Final Exam Solutions

Exercise 1.

a) The transition probabilities are as follows:

$$p_{0,+1} = p_{0,-1} = \frac{1}{2}, \quad p_{0,j} = 0 \quad \forall j \notin \{-1,+1\}$$

and for $i \neq 0$, we have

$$p_{i,2i} = p_{i,0} = \frac{1}{2}, \quad p_{i,j} = 0 \quad \forall j \notin \{0, 2i\}$$

b) As there are only two ways to come back to state 0 in $n \geq 2$ steps (either via the negative numbers, or via the positive numbers), we obtain:

$$f_{00}^{(1)} = 0 \quad \text{and} \quad f_{00}^{(n)} = 2 \cdot \frac{1}{2^n} = \frac{1}{2^{n-1}} \quad \text{for} \quad n \geq 2$$

c) Part b) implies that

$$f_{00} = \sum_{n\geq 1} f_{00}^{(n)} = \sum_{n\geq 2} \frac{1}{2^{n-1}} = 1$$

so state 0 is recurrent.

d) The set of states $S = \{0, \pm 2^k, k \geq 0\}$ forms a single recurrent class. All other states in $\mathbb{Z}$ are transient (as for each such state, there is a positive probability (actually a probability 1) of no coming back).

e) Solving the equation $\pi = \pi P$ gives $\pi_i = 0$ for all $i \notin S$ and :

$$\pi_{-2^{k+1}} = \frac{1}{2}\,\pi_{-2^k}, \quad \pi_{-1} = \frac{1}{2}\,\pi_0 = \pi_{+1}, \quad \pi_{+2^{k+1}} = \frac{1}{2}\,\pi_{+2^k}$$

for every $k \geq 0$, so

$$\pi_{-2^k} = \pi_{+2^k} = \frac{1}{2^{k+1}}\,\pi_0$$

and

$$\pi_0 \left(1 + 2 \sum_{k\geq 0} \frac{1}{2^{k+1}} \right) = 1 \quad \text{so} \quad \pi_0 = \frac{1}{3}$$

implying

$$\pi_{-2^k} = \pi_{+2^k} = \frac{1}{3}\,\frac{1}{2^{k+1}} \quad \forall k \geq 0$$

f) No, detailed balance is not satisfied, as there is for example a non-zero probability to go from state +1 to state +2, but no probability to go in the reverse direction.

g) Yes, π is also a limiting distribution, as the chain is aperiodic (the probability to come back to state 0 in n steps is strictly positive for every $n \geq 2$).

Exercise 2.

a) The transition matrix of the chain (with the ordering of the states $S = \{a, b, c\}$, as in the problem set) is given by:

$$P = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/5 & 2/5 & 2/5 \\ 1/3 & 2/3 & 0 \end{pmatrix}$$

This chain is ergodic (because finite, irreducible and aperiodic), so it admits a unique stationary and limiting distribution π. Moreover, one can try to solve directly the detailed balance equation:

$$\pi_a \, \frac{1}{2} = \pi_b \, \frac{1}{5} \quad \text{and} \quad \pi_a \, \frac{1}{2} = \pi_c \, \frac{1}{3} \qquad \text{leading to} \qquad \pi_b = \frac{5}{2} \, \pi_a \quad \text{and} \quad \pi_c = \frac{3}{2} \, \pi_a$$

so

$$\pi_a \left(1 + \frac{5}{2} + \frac{3}{2}\right) = 1 \quad \text{implying} \quad \pi_a = \frac{1}{5}$$

which provides the answer to the question (and notice that $\pi_b = \frac{1}{2}$, $\pi_c = \frac{3}{10}$).

b) Let us compute the eigenvalues $\lambda_0, \lambda_1, \lambda_2$ of P:

$$\lambda_0 = 1, \quad \text{Tr}(P) = \frac{2}{5} = 1 + \lambda_1 + \lambda_2, \quad \det(P) = \frac{1}{15} = \lambda_1 \cdot \lambda_2$$

