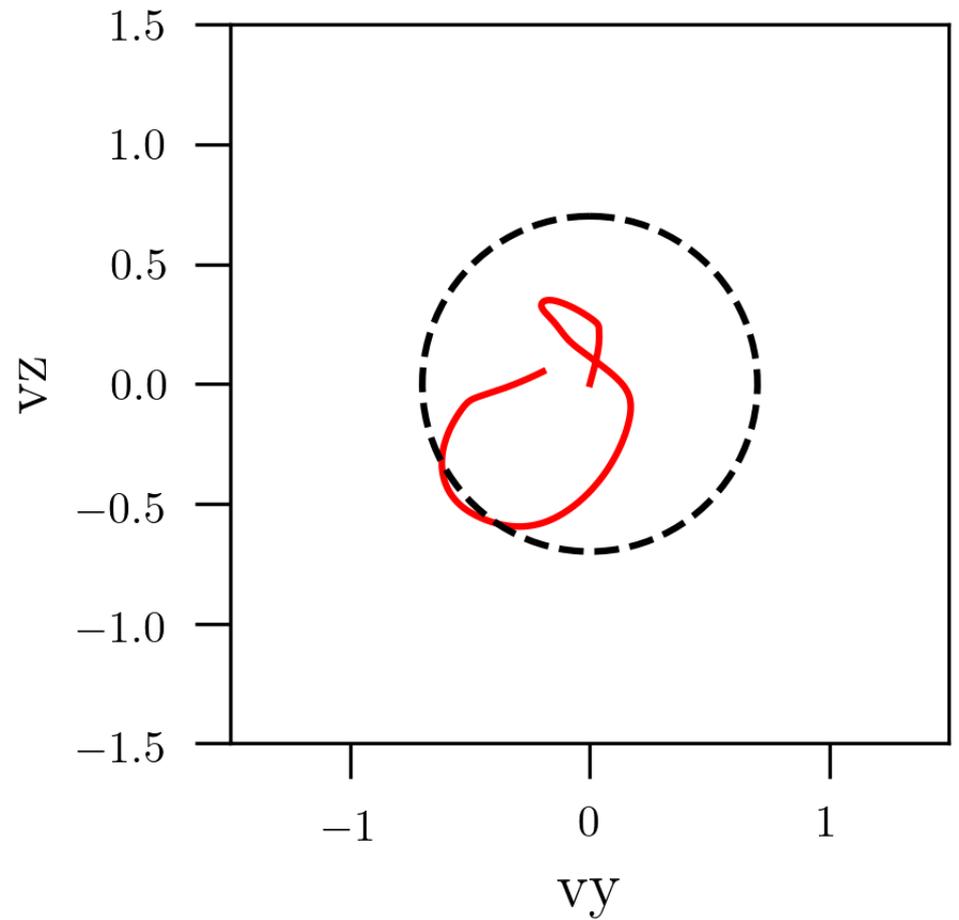
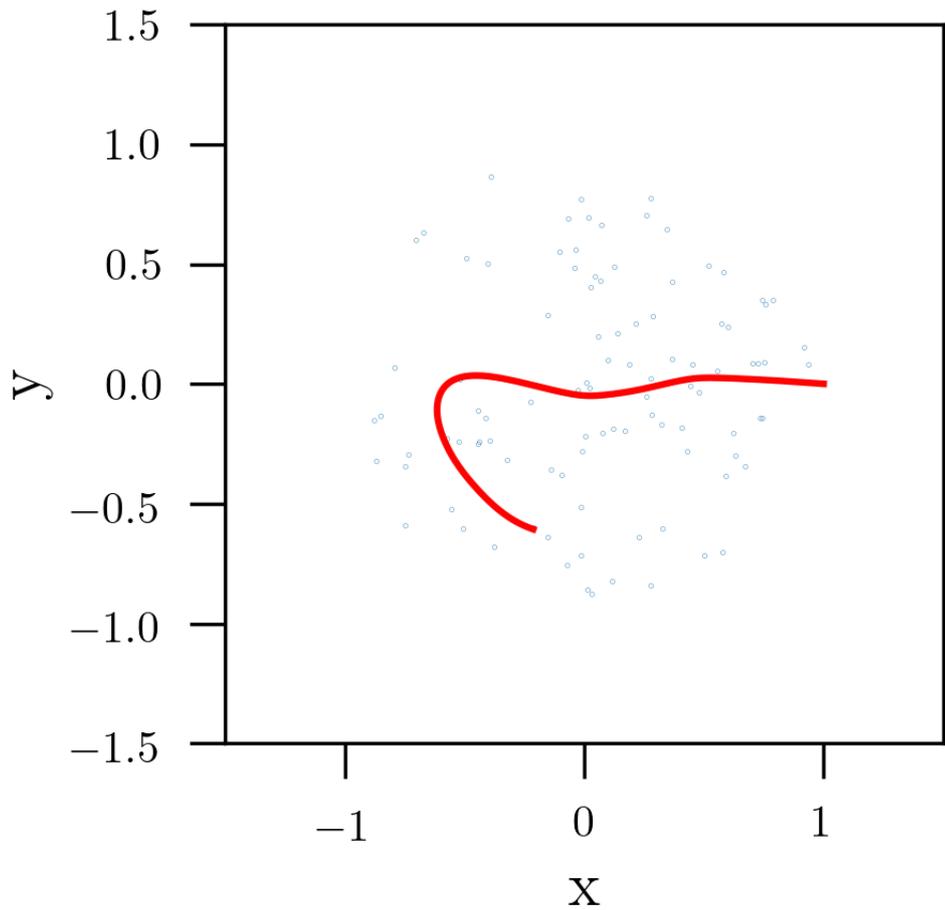


$N = 100$ Time = 2.00



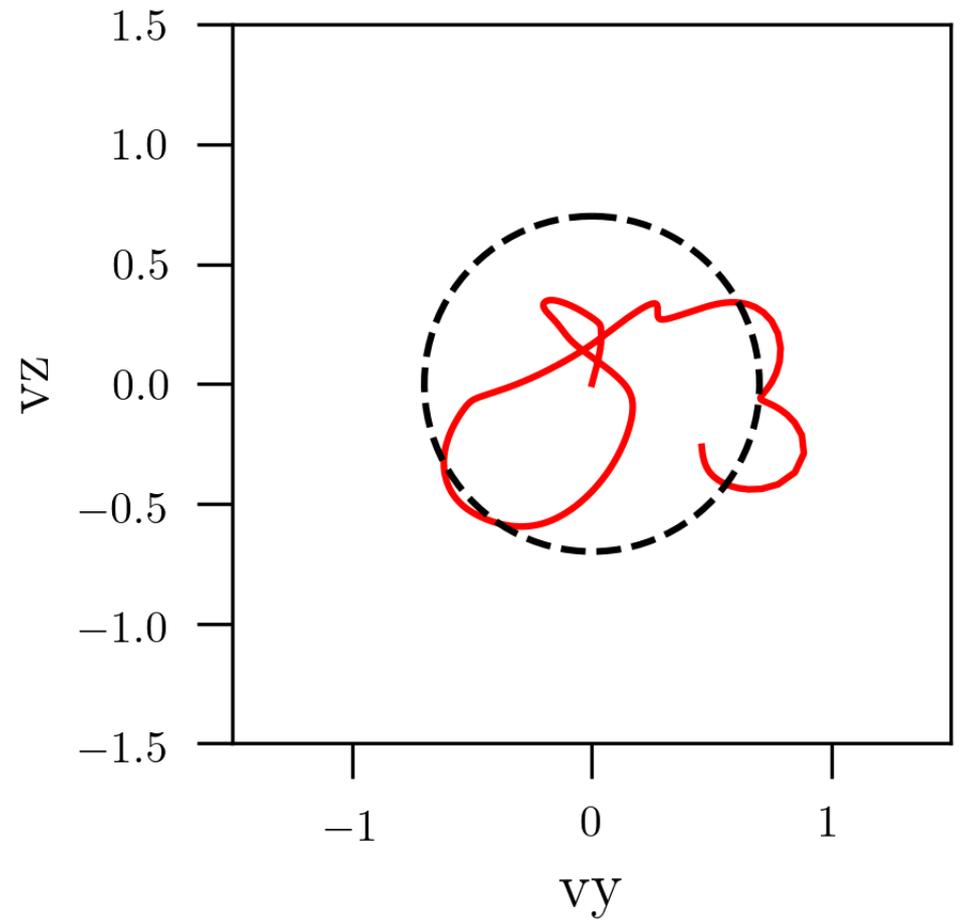
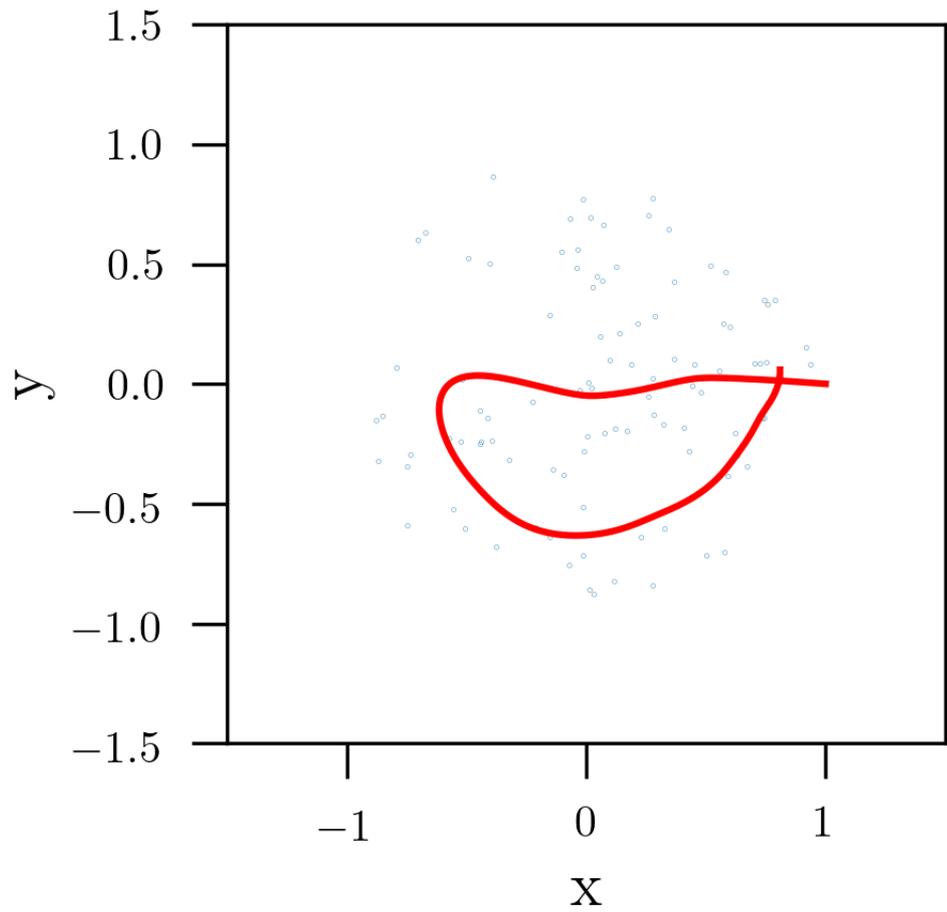
$t_{\text{relax}} = 3$

