Exercise Set 2
Quantum Computation

Exercise 1 *Boolean functions and classical circuits*

(a) Build a classical circuit that computes the Boolean function $f : \{0,1\}^4 \to \{0,1\}^2$ defined as follows:

$$f(x_1, x_2, x_3, x_4) = \begin{cases} (1,1) & \text{if } x_1 = x_2 \text{ and } x_3 = x_4 \\ (1,0) & \text{if } x_1 = x_2 \text{ and } x_3 \neq x_4 \\ (0,1) & \text{if } x_1 \neq x_2 \text{ and } x_3 = x_4 \\ (0,0) & \text{if } x_1 \neq x_2 \text{ and } x_3 \neq x_4 \end{cases}$$

Indication: Start by building a circuit for $f : \{0,1\}^2 \to \{0,1\}$ such that $f(x,y) = 1$ if and only if $x = y$: this circuit can then be used as a sub-circuit of the desired circuit.

(b) More difficult (?) case: build a circuit for

$$f(x_1, x_2, x_3, x_4) = \begin{cases} (1,1) & \text{if } x_1 = x_2 \text{ and } x_3 = x_4 \\ (0,1) & \text{if } x_1 = x_2 \text{ and } x_3 \neq x_4 \\ (1,0) & \text{if } x_1 \neq x_2 \text{ and } x_3 = x_4 \\ (0,0) & \text{if } x_1 \neq x_2 \text{ and } x_3 \neq x_4 \end{cases}$$

Exercise 2 *NOT, C-NOT, CC-NOT gates*

In the course, we have shown that {NOT, C-NOT, CC-NOT} is a universal set of gates, as these gates can be used to build each gate in the set {AND, OR, NOT, COPY}, which is a universal set of gates, according to Post's theorem.

In this exercise, we ask you to prove the reverse statement, namely, that each gate from the set {NOT, C-NOT, CC-NOT} can be constructed with gates in the set {AND, OR, NOT, COPY}.

Exercise 3 *Production of Bell states*

Let $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ be a composite Hilbert-Space. A state $|\psi\rangle \in \mathcal{H}$ is called a *product state* if there exist two states $|\psi_1\rangle \in \mathcal{H}_1,\ |\psi_2\rangle \in \mathcal{H}_2$ such that $|\psi\rangle$ can be written as the product

$$|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle \quad . \tag{1}$$

Whenever a state $|\psi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2$ cannot be written in such a way, we say it is *entangled.*

(a) 1. Show that a state $|\psi\rangle = \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle$ is a product state if and only if $\det\begin{pmatrix}\alpha_{00} & \alpha_{01}\\ \alpha_{10} & \alpha_{11}\end{pmatrix} = 0$.

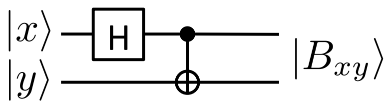
2. Which of the following states are product states / entangled states?
(i) $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ (ii) $\frac{1}{\sqrt{2}}(|00\rangle + |01\rangle)$ (iii) $\frac{1}{2}(|00\rangle + |01\rangle - |10\rangle + |11\rangle)$

(iv) $\frac{1}{2}(|00\rangle - |01\rangle - |10\rangle + |11\rangle)$ (v) $\frac{1}{2}(|00\rangle + |01\rangle - |10\rangle - |11\rangle)$

(b) Compute the four Bell states using the following identity with Dirac's notation. Do not use the component and matrix representations.

$$|B_{xy}\rangle = (CNOT)\,(H \otimes I)\;|x\rangle \otimes |y\rangle$$

where $x, y \in \{0, 1\}$ and $|B_{xy}\rangle$ are the Bell states. The circuit is the following:



Hint: $H\,|x\rangle = \frac{1}{\sqrt{2}}\,|0\rangle + \frac{1}{\sqrt{2}}(-1)^x\,|1\rangle$.

(c) Prove that the Bell states form an orthonormal basis of $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2$

(d) Which circuit U realizes the reverse operation: $|x\rangle \otimes |y\rangle = U\,|B_{xy}\rangle$?

(e) (*Coding Exercise in qBraid*) In this exercise, you will learn how to use the Python package Qiskit to build and evaluate quantum circuits. To get started, open the Jupyter notebook "Worksheet 2" (available on Moodle) in qBraid. The notebook guides you through the preparation, visualization, and measurement of a quantum circuit using a Bell state as an example. You will then apply these methods to design circuits that create other entangled states and verify their correctness.

Exercise 4 *Matrix representation of a few gates / circuits*

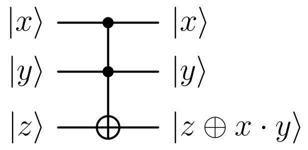
As a reminder, the matrix representation of the CNOT gate in the computational basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ of $\mathbb{C}^4$ (where the control bit is the first one) is given by

$$CNOT = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\end{pmatrix}$$

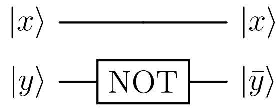
which is a unitary matrix (and also a permutation matrix, with $CNOT^{-1} = CNOT$).

Give the matrix representation in the computational basis of the following four circuits:

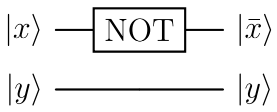
(a) CCNOT or Toffoli gate:



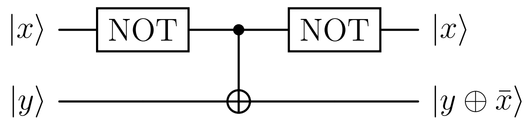
(b) NOT_y gate:



(c) NOT_x gate:



(d) A combination of gates:



Are all these matrices also permutation matrices? And are they all equal to their inverse?

Exercise 5 *Hadamard Transform of NOT-Gate*

Consider the following 1-qubit circuit: HXH (where we recall that X is an equivalent notation for the NOT gate).

(a) Is this product unitary ? Why ?

(b) Compute the output for a generic input state $|\varphi\rangle = \alpha_0 |0\rangle + \alpha_1 |1\rangle$.

(c) What do you find in the particular cases $|\varphi\rangle = |0\rangle, |1\rangle, |+\rangle, |-\rangle$?