Solving for λ_1, λ_2 gives $\lambda_1 = \frac{-3+\sqrt{7/3}}{10}$, $\lambda_2 = \frac{-3-\sqrt{7/3}}{10}$. So the spectral gap of the chain is given by

$$\gamma = 1 - |\lambda_2| = \frac{7 - \sqrt{7/3}}{10}$$

Using the bound seen in class (remembering that the chain starts in a corner (state c)), we obtain

$$|\mathbb{P}(X_n = a | X_0 = c) - \pi_a| \leq \|P_c^n - \pi\|_{\text{TV}} \leq \frac{1}{2\sqrt{\pi_c}} \, \exp(-\gamma n) = \sqrt{\frac{5}{6}} \, \exp\left(-\frac{7 - \sqrt{7/3}}{10} \cdot n\right)$$

c) The target distribution $\widetilde{\pi} = \left(\frac{1}{9}, \frac{4}{9}, \frac{4}{9}\right)$. Let us first compute the 3 ratios:

$$X = \frac{\widetilde{\pi}_b \, p_{ba}}{\widetilde{\pi}_a \, p_{ab}} = \frac{8}{5}, \quad Y = \frac{\widetilde{\pi}_c \, p_{ca}}{\widetilde{\pi}_a \, p_{ac}} = \frac{8}{3} \quad \text{and} \quad Z = \frac{\widetilde{\pi}_c \, p_{cb}}{\widetilde{\pi}_b \, p_{bc}} = \frac{5}{3}$$

Then the Metropolis-Hastings transition probabilities are given by

$\widetilde{p}_{ab} = p_{ab} \, \min(1, X) = p_{ab} = \frac{1}{2}$ $\quad$ $\widetilde{p}_{ac} = p_{ac} \, \min(1, Y) = p_{ac} = \frac{1}{2}$ $\quad$ $\widetilde{p}_{aa} = 0$

$\widetilde{p}_{ba} = p_{ba} \, \min(1, 1/X) = p_{ba}/X = \frac{1}{8}$ $\quad$ $\widetilde{p}_{bc} = p_{bc} \, \min(1, Z) = p_{bc} = \frac{2}{5}$ $\quad$ $\widetilde{p}_{bb} = 1 - \widetilde{p}_{ba} - \widetilde{p}_{bc} = \frac{19}{40}$

$\widetilde{p}_{ca} = p_{ca} \, \min(1, 1/Y) = p_{ca}/Y = \frac{1}{8}$ $\quad$ $\widetilde{p}_{cb} = p_{cb} \, \min(1, 1/Z) = p_{cb}/Z = \frac{2}{5}$ $\quad$ $\widetilde{p}_{cc} = 1 - \widetilde{p}_{ca} - \widetilde{p}_{cb} = \frac{19}{40}$

i.e.,

$$\widetilde{P} = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/8 & 19/40 & 2/5 \\ 1/8 & 2/5 & 19/40 \end{pmatrix}$$

(whose stationary distribution is indeed $\widetilde{\pi}$).

Exercise 3.

a)

$$H(0,0)=0, \quad H(0,1)=1+J, \quad H(1,0)=1+J, \quad H(1,1)=1+1+J\cdot 0=2$$

Fix a site i and let $j \neq i$. We have for the conditional probabilities:

$$\pi(x_i \mid x_j) = \frac{e^{-H(x_i,x_j)}}{e^{-H(0,x_j)}+e^{-H(1,x_j)}}$$

- For $x_j = 0$:

$$u = \pi(x_i = 1 \mid x_j = 0) = \frac{1}{1+e^{1+J}}, \quad 1-u = \pi(x_i = 0 \mid x_j = 0) = \frac{e^{1+J}}{1+e^{1+J}}$$

- For $x_j = 1$:

$$v = \pi(x_i = 0 \mid x_j = 1) = \frac{e^{1-J}}{1+e^{1-J}}, \quad 1-v = \pi(x_i = 1 \mid x_j = 1) = \frac{1}{1+e^{1-J}}$$

b) By the description given (site $i \in \{1,2\}$ chosen uniformly at random), $w = 1/2$, so we obtain

- Transitions $00 \to x_1x_2$

$$P_{00\to00} = \frac{1-u}{2}+\frac{1-u}{2} = 1-u, \quad P_{00\to01} = \frac{u}{2}, \quad P_{00\to10} = \frac{u}{2}, \quad P_{00\to11} = 0$$

- Transitions $01 \to x_1x_2$

$$P_{01\to00} = \frac{1-u}{2}, \quad P_{01\to01} = \frac{u}{2}+\frac{v}{2}, \quad P_{01,10} = 0, \quad P_{01\to11} = \frac{1-v}{2}$$