$N = 100$ Time = 4.00



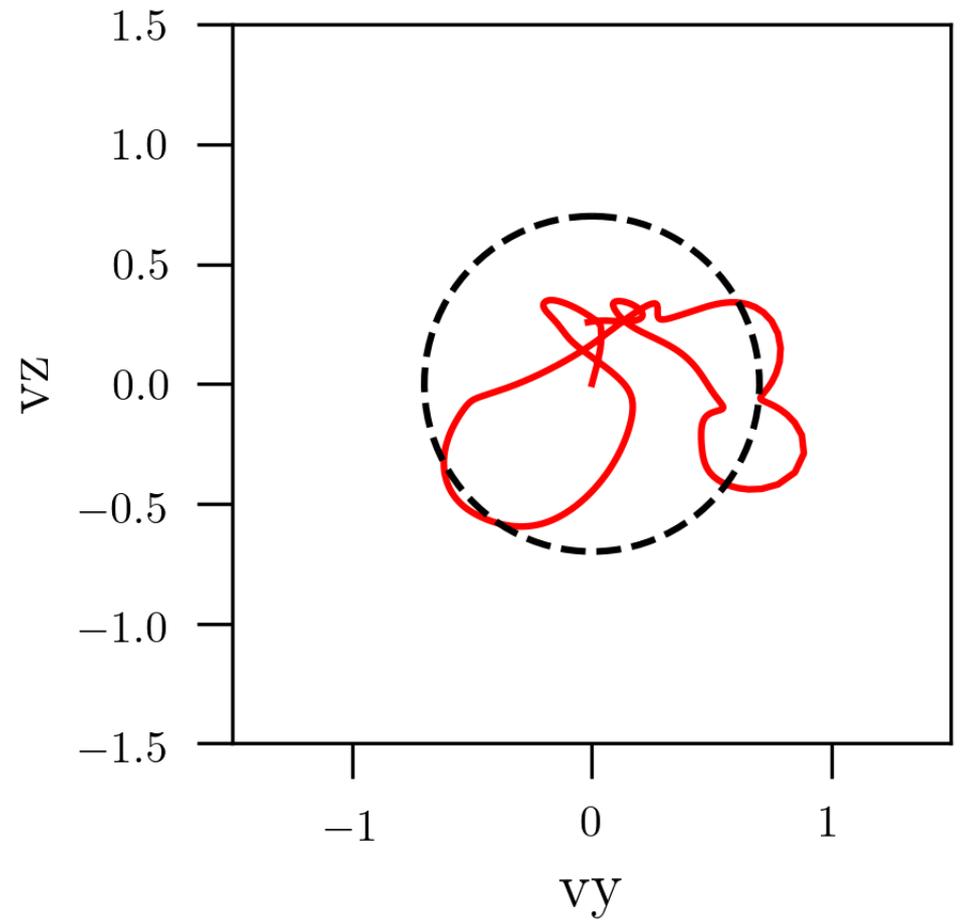
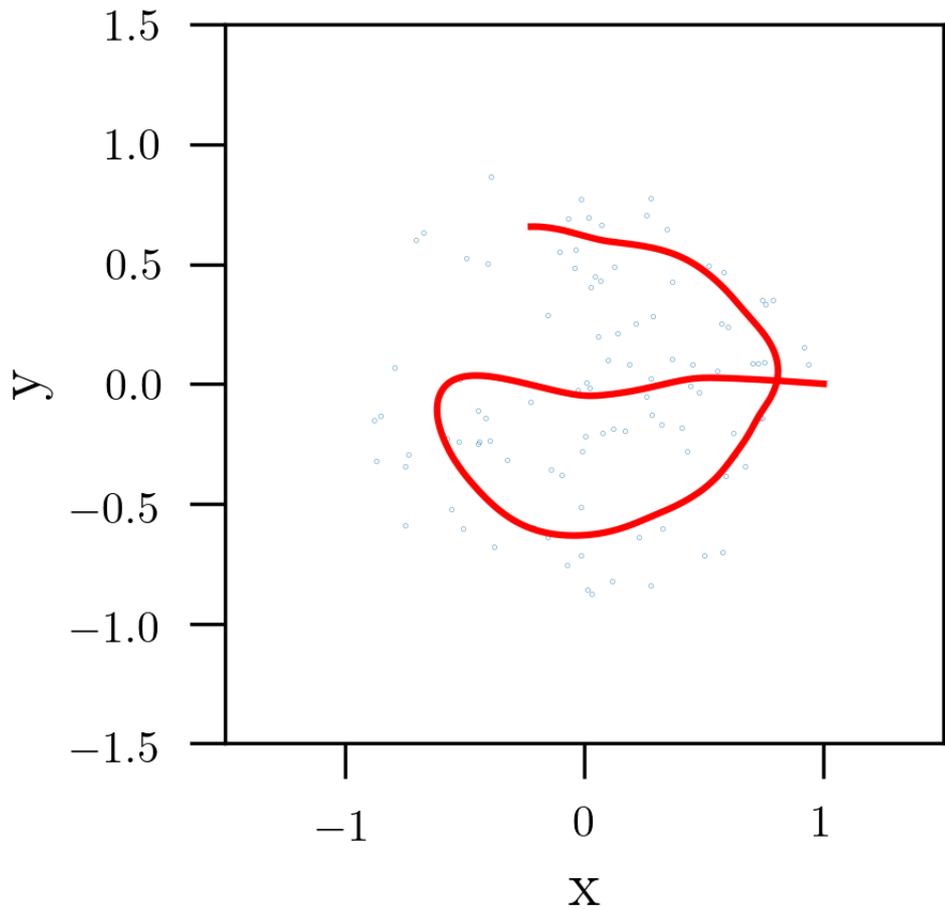
$t_{\text{relax}} = 3$

