Solution 11
Introduction to Quantum Information Processing

Exercise 1 *Adapting the Three-Qubit Repetition Code to Correct Phase-Flip Errors*

a) We first note the effect of a phase-flip (also called "Z error") on, say, the first physical qubit:

$$(Z_1 \otimes \mathbf{I}_2 \otimes \mathbf{I}_3)(\alpha |+++\rangle + \beta |---\rangle) = \alpha |-++\rangle + \beta |+--\rangle . \tag{1}$$

It effectively flips the first qubit in the Hadamard basis. So, we want to build an observable that is invariant towards the initial state but detects phase-flips. Using that $|\pm\rangle$ are eigenvectors of the X Pauli matrix, with eigenvalues ± 1 we can build the following observable

$$X_1 X_2, \qquad X_2 X_3,$$

which will detect single phase-flips.

b) Indeed, we have the following outcomes when making the two measurements,

$$\begin{aligned}
\alpha |+++\rangle + \beta |---\rangle &\longrightarrow \quad \text{Outcome } (+1, +1) && (2)\\
\alpha |-++\rangle + \beta |+--\rangle &\longrightarrow \quad \text{Outcome } (-1, +1) && (3)\\
\alpha |+-+\rangle + \beta |-+-\rangle &\longrightarrow \quad \text{Outcome } (-1, -1) && (4)\\
\alpha |++-\rangle + \beta |--+\rangle &\longrightarrow \quad \text{Outcome } (+1, -1). && (5)
\end{aligned}$$

c) A majority decoding can be used to retrieve the original message after a single Z error occurred. Based on the outcome of our measurements, we can conclude which qubit has been flipped and apply the corresponding operation to restore the original message,

$$\begin{aligned}
&\text{Outcome } (+1, +1) \longrightarrow \text{no error} && (6)\\
&\text{Outcome } (-1, +1) \longrightarrow \text{error on qubit 1} \longrightarrow \text{Apply } Z_1 && (7)\\
&\text{Outcome } (-1, -1) \longrightarrow \text{error on qubit 2} \longrightarrow \text{Apply } Z_2 && (8)\\
&\text{Outcome } (+1, -1) \longrightarrow \text{error on qubit 3} \longrightarrow \text{Apply } Z_3. && (9)
\end{aligned}$$

d) We can use Hadamard gate to convert the Hadamard basis to the computational basis, and then use the class circuit to detect the errors. The circuit is given in Figure 1. After the error correction, we should apply Hadamard gates again to return to the Hadamard basis.

e) It can directly be checked that $X_L = Z_1 \otimes Z_2 \otimes Z_3$ and $Z_L = X_1 \otimes X_2 \otimes X_3$

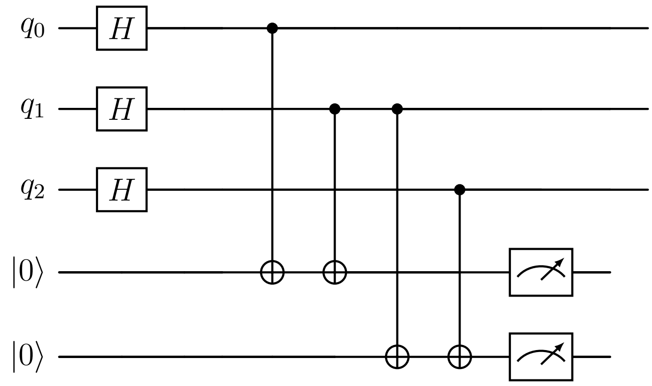

Figure 1: question d

Exercise 2 *Syndrome measurement for 2 qubits*

a) Let's assume that two-qubit data state is in the most general form

$$|\psi\rangle_{q_1 q_2} = \alpha |00\rangle + \beta |01\rangle + \gamma |10\rangle + \delta |11\rangle . \tag{10}$$

Measuring $S_Z = Z_1 \otimes Z_2$ can give two outcomes, $+1$ or -1.

- Outcome $+1$: postmeasurement state is either $|00\rangle$ or $|11\rangle$, so it occurs with probability $|\langle 00|\psi\rangle_{q_1 q_2}|^2 + |\langle 11|\psi\rangle_{q_1 q_2}|^2 = |\alpha|^2 + |\delta|^2$
- Outcome -1: postmeasurement state is either $|01\rangle$ or $|10\rangle$, so it occurs with probability $|\langle 01|\psi\rangle_{q_1 q_2}|^2 + |\langle 10|\psi\rangle_{q_1 q_2}|^2 = |\beta|^2 + |\gamma|^2$

