
Exercise Set 1 : Solution
Quantum Computation

Exercise 1 *Dirac's notation for vectors and matrices*

(a) If $|w\rangle$ is a vector and α is a scalar, then

$$(\alpha |w\rangle)^\dagger = \langle w| \bar{\alpha} = \bar{\alpha} \langle w|$$

(you can check this in components). Moreover, we have linearity of transposition and complex conjugation :

$$(\alpha |v\rangle + \beta |w\rangle)^\dagger = (\alpha |v\rangle)^\dagger + (\beta |w\rangle)^\dagger.$$

(b) Then we get

$$\langle v| = (|v\rangle)^\dagger = (v_1 |e_1\rangle + v_2 |e_2\rangle + \dots + v_N |e_N\rangle)^\dagger = \bar{v}_1 \langle e_1| + \bar{v}_2 \langle e_2| + \dots + \bar{v}_N \langle e_N|.$$

(c) If $\langle v| = \sum_{i=1}^N \bar{v}_i \langle e_i|$ and $|w\rangle = \sum_{j=1}^N w_j |e_j\rangle$, then

$$\langle v|w\rangle = \sum_{i=1}^N \sum_{j=1}^N \bar{v}_i w_j \langle e_i|e_j\rangle = \sum_{i=1}^N \sum_{j=1}^N \bar{v}_i w_j \delta_{ij} = \sum_{i=1}^N \bar{v}_i w_i.$$

(d) For $\vec{v} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$, we have $\|\vec{v}\|^2 = \vec{v}^\dagger \cdot \vec{v}$, so $\|\vec{v}\|^2 = \bar{\alpha} \alpha + \bar{\beta} \beta$. On the other hand, $\langle v|v\rangle = \bar{\alpha} \alpha + \bar{\beta} \beta$ also by (c).

(e) Using components we have :

$$|e_k\rangle \langle e_l| = \text{k-th pos} \left\{ \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \underbrace{(0 \dots 0 1 0 \dots 0)}_{\text{l-th pos}} = \begin{pmatrix} 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & 1 & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

(f) Thus,

$$A = \sum_{k,l} a_{kl} |e_k\rangle \langle e_l|.$$

So,

$$\langle e_i| A |e_j\rangle = \sum_{k,l} a_{kl} \langle e_i|e_k\rangle \langle e_l|e_j\rangle = \sum_{k,l} a_{kl} \delta_{ik} \delta_{lj} = a_{ij}.$$

(g) From part (e), we have

$$I = \sum_{i=1}^N |e_i\rangle \langle e_i|.$$

Indeed, $|e_i\rangle \langle e_i|$ is the matrix with 1 at the i -th row and i -th column and zeros elsewhere. This is called the closure relation, and it is valid for any orthonormal basis, as the following computation shows : if $\{|\varphi_i\rangle\}_{i=1\dots N}$ are orthonormal, there exists a unitary basis change (a “rotation”) such that

$$|\varphi_i\rangle = U |e_i\rangle \quad \text{and therefore also} \quad \langle \varphi_i| = \langle e_i| U^\dagger.$$

Then from $I = \sum_{i=1}^N |e_i\rangle \langle e_i|$, we get : $UIU^\dagger = \sum_{i=1}^N U |e_i\rangle \langle e_i| U^\dagger$, so

$$I = \sum_{i=1}^N |\varphi_i\rangle \langle \varphi_i|.$$

(h) From $\alpha_i |\varphi_i\rangle = A |\varphi_i\rangle$, we get directly

$$\sum_{i=1}^N \alpha_i |\varphi_i\rangle \langle \varphi_i| = \sum_{i=1}^N A |\varphi_i\rangle \langle \varphi_i| = A \sum_{i=1}^N |\varphi_i\rangle \langle \varphi_i| = AI = A.$$

Exercise 2 Matrices in Hilbert Spaces

(a) Let $|v\rangle \in \mathbb{C}^N$ be a nonzero eigenvector of H with an eigenvalue $\lambda \in \mathbb{C}$, so that $H|v\rangle = \lambda|v\rangle$. Then

$$\lambda \|v\|^2 = \langle v| H |v\rangle = \langle v| H^\dagger |v\rangle = \bar{\lambda} \|v\|^2. \quad (1)$$

Since $|v\rangle \neq 0$ its norm does not vanish. We get that $\lambda = \bar{\lambda}$, or equivalently $\Im[\lambda] = 0$.