$N = 100$ Time = 6.00



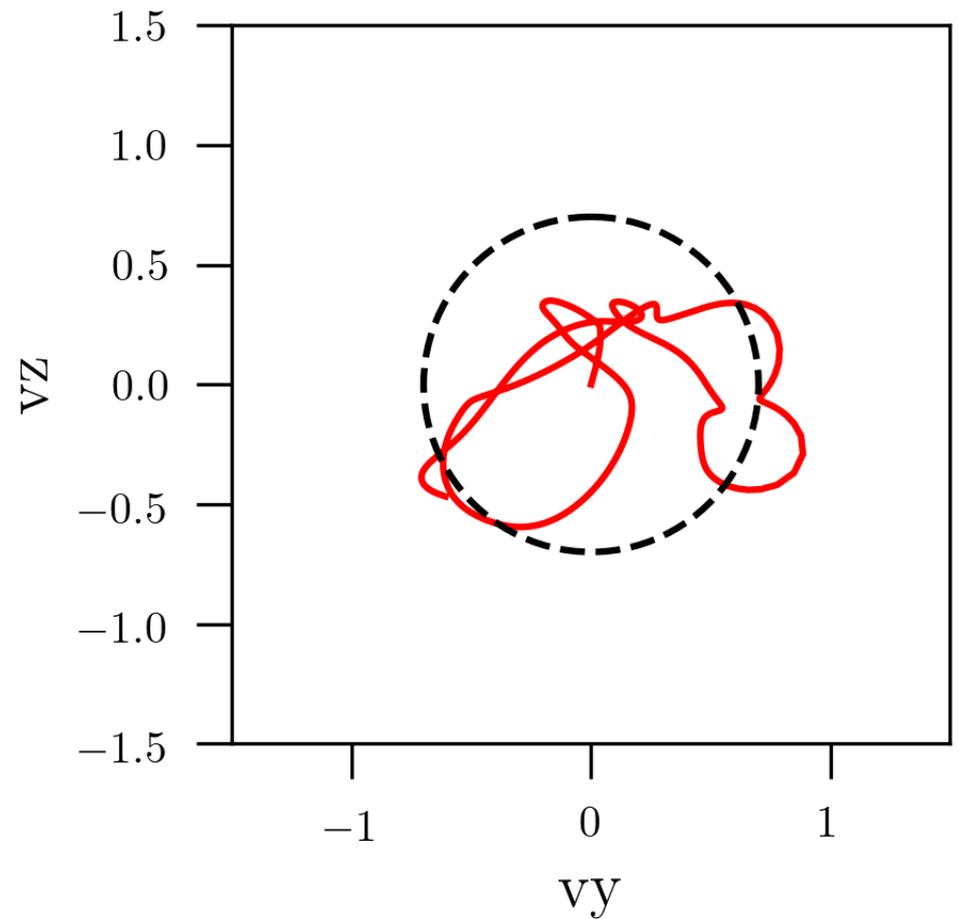
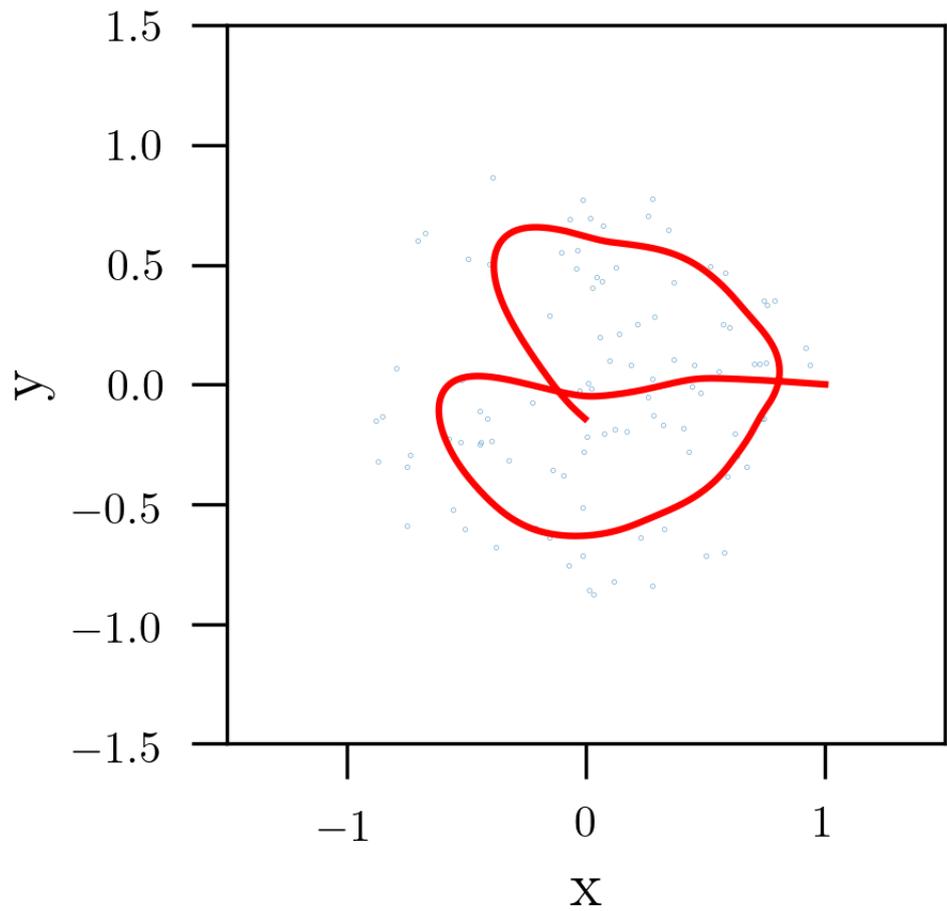
$t_{\text{relax}} = 3$

$N = 100$ Time = 8.00



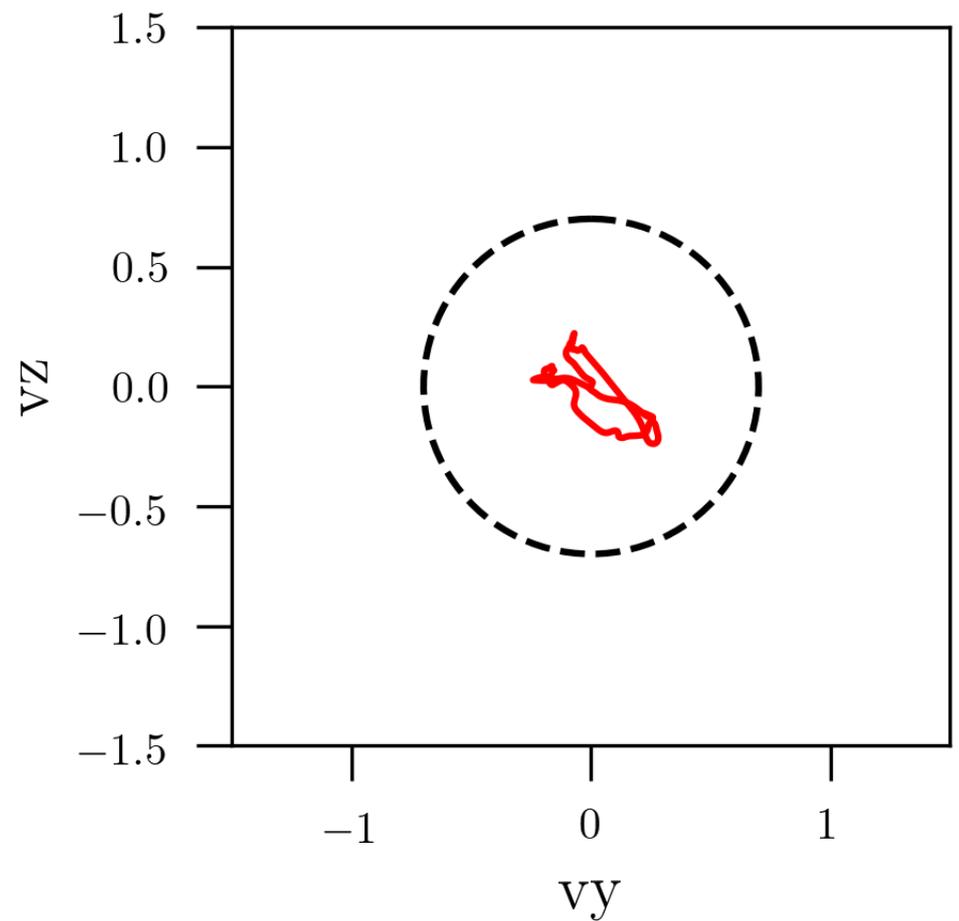
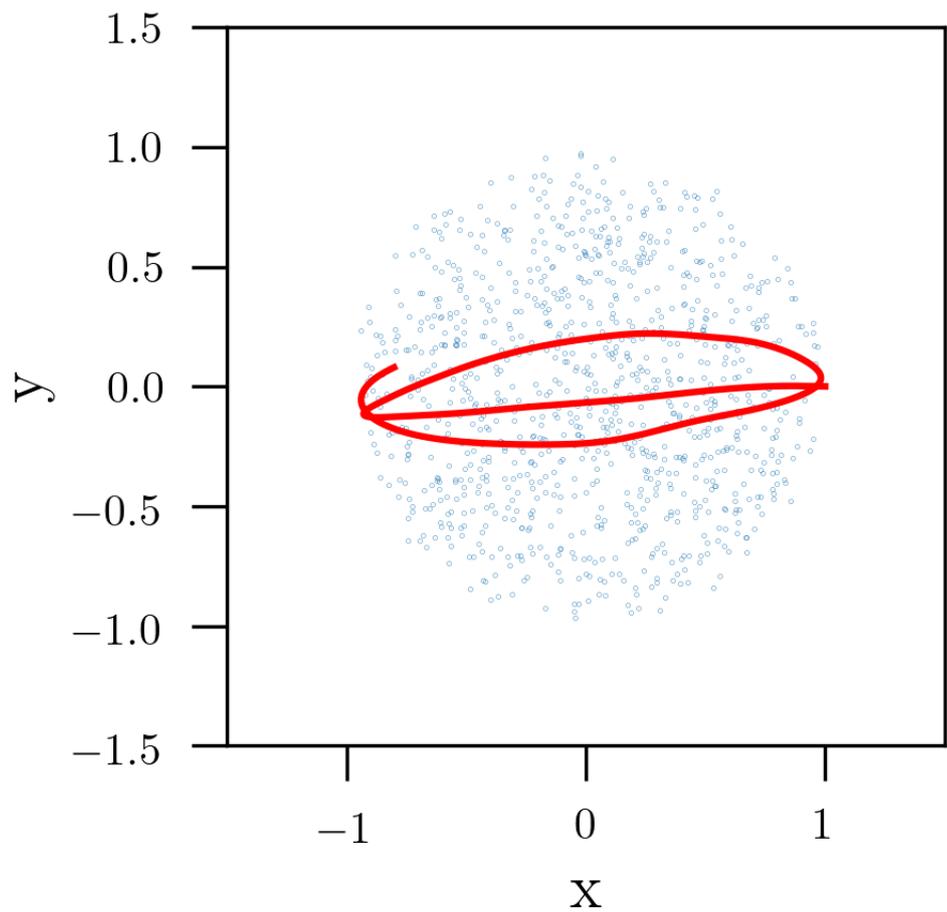
$t_{\text{relax}} = 3$