We want to show that the same outcomes with the same probabilities can be obtained by running the given circuits, and measuring on an ancilla qubit, thus allowing us not to destroy the information state $|\psi\rangle_{q_1 q_2}$! Let's look at the state obtained after running the first circuit on $|\psi_{in}\rangle = |\psi\rangle_{q_1 q_2} \otimes |0\rangle$; it is given by,

$$\begin{aligned}
\text{CNOT}_{2\to3}\text{CNOT}_{1\to3} |\psi_{in}\rangle &= \text{CNOT}_{2\to3}\text{CNOT}_{1\to3}(\alpha |00\rangle + \beta |01\rangle + \\
&\qquad + \gamma |10\rangle + \delta |11\rangle)) \otimes |0\rangle && (11) \\
&= \text{CNOT}_{2\to3}(\alpha |00\rangle + \beta |01\rangle) \otimes |0\rangle + \\
&\qquad + (\gamma |10\rangle + \delta |11\rangle) \otimes |1\rangle && (12) \\
&= \alpha |000\rangle + \beta |011\rangle + \gamma |101\rangle + \delta |110\rangle && (13) \\
&= (\alpha |00\rangle + \delta |11\rangle) \otimes |0\rangle + (\gamma |10\rangle + \beta |01\rangle) \otimes |1\rangle && (14)
\end{aligned}$$

If we measure m_Z in the computational basis now, the two possible outcomes are ± 1 with respective probabilities $|\alpha|^2 + |\delta|^2$ and $|\gamma|^2 + |\beta|^2$. Note that we used the same remark as in Homework 5 to compute the probabilities, with the projectors $\mathbf{I}_1 \otimes \mathbf{I}_2 \otimes |0\rangle\langle 0|$ and $\mathbf{I}_1 \otimes \mathbf{I}_2 \otimes |1\rangle\langle 1|$, respectively.

Remark : To compute the probability, one can use the formula $\langle \phi | P | \phi \rangle = \langle \phi | P^2 | \phi \rangle = \| P |\phi\rangle \|^2$ *where* $|\phi\rangle$ *is the state prior to measurement and* P *is some projector.*

We see that in both cases, we measure the same outcomes with the same probabilities, so the two measurements are equivalent.

A similar analysis can be done for the S_X measurement, where the state at the end of the circuit is,

$$\begin{aligned}
H_3\mathrm{CNOT}_{3\to2}\mathrm{CNOT}_{3\to1}H_3\,|\psi_{in}\rangle &= H_3\mathrm{CNOT}_{3\to2}\mathrm{CNOT}_{3\to1}H_3(\alpha\,|00\rangle + \qquad (15)\\
&\quad + \beta\,|01\rangle + \gamma\,|10\rangle + \delta\,|11\rangle) \otimes |0\rangle\\
&= \frac{1}{\sqrt{2}}H_3\mathrm{CNOT}_{3\to2}\mathrm{CNOT}_{3\to1}(\alpha\,|00\rangle + \beta\,|01\rangle + \qquad (16)\\
&\quad + \gamma\,|10\rangle + \delta\,|11\rangle) \otimes (|0\rangle + |1\rangle)\\
&= \frac{1}{\sqrt{2}}H_3\Big((\alpha\,|00\rangle + \beta\,|01\rangle + \gamma\,|10\rangle + \delta\,|11\rangle) \otimes |0\rangle + \qquad (17)\\
&\quad + (\alpha\,|11\rangle + \beta\,|10\rangle + \gamma\,|01\rangle + \delta\,|00\rangle) \otimes |1\rangle\\
&= \frac{1}{2}\Big[\big((\alpha+\delta)(|00\rangle + |11\rangle) + (\beta+\gamma)(|01\rangle + |10\rangle)\big) \otimes |0\rangle + \qquad (18)\\
&\quad \big((\alpha-\delta)(|00\rangle + |11\rangle) + +(\beta-\gamma)(|01\rangle + |10\rangle)\big) \otimes |1\rangle\Big]
\end{aligned}$$

One can show that the outcomes and their respective probabilities are the same whether we measure X_1X_2 on $|\psi\rangle_{q_1q_2}$ or whether we measure m_X after running the circuit, and are given by,

$$\Pr[\text{Outome is } +1] = \frac{1}{2}(|\alpha+\delta|^2 + |\beta+\gamma|^2) \qquad (19)$$

$$\Pr[\text{Outome is } +1] = \frac{1}{2}(|\alpha-\delta|^2 + |\beta-\gamma|^2). \qquad (20)$$

b) As shown previously, the possible outcomes after measuring S_Z are ±1. Each outcome can occur in two cases:

- If q_1 and q_2 are identical, then

$$S_Z\,|\psi\rangle_{q_1q_2} = \begin{cases}(+1)(+1)\,|00\rangle\\(-1)(-1)\,|11\rangle\end{cases} = |\psi\rangle_{q_1q_2} \qquad (21)$$

- If q_1 and q_2 are different, then

$$S_Z\,|\psi\rangle_{q_1q_2} = \begin{cases}(+1)(-1)\,|01\rangle\\(-1)(+1)\,|10\rangle\end{cases} = -\,|\psi\rangle_{q_1q_2} \qquad (22)$$

