Exercise Set 1
Quantum Computation

Exercise 1 *Dirac's notation for vectors and matrices*

Let $\mathcal{H} = \mathbb{C}^N$ be a vector space of N dimensional vectors with complex components. Let

$$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_N \end{pmatrix}$$

be a column vector. We define its "conjugate" as

$$\vec{v}^\dagger = (\overline{v}_1, \ldots, \overline{v}_N)$$

where $\overline{z}$ stands for the complex conjugate of $z \in \mathbb{C}$. So $\vec{v}^\dagger$ is obtained by transposition and complex conjugation (if the components are real this reduces to the usual transposed vector). The inner or scalar product is by definition

$$\vec{v}^\dagger \cdot \vec{w} = \overline{v}_1\, w_1 + \ldots + \overline{v}_N\, w_N$$

and the norm is

$$\|\vec{v}\|^2 = \vec{v}^\dagger \cdot \vec{v} = \overline{v}_1\, v_1 + \ldots + \overline{v}_N\, v_N = |v_1|^2 + \ldots + |v_N|^2$$

In Dirac's notation we write $\vec{v} = |v\rangle$ and $\vec{v}^\dagger = \langle v|$. Therefore the inner product becomes

$$\langle v|w\rangle = \overline{v}_1\, w_1 + \ldots + \overline{v}_N\, w_N$$

The canonical orthonormal basis vectors are by definition

$$\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \vec{e}_N = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

In Dirac's notation the orthonormality of the basis vectors is expressed as

$$\langle e_i|e_j\rangle = \delta_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j. \end{cases}$$

The expansion of a vector on this basis is

$$|v\rangle = v_1\,|e_1\rangle + v_2\,|e_2\rangle + \ldots + v_N\,|e_N\rangle$$

Now you will check a few easy facts of linear algebra and translate them in Dirac's notation.

(a) Check from the definitions above that if $|v\rangle = \alpha\,|v'\rangle + \beta\,|v''\rangle$ then

$$\langle v| = \overline{\alpha}\ \langle v'| + \overline{\beta}\ \langle v''|\,.$$

(b) In particular deduce that if $|v\rangle = v_1\,|e_1\rangle + v_2\,|e_2\rangle + \ldots + v_N\,|e_N\rangle$ then

$$\langle v| = \overline{v}_1\ \langle e_1| + \overline{v}_2\ \langle e_2| + \ldots + \overline{v}_N\ \langle e_N|\,.$$

(c) Show directly in Dirac notation that

$$\langle v|w\rangle = \overline{v}_1\, w_1 + \ldots + \overline{v}_N\, w_N.$$

(d) Deduce that $\sqrt{\langle v|v\rangle} = \|v\|$.

(e) Consider the ket-bra expression $|e_i\rangle\,\langle e_j|$ for canonical basis vectors. Write this as an $N \times N$ matrix.

(f) Consider now an $N \times N$ matrix A with complex matrix elements $a_{ij}\,;\ i = 1\ldots N\,;$ $j = 1\ldots N$. Deduce from the above question that

$$A = \sum_{i,j=1}^{N} a_{ij}\,|e_i\rangle\,\langle e_j|$$

and also that

$$a_{ij} = \langle e_i|\,A\,|e_j\rangle\,.$$

(g) In particular verify that the identity matrix satisfies :

$$I = \sum_{i=1}^{N} |e_i\rangle\,\langle e_i|\,.$$

This is called the closure relation. Deduce that in fact this relation is valid for any orthonormal basis of vectors $|\varphi_i\rangle$, $i = 1, \ldots, N$.

(h) (Spectral theorem) Let $A = A^\dagger$. This is called a *Hermitian matrix.* An important theorem of linear algebra states that : “any Hermitian matrix has N orthonormal eigenvectors”. Let $|\varphi_i\rangle\,, \alpha_i,\ i = 1, \ldots, N$ be the eigenvectors and eigenvalues of A, *i.e.*,

$$A\,|\varphi_i\rangle = \alpha_i\,|\varphi_i\rangle\,.$$

Prove directly in Dirac’s notation that

$$A = \sum_{i=1}^{N} \alpha_i\,|\varphi_i\rangle\,\langle\varphi_i|\,.$$

This “expansion” is often called the spectral expansion (or theorem).

Exercise 2 *Matrices in Hilbert Spaces*

Let $\mathcal{H} = \mathbb{C}^N$ be a Hilbert space.

(a) Let H be a Hermitian matrix on $\mathcal{H}$, meaning $H^\dagger = H$. Show that all the eigenvalues of H are real.