$N = 100$ Time = 10.00



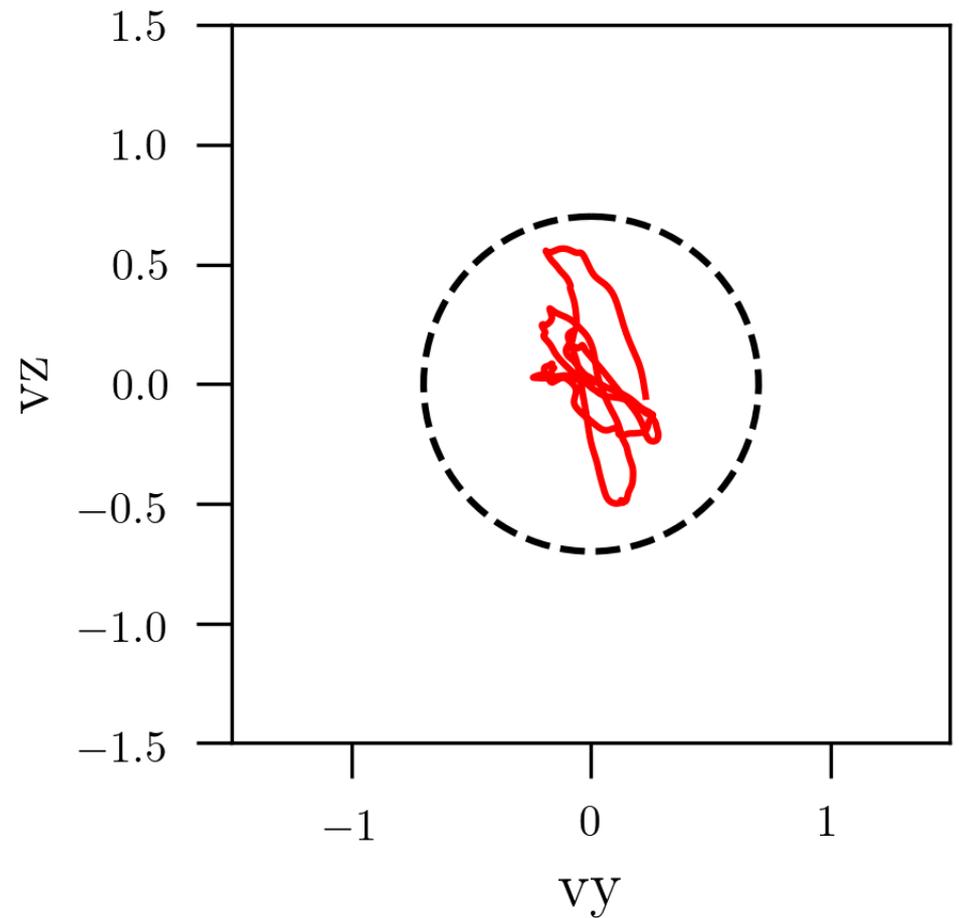
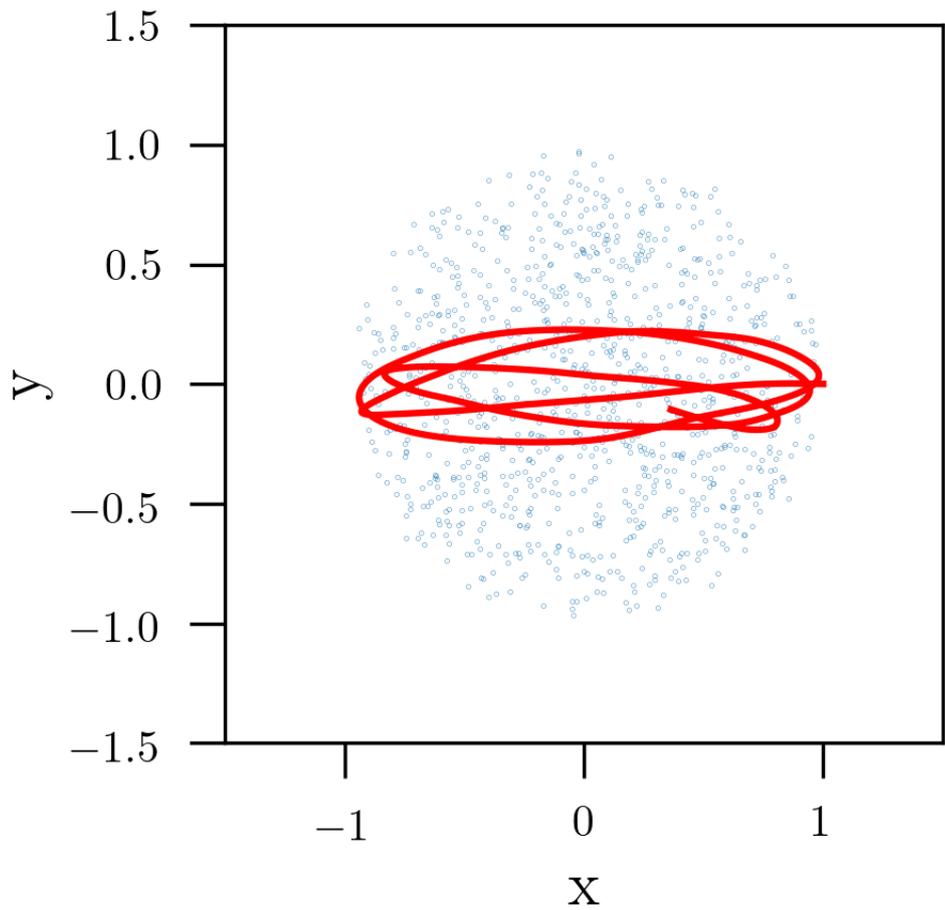
$t_{\text{relax}} = 3$

$N = 1000$ Time = 10.00



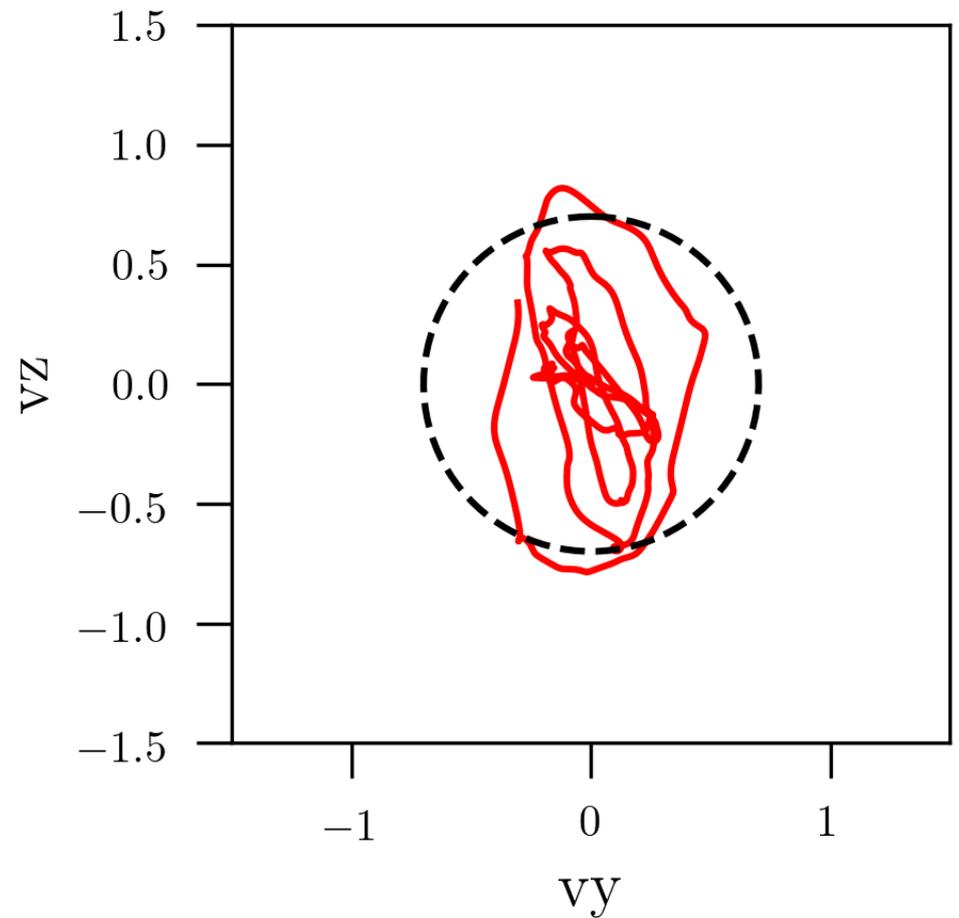
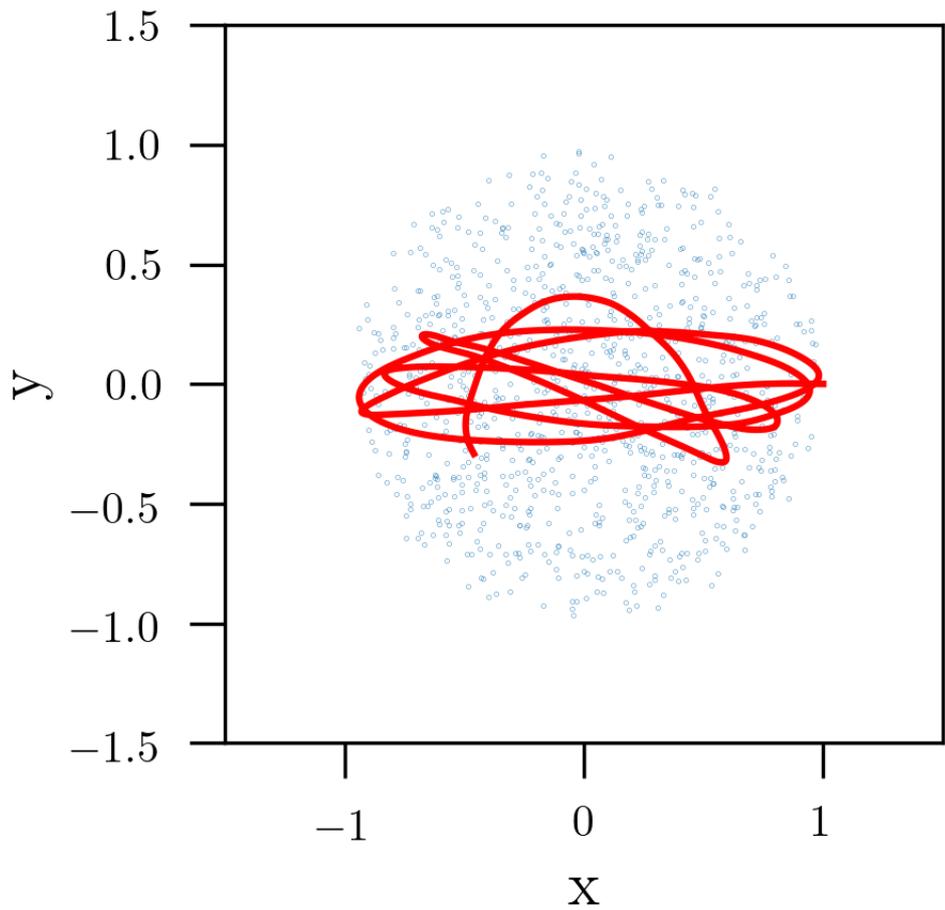
$t_{\text{relax}} = 20$

