Final Exam

Please pay attention to the presentation of your answers **(2 points)**.

Exercise 1. (22 points)

Let $(Z_n,\, n \geq 1)$ be a sequence of i.i.d. random variables such that $\mathbb{P}(Z_n = +1) = \mathbb{P}(Z_n = -1) = \frac{1}{2}$, $\forall n \geq 1$.

Let also $X = (X_n,\, n \geq 0)$ be the Markov chain with state space $S = \mathbb{Z}$ defined recursively as

$$X_0 = x_0 \in \mathbb{Z}, \quad X_{n+1} = \begin{cases} X_n \cdot (1 + Z_{n+1}), & \text{if } X_n \neq 0, \\ Z_{n+1}, & \text{if } X_n = 0, \end{cases} \qquad n \geq 0$$

a) Compute the transition probabilities of the chain X.

b) Compute $f_{00}^{(n)} = \mathbb{P}(T_0 = n \,|\, X_0 = 0)$ for $n \geq 1$.

Reminder: $T_0 = \inf\{n \geq 1 : X_n = 0\}$ is the first return time of the chain to state 0.

c) Is state 0 transient or recurrent ? Use part b) to justify your answer.

d) Which states of the chain are transient, which are recurrent?

e) Compute the unique stationary distribution π of the chain.

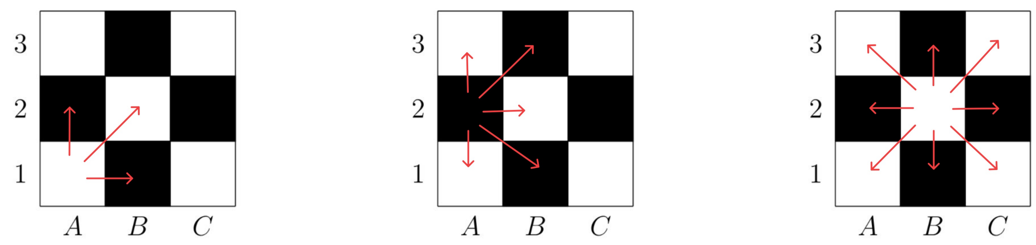
f) Does π satisfy detailed balance? Justify your answer.

g) Is π also a limiting a distribution? Justify your answer.

Exercise 2. (22 points)

Preliminary note: Even though the questions below only require short answers, please explain in detail the reasoning leading to each answer!

A king is moving randomly on a 3×3 chessboard. For each type of departing square (corner, border or center), here are the possible moves of the king:



The king starts from a corner of the chessboard and then at each step, choses uniformly one of the possible moves.

Hint: In order to solve this exercise, it is possible (and also strongly advised) to consider the state space $S = \{a, b, c\}$, where:

- a is the state corresponding to the central square of the chessboard;
- b is the state corresponding to being in one of the four black squares on the borders of the chessboard;
- c is the state corresponding to being in one of the four white squares in the corners of the chessboard.

a) On the long run, what is the probability π_a that one finds the king in the central square?

b) Let X_n denote the position of the king at time n. Find a tight upper bound (for large n) on

$$|\mathbb{P}(X_n = a \,|\, X_0 = c) - \pi_a|$$

c) Consider now the Markov chain described above as the base chain used for the Metropolis-Hastings algorithm whose target distribution $\widetilde{\pi}$ is the uniform distribution on the chessboard (i.e., each square should have the same probability). Write down the transition matrix $\widetilde{P}$ of the Metropolis-Hastings chain.

Exercise 3. (24 points)

Let the state space be $\mathcal{S} = \{0,1\}^2$. For $(x_1, x_2) \in \mathcal{S}$ define the energy function of a 'spin system' on two sites

$$H(x_1, x_2) = x_1 + x_2 + J\,\mathbb{1}_{\{x_1 \neq x_2\}}, \qquad J \geq 0$$

and the associated Gibbs distribution

$$\pi(x_1, x_2) = \frac{1}{Z}\exp(-H(x_1, x_2))$$

We consider the Markov chain (also called heat-bath dynamics) defined as follows:

- at each step, choose a site $i \in \{1,2\}$ uniformly at random;
- update x_i by sampling from the conditional distribution $\pi(x_i \mid x_j)$, where $j \neq i$. This means that for a given value of x_j, $j \neq i$, we sample a value $x_i \sim \pi(\dots \mid x_j)$ from the conditional distribution induced by the joint distribution $\pi(x_1, x_2)$; and assign x_i to site i.

a) Compute explicitely the energy of the four states $00, 01, 10, 11 \in \mathcal{S}$ and each conditional probability

$$u = \pi(x_i = 1 \mid x_j = 0), \quad 1 - u = \pi(x_i = 0 \mid x_j = 0),$$
$$v = \pi(x_i = 0 \mid x_j = 1), \quad 1 - v = \pi(x_i = 1 \mid x_j = 1),$$

b) The probability of a transition $x_1x_2 \to y_1y_2$ for the heat bath dynamics defined above is of the form:

$$P_{x_1x_2 \to y_1y_2} = w\,\mathbb{1}_{x_1 = y_1}\,\pi(y_2 \mid x_1) + (1 - w)\,\mathbb{1}_{x_2 = y_2}\,\pi(y_1 \mid x_2)$$

- What is the value of the probability weight w?
- Using the state ordering $(00, 01, 10, 11)$ for rows and columns, write down explicitly the transition matrix P of the Markov chain. Give you results entirely in terms of u and v.

Note: Check that your result satisfies the vector equation:

(first row + fourth row) = (second row + third row)

c) Compute the four eigenvalues of P.
Hint: These can be quickly computed by using that $e_0 = (0, 1, -1, 0)$ is an eigenvector and also the 'Note' above.

d) Deduce the spectral gap and an upper bound on the mixing time.

e) With two or three short sentences: discuss the behaviour of the mixing time as a function of $J \geq 0$ as J becomes large. In particular explain what happens to the chain for $J = +\infty$ and draw the state graph and edge probabilities for this case.

f) Let $\alpha \in (0,1)$ and define a modified chain with transition matrix $P^{(\alpha)} = \alpha I + (1 - \alpha)P$.

- In words: what does this amount to do to the state graph?
- What is the new spectral gap? Does the modified chain converge quicker/slower (as a function of α) to the stationary distribution?