(b) Let U be a unitary matrix on $\mathcal{H}$, meaning that U is invertible and $U^\dagger = U^{-1}$. Show that all the eigenvalues λ of U lie in the unit circle, i.e. $|\lambda| = 1$.

(c) Let P be a projection, meaning $P^2 = P$. Show that the eigenvalues of P are either 0 or 1 .

(d) Let A be a positive-semidefinite matrix, meaning that for all $|v\rangle \in \mathcal{H}$,

$$\langle v| A |v\rangle \geq 0 \, . \tag{1}$$

Show that the eigenvalues of A are non-negative.

Exercise 3 *Tensor Product in Dirac's notation*

Let $\mathcal{H}_1 = \mathbb{C}^N$ and $\mathcal{H}_2 = \mathbb{C}^M$ be N and M dimensional Hilbert spaces. The tensor product space $\mathcal{H}_1 \otimes \mathcal{H}_2$ is a new Hilbert space formed by "pairs of vectors" denoted as $|v\rangle_1 \otimes |w\rangle_2 \equiv |v, w\rangle$ with the properties :

- $\big(\alpha |v\rangle_1 + \beta |v'\rangle_1\big) \otimes |w\rangle_2 = \alpha |v\rangle_1 \otimes |w\rangle_2 + \beta |v'\rangle_1 \otimes |w\rangle_2$,
- $|v\rangle_1 \otimes \big(\alpha |w\rangle_2 + \beta |w'\rangle_2\big) = \alpha |v\rangle_1 \otimes |w\rangle_2 + \beta |v\rangle_1 \otimes |w'\rangle_2$,
- $\big(|v\rangle_1 \otimes |w\rangle_2\big)^\dagger = \langle v|_1 \otimes \langle w|_2$,
- $\langle v, w|v', w'\rangle = \langle v|v'\rangle_1 \langle w|w'\rangle_2$.

(a) Show that for any two vectors of $\mathcal{H}_1$ and $\mathcal{H}_2$ expanded on two basis, $|v\rangle_1 = \sum_{i=1}^N v_i |e_i\rangle_1$ and $|w\rangle_2 = \sum_{j=1}^M w_j |f_j\rangle_2$, then

$$|v\rangle_1 \otimes |w\rangle_2 = \sum_{i=1}^N \sum_{j=1}^M v_i w_j |e_i\rangle_1 \otimes |f_j\rangle_2 \, .$$

(b) Show that if $\{|e_i\rangle_1 ; i = 1 \dots N\}$ and $\{|f_j\rangle_2 ; j = 1 \dots M\}$ are orthonormal, then $|e_i\rangle_1 \otimes |f_j\rangle_2 \equiv |e_i, f_j\rangle$ is an orthonormal basis of $\mathcal{H}_1 \otimes \mathcal{H}_2$. What is the dimension of $\mathcal{H}_1 \otimes \mathcal{H}_2$?

(c) Any vector $|\Psi\rangle$ of $\mathcal{H}_1 \otimes \mathcal{H}_2$ can be expanded on the basis $|e_i\rangle_1 \otimes |f_j\rangle_2 \equiv |e_i, f_j\rangle \, , i = 1 \dots N, j = 1 \dots M$,

$$|\Psi\rangle = \sum_{i=1,j=1}^{N,M} \psi_{ij} |e_i, f_j\rangle \, .$$

If A is a matrix acting on $\mathcal{H}_1$ and B is a matrix acting on $\mathcal{H}_2$, the tensor product $A \otimes B$ is defined as

$$A \otimes B |\Psi\rangle = \sum_{i,j} \psi_{ij} A |e_i\rangle_1 \otimes B |f_j\rangle_2 \, .$$

Check that the matrix elements of $A \otimes B$ in the basis $|e_i, f_j\rangle$ are :

$$\langle e_i, f_j| A \otimes B |e_k, f_l\rangle = a_{ik} b_{jl}.$$

(d) Let $\mathcal{H}_1 = \mathbb{C}^2$, $\mathcal{H}_2 = \mathbb{C}^2$. Take $A_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $B_2 = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$, $|v\rangle_1 = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$, $|w\rangle_2 = \begin{pmatrix} \gamma \\ \delta \end{pmatrix}$. From the defining properties of the tensor product deduce the the following formulas :

$$|v\rangle_1 \otimes |w\rangle_2 = \begin{pmatrix} \alpha\gamma \\ \alpha\delta \\ \beta\gamma \\ \beta\delta \end{pmatrix}, \qquad A_1 \otimes B_2 = \begin{pmatrix} ae & af & be & bf \\ ag & ah & bg & bh \\ ce & cf & de & df \\ cg & ch & dg & dh \end{pmatrix}.$$

These are often useful in order to do calculations in components.