$N = 1000$ Time = 20.00



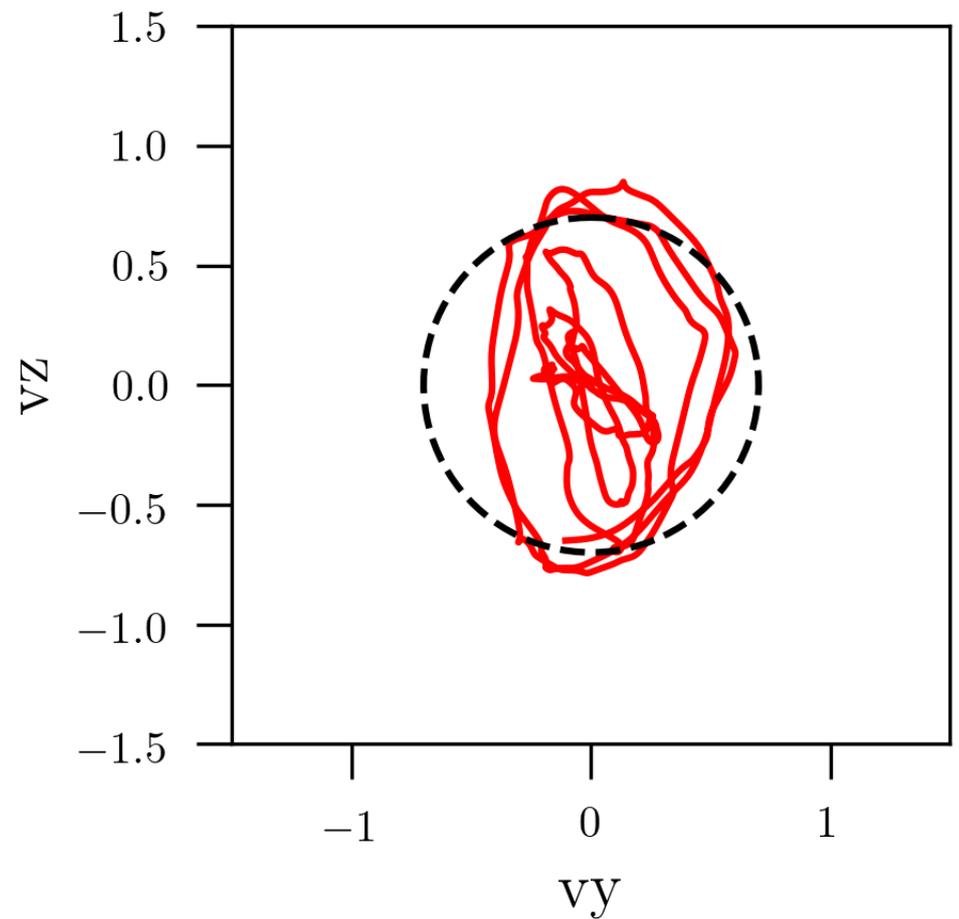
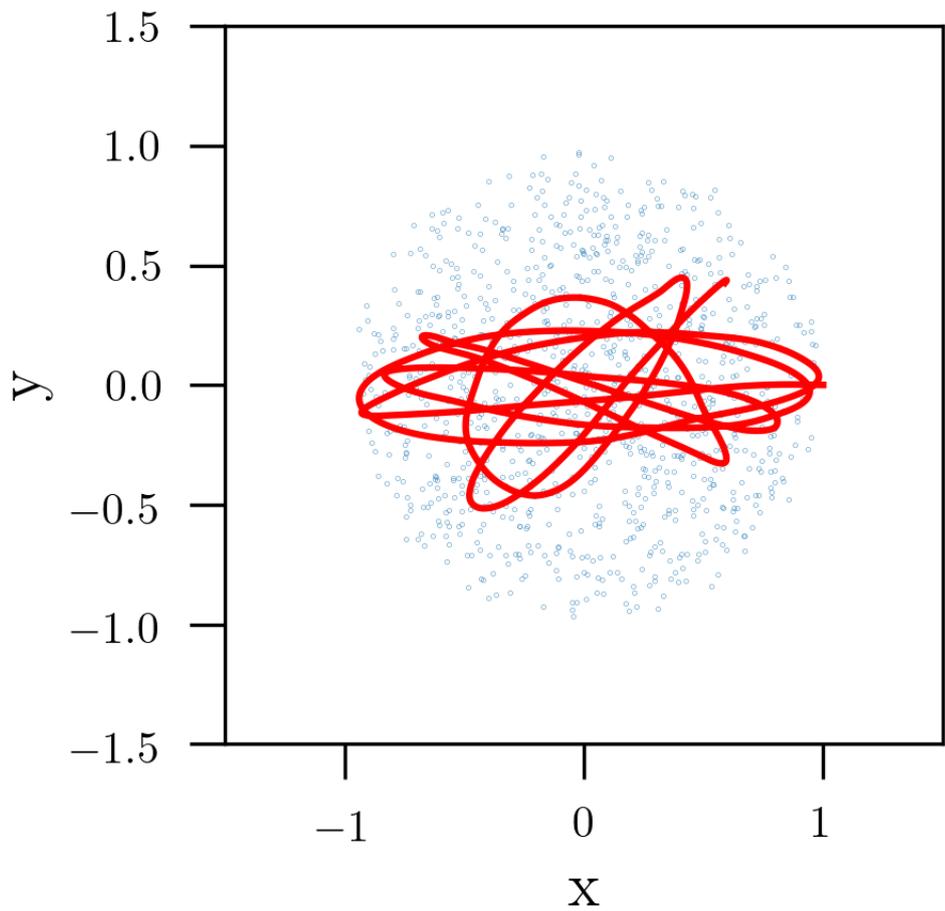
$t_{\text{relax}} = 20$