(b) Let $|v\rangle$ be a nonzero eigenvector of U with an eigenvalue $\lambda \in \mathbb{C}$. Then

$$\langle v|v\rangle = (\langle v| U^\dagger)(U |v\rangle) = \langle v| \bar{\lambda} \lambda |v\rangle = \bar{\lambda} \lambda \langle v|v\rangle. \quad (2)$$

Since $|v\rangle \neq 0$ its norm does not vanish and we get $|\lambda|^2 = 1$. This implies that $|\lambda| = 1$ as required.

(c) Let $|v\rangle$ be a nonzero eigenvector of P with an eigenvalue $\lambda \in \mathbb{C}$. Then

$$\lambda |v\rangle = P |v\rangle = P^2 |v\rangle = \lambda^2 |v\rangle. \quad (3)$$

Since $|v\rangle$ is not the zero vector we get $\lambda^2 = \lambda$, which implies that $\lambda \in \{0, 1\}$.

(d) Let $|v\rangle$ be a nonzero eigenvector of A with an eigenvalue $\lambda \in \mathbb{C}$. Then

$$0 \leq \langle v| A |v\rangle = \lambda \|v\|^2. \quad (4)$$

Since $|v\rangle \neq 0$ its norm does not vanish, and we get that $\lambda \geq 0$.

Exercise 3 *Tensor Product in Dirac's notation*

(a) By distributivity of the tensor product (first two properties), it follows that :

$$|v\rangle_1 \otimes |w\rangle_2 = \left(\sum_{i=1}^N v_i |e_i\rangle_1 \right) \otimes \left(\sum_{j=1}^M w_j |f_j\rangle_2 \right) = \sum_{i=1}^N \sum_{j=1}^M v_i w_j |e_i\rangle_1 \otimes |f_j\rangle_2.$$

(b) Take two vectors $|e_i, f_j\rangle$ and $|e_k, f_l\rangle$ of $\mathcal{H}_1 \otimes \mathcal{H}_2$. Then by definition of the inner product :

$$\langle e_k, f_l | e_i, f_j \rangle = \langle e_k | e_i \rangle \cdot \langle f_l | f_j \rangle = \delta_{ki} \cdot \delta_{lj}.$$

So this equals one if and only if $(k, l) = (i, j)$ and zero otherwise. This means that $\{ |e_i, f_j\rangle ; i = 1 \dots N ; j = 1 \dots M \}$ is an orthonormal basis of $\mathcal{H}_1 \otimes \mathcal{H}_2$. The dimension equals the number of basis vectors, so is $N \cdot M$, the product of $\dim \mathcal{H}_1$ and $\dim \mathcal{H}_2$.

(c) We apply the definition

$$A \otimes B |\Psi\rangle = \sum_{i,j} \psi_{ij} A |e_i\rangle_1 \otimes B |f_j\rangle_2$$

to $|\Psi\rangle = |e_k, f_l\rangle$. So $\psi_{ij} = 1$ for $(i, j) = (k, l)$ and 0 otherwise. This means :

$$A \otimes B |e_k, f_l\rangle = A |e_k\rangle \otimes B |f_l\rangle$$

and multiplying by $\langle e_i, f_j |$, we find :

$$\begin{aligned} \langle e_i, f_j | A \otimes B |e_k, f_l\rangle &= (\langle e_i | \otimes \langle f_j |) (A |e_k\rangle \otimes B |f_l\rangle) \\ &= \langle e_i | A |e_k\rangle \langle f_j | B |f_l\rangle = a_{ik} b_{jl}. \end{aligned}$$

(d) The formulas follow by translating the formulas found in (a) and (c) to the component notation.