- Transitions $10 \to x_1x_2$

$$P_{10\to00} = \frac{1-u}{2}, \quad P_{10\to01} = 0, \quad P_{10\to10} = \frac{v}{2}+\frac{u}{2}, \quad P_{10\to11} = \frac{1-v}{2}$$

- Transitions $11 \to x_1x_2$

$$P_{11\to11} = 1-v, \quad P_{11\to01} = \frac{v}{2}, \quad P_{11\to10} = \frac{v}{2}, \quad P_{11\to00} = 0$$

Putting this together with the order $00, 01, 10, 11$

$$P = \begin{pmatrix} 1-u & \frac{u}{2} & \frac{u}{2} & 0 \\ \frac{1-u}{2} & \frac{u+v}{2} & 0 & \frac{1-v}{2} \\ \frac{1-u}{2} & 0 & \frac{u+v}{2} & \frac{1-v}{2} \\ 0 & \frac{v}{2} & \frac{v}{2} & 1-v \end{pmatrix}$$

c) We have

$$Pe_0 = \frac{u+v}{2}e_0$$

Hence $\frac{u+v}{2}$ is an eigenvalue with eigenvector e_0.

We then note that the sum of (first + fourth) rows and the sum of (second + third) rows are equal vectors. Thus (first + fourth - second - third) rows equals the zero vector. Thus $\det P = 0$ and there is a 0 eigenvalue.

Moreover, we know there must be the 1 eigenvalue (since sums of rows is equal to one for a stochastic matrix).

The fourth eigenvalue can be found by looking at the trace $\mathrm{Tr}P = 2$:

$$0 + 1 + \frac{u+v}{2} + \lambda = 2$$

so $\lambda = 1 - \frac{u+v}{2} = \frac{1-u}{2} + \frac{1-v}{2}$.

Summarizing, the eigenvalues are:

$$\lambda_1 = 1, \quad \lambda_2 = \frac{u+v}{2}, \quad \lambda_3 = \frac{1-u}{2} + \frac{1-v}{2}, \quad \lambda_4 = 0$$

They are all non-negative.

d) In fact $J \geq 0$ implies $\lambda_3 \geq \lambda_2$. Indeed: $1 - \frac{u+v}{2} \geq \frac{u+v}{2}$ is equivalent to $1 \geq u + v$ which means

$$(1 + e^{1+J})(1 + e^{1-J}) \geq 1 + e^{1-J} + e^{1-J}(1 + e^{1+J})$$

equivalent to

$$1 + e^{1+J} + e^{1-J}(1 + e^{1+J}) \geq 1 + e^{1-J} + e^{1-J}(1 + e^{1+J})$$

in other words $e^{1+J} \geq e^{1-J}$, true for $J \geq 0$.

Therefore the spectral gap is

$$\gamma = 1 - \left(1 - \frac{u+v}{2}\right) = \frac{u+v}{2}$$

For $J = 0$ it is $1/2$ and for $J >> 1$ it behaves like $O(e^{-J})$. A standard bound seen in class (for given ϵ TV distance)

$$T_{\text{mix}} \leq \frac{1}{\gamma} \log\left(\frac{1}{\varepsilon\, \pi_{\text{min}}}\right), \qquad \pi_{\text{min}} := \min_{(x_1,x_2)\in\mathcal{S}} \pi(x_1, x_2).$$

So

$$T_{\text{mix}} = O\left(\frac{2}{u+v}\right)$$

e) The mixing time is minimal for $J = 0$ and grows like $\Omega(e^J)$ for $J \to +\infty$. This is intuitive since the chain has more and more difficulty to jump from state 00 to 01 or 10. These two states act like a bottleneck for paths going from 00 to 11.

At $J = +\infty$ the states 00 and 11 become absorbing. The states 10 and 01 become transient. The chain is not irreducible.

f) Lazy version $P^{(\alpha)} = \alpha I + (1-\alpha)P$: this increases the self-loop probabilities and decreases accordingly the edge probabilities. The eigenvalues transform as $\lambda \longmapsto \alpha + (1-\alpha)\lambda$. In particular the top eigenvalue stays 1, and the since all eigenvalues are positive their ordering does not change, and the new spectral gap is given by

$$\gamma^{(\alpha)} = 1 - \left(\alpha + (1-\alpha)\left(1 - \frac{u+v}{2}\right)\right) = (1-\alpha)\left(\frac{u+v}{2}\right)$$