$N = 1000$ Time = 30.00



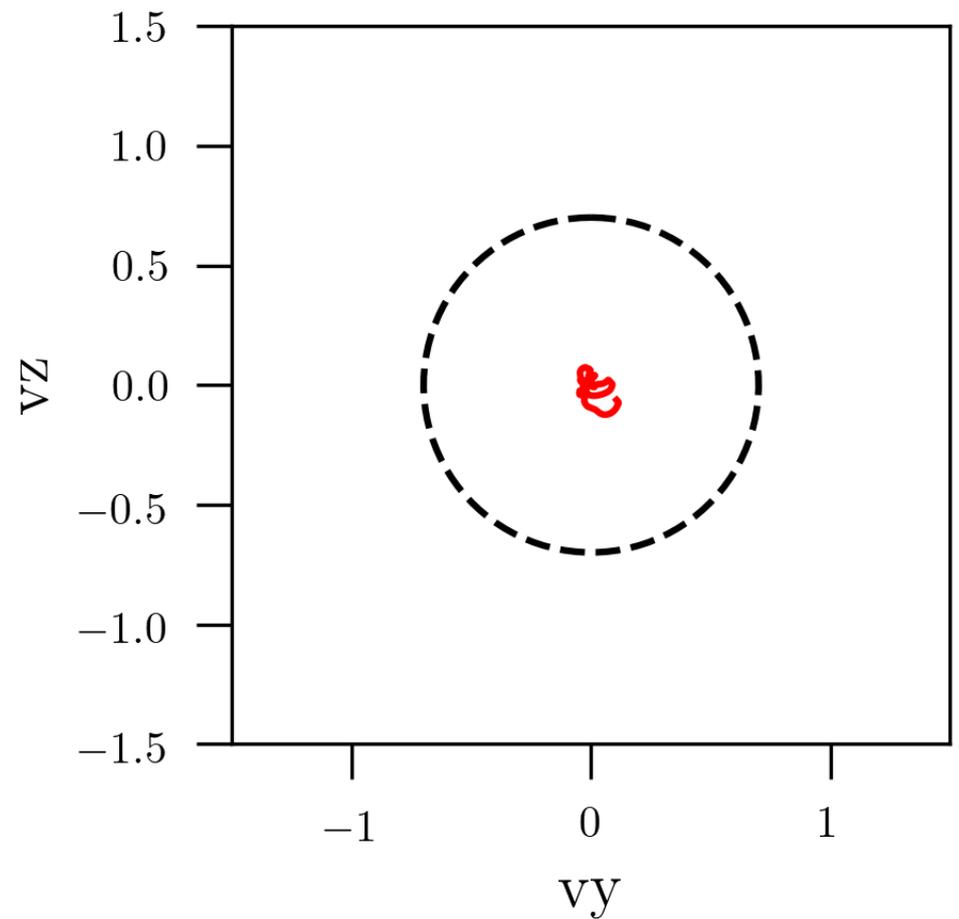
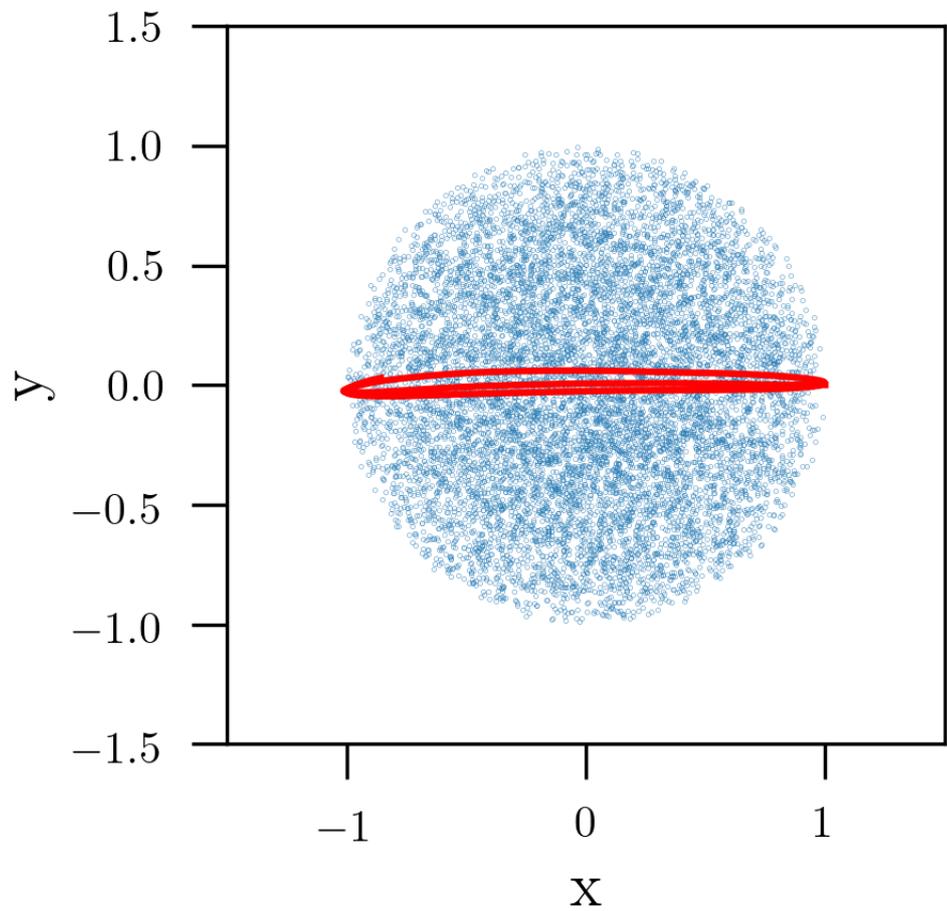
$t_{\text{relax}} = 20$

$N = 1000$ Time = 40.00



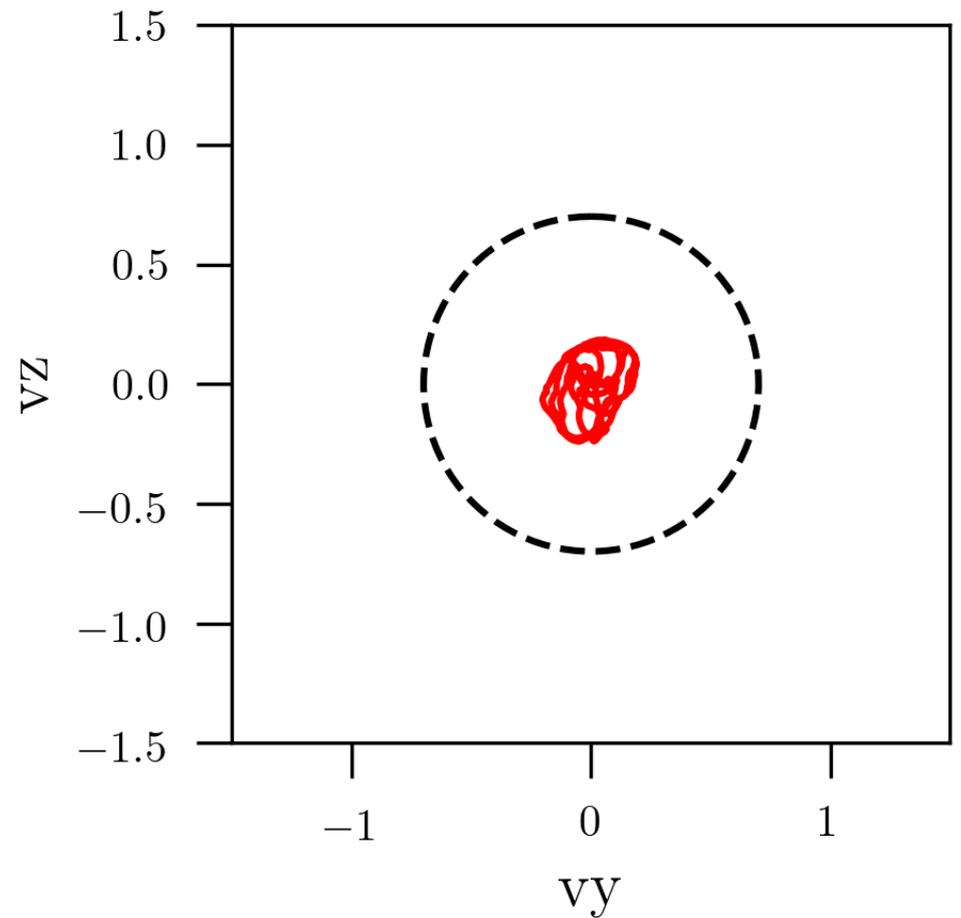
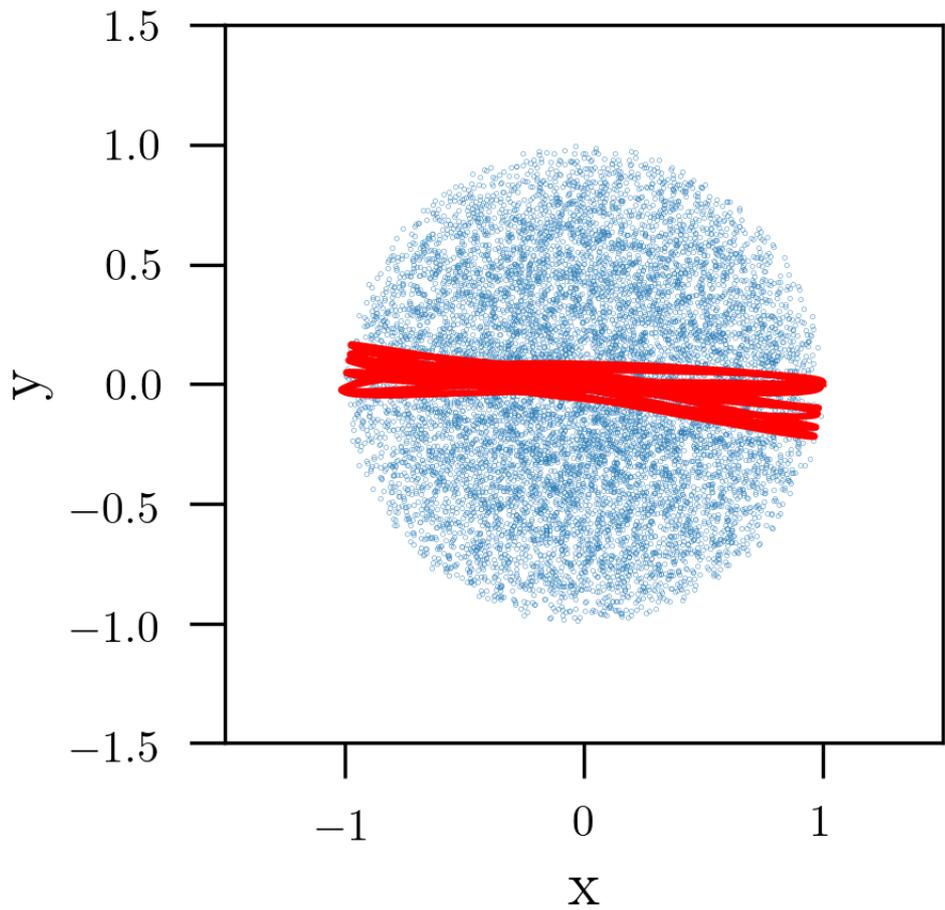
$t_{\text{relax}} = 20$

$N = 10000$ Time = 10.00



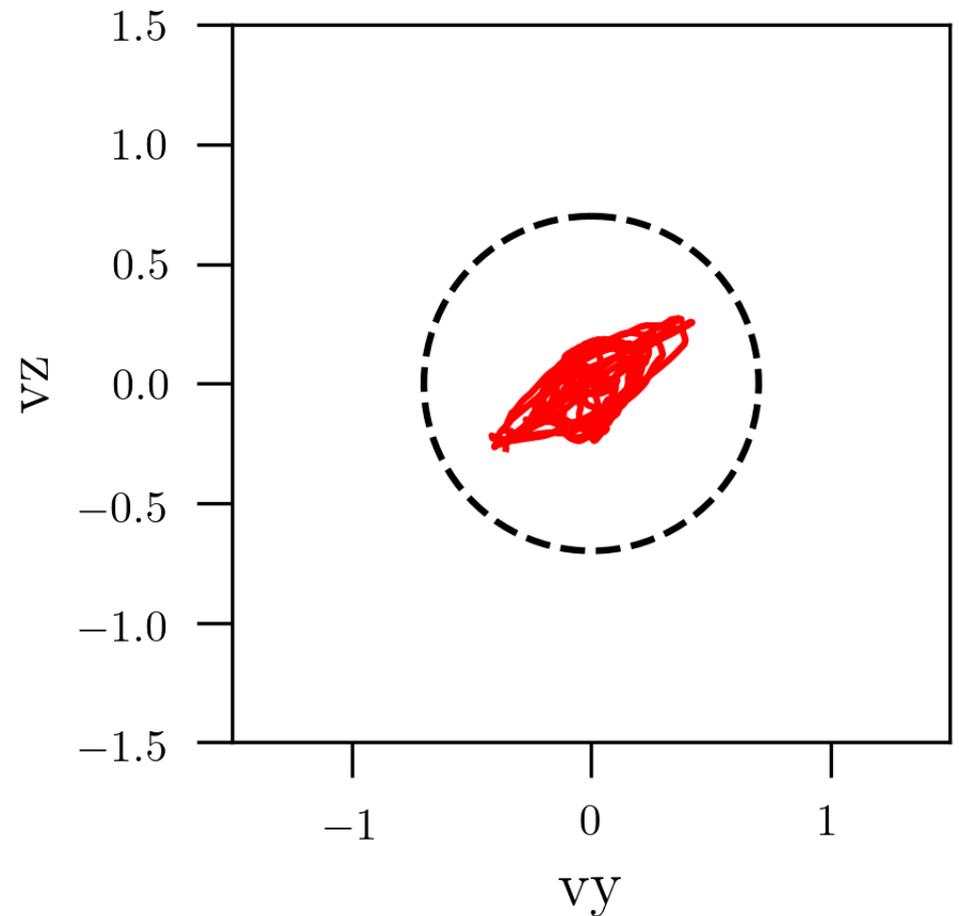
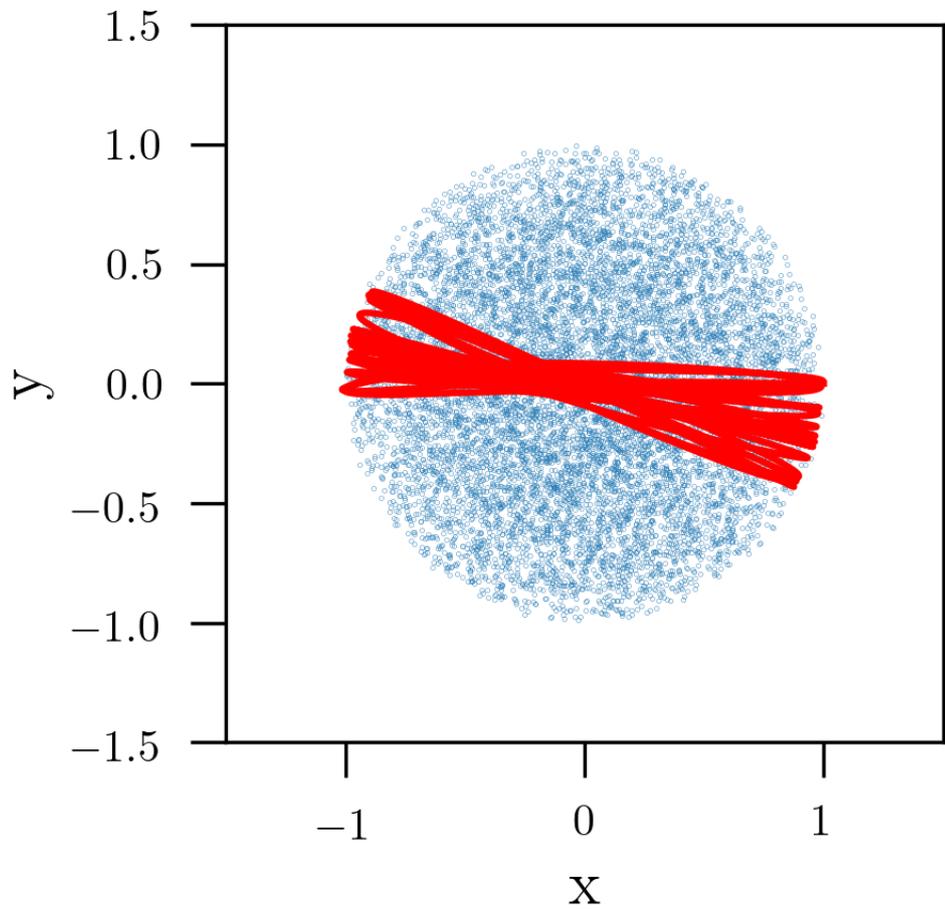
$t_{\text{relax}} = 150$

$N = 10000$ Time = 40.00



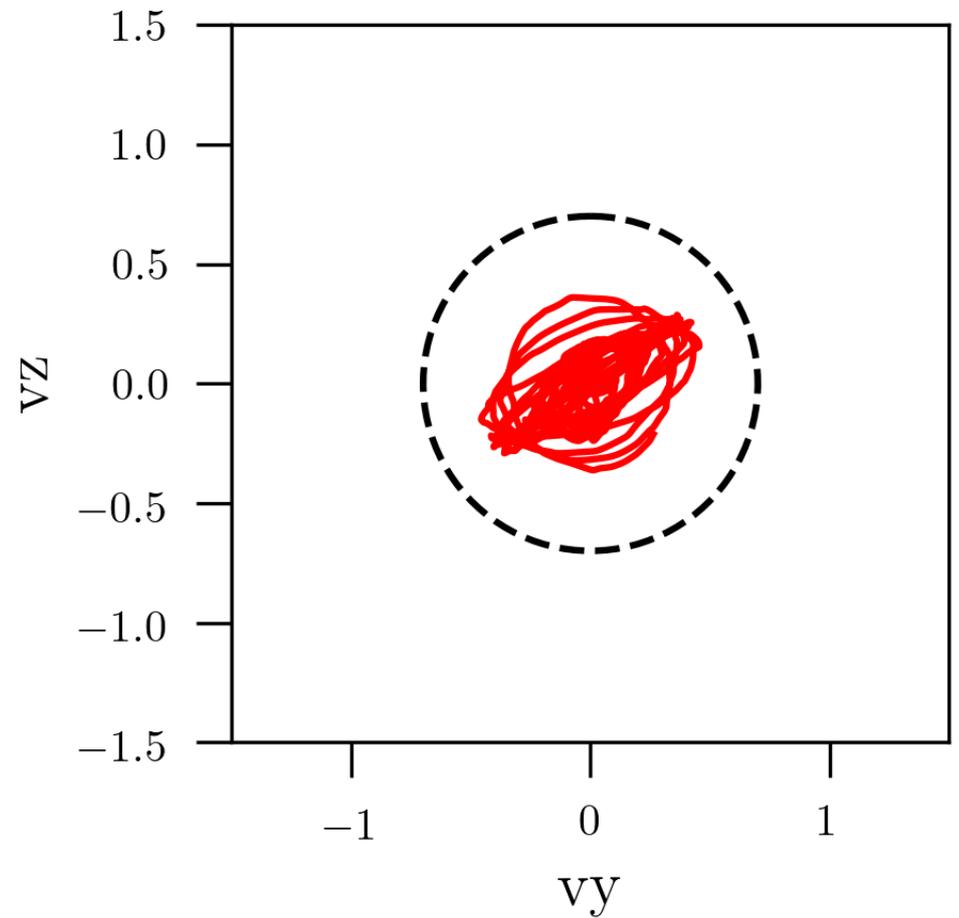
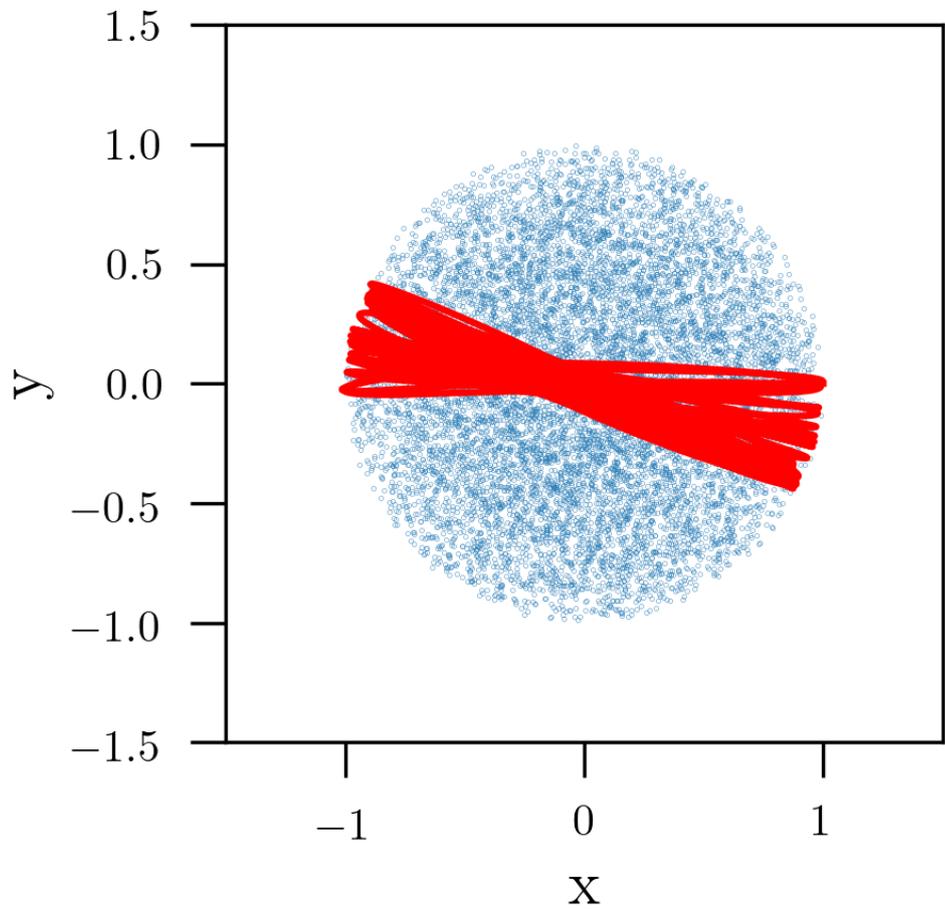
$t_{\text{relax}} = 150$

$N = 10000$ Time = 80.00



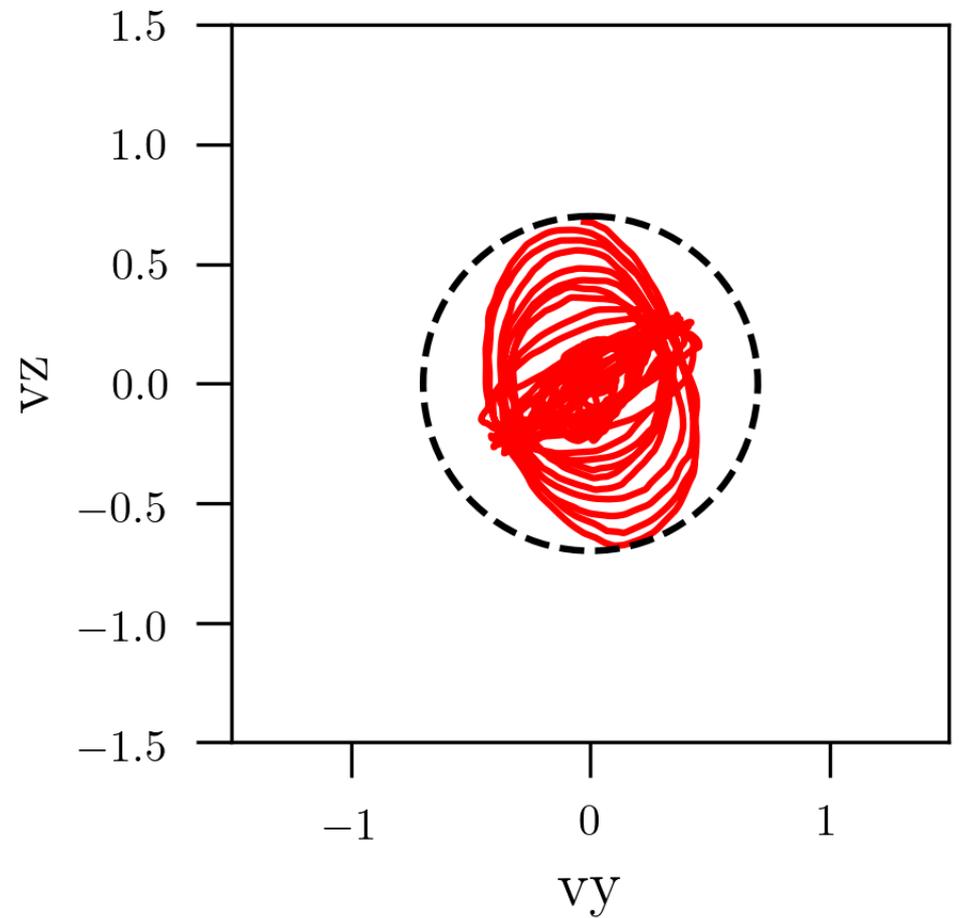
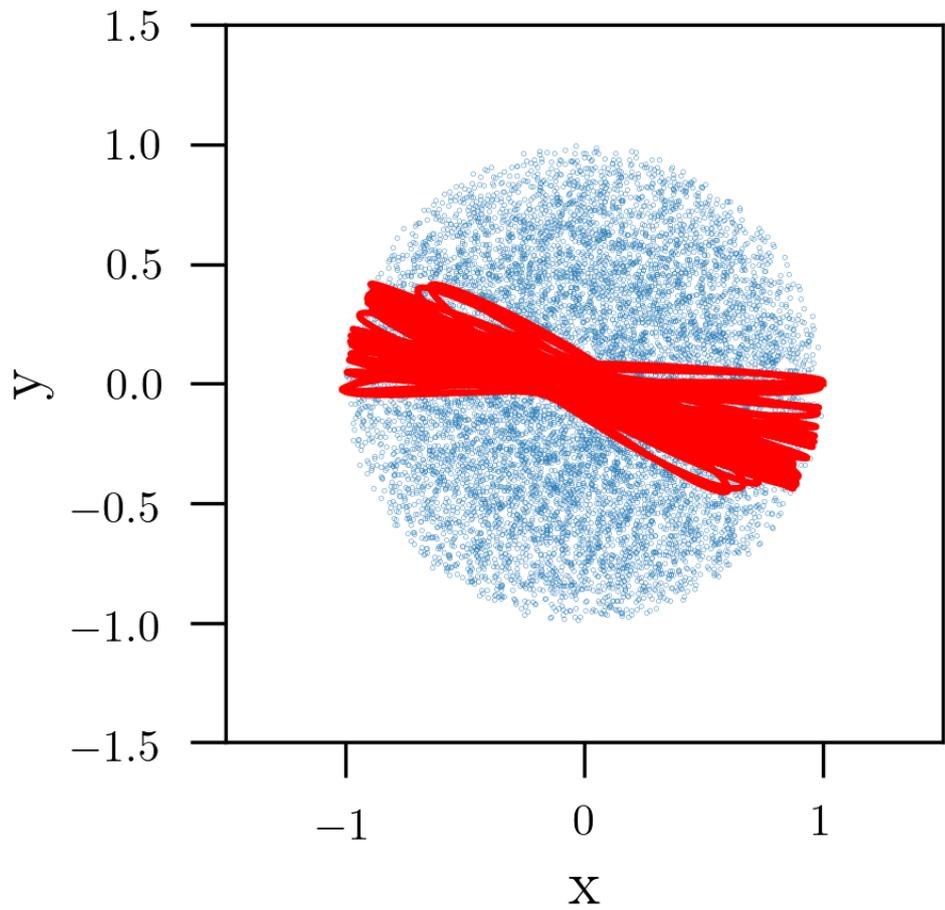
$t_{\text{relax}} = 150$

$N = 10000$ Time = 120.00



$t_{\text{relax}} = 150$

$N = 10000$ Time = 160.00



$t_{\text{relax}} = 150$

The End