We deduce that the outcome of the measurement tells us if the two states are identical. If one uses a repetition encoding, such measurement can be used to detect single bit-flip errors (as seen in class).

c) In a similar manner, the two possible outcomes after measuring S_X are ±1, with each outcome occurring in two cases:

- If q_1 and q_2 are identical, then

$$S_X \left|\psi\right\rangle_{q_1 q_2} = \begin{cases} (+1)(+1)\left|++\right\rangle \\ (-1)(-1)\left|--\right\rangle \end{cases} = \left|\psi\right\rangle_{q_1 q_2} \tag{23}$$

- If q_1 and q_2 are different, then

$$S_Z \left|\psi\right\rangle_{q_1 q_2} = \begin{cases} (+1)(-1)\left|+-\right\rangle \\ (-1)(+1)\left|-+\right\rangle \end{cases} = -\left|\psi\right\rangle_{q_1 q_2} \tag{24}$$

where we used that $X\left|\pm\right\rangle = \pm 1\left|\pm\right\rangle$. Since a phase-flip acts as follows $\left|\pm\right\rangle \to \left|\mp\right\rangle$, upon measuring the value -1, we can conclude that a single phase-flip error occurred.

d) The Bell states in the Hadamard basis read,

$$\left|\Phi^{\pm}\right\rangle = \frac{\left|+\pm\right\rangle + \left|-\mp\right\rangle}{\sqrt{2}}, \quad \left|\Psi^{\pm}\right\rangle = \frac{\left|\pm+\right\rangle - \left|\mp-\right\rangle}{\sqrt{2}}. \tag{25}$$

Remarkably, they remain highly symmetric also in the Hadamard basis.

e) We compute the states obtained after applying, respectively, the S_Z and S_X and operator to the 4 Bell states. First for S_Z, we get,

$$\begin{aligned} S_Z\left|\Phi^{\pm}\right\rangle &= Z_1 Z_2 \frac{\left|00\right\rangle \pm \left|11\right\rangle}{\sqrt{2}} = \frac{(+1)(+1)\left|00\right\rangle \pm (-1)(-1)\left|11\right\rangle}{\sqrt{2}} = \frac{\left|00\right\rangle \pm \left|11\right\rangle}{\sqrt{2}} \\ &= \left|\Phi^{\pm}\right\rangle \end{aligned} \tag{26}$$

$$\begin{aligned} S_Z\left|\Psi^{\pm}\right\rangle &= Z_1 Z_2 \frac{\left|01\right\rangle \pm \left|10\right\rangle}{\sqrt{2}} = \frac{(+1)(-1)\left|01\right\rangle \pm (-1)(+1)\left|10\right\rangle}{\sqrt{2}} = -\frac{\left|01\right\rangle \pm \left|10\right\rangle}{\sqrt{2}} \\ &= -\left|\Psi^{\pm}\right\rangle. \end{aligned} \tag{27}$$

Similarly for S_X, we get,

$$\begin{aligned} S_X\left|\Phi^{\pm}\right\rangle &= X_1 X_2 \frac{\left|+\pm\right\rangle + \left|-\mp\right\rangle}{\sqrt{2}} = \frac{(+1)(\pm 1)\left|+\pm\right\rangle + (-1)(\mp 1)\left|-\mp\right\rangle}{\sqrt{2}} = \pm\frac{\left|+\pm\right\rangle + \left|-\mp\right\rangle}{\sqrt{2}} \\ &= \pm\left|\Phi^{\pm}\right\rangle \end{aligned} \tag{28}$$

$$\begin{aligned} S_Z\left|\Psi^{\pm}\right\rangle &= X_1 X_2 \frac{\left|\pm+\right\rangle - \left|\mp-\right\rangle}{\sqrt{2}} = \frac{(\pm 1)(+1)\left|\pm+\right\rangle - (\mp 1)(-1)\left|\mp-\right\rangle}{\sqrt{2}} = \pm\frac{\left|\pm+\right\rangle - \left|\mp-\right\rangle}{\sqrt{2}} \\ &= \pm\left|\Psi^{\pm}\right\rangle. \end{aligned} \tag{29}$$

We see that the Bell states are eigenstates of S_Z and S_Z with eigenvalues ± 1.

f) The state of the system after one cycle is given by,

$$\begin{aligned}
|\chi\rangle &= H_4\mathrm{CNOT}_{4\to2}\mathrm{CNOT}_{4\to1}H_4\mathrm{CNOT}_{2\to3}\mathrm{CNOT}_{1\to3}(\alpha\,|00\rangle + \beta\,|01\rangle + \\
&\qquad\qquad + \gamma\,|10\rangle + \delta\,|11\rangle)\otimes|00\rangle && (30)\\
&= H_4\mathrm{CNOT}_{4\to2}\mathrm{CNOT}_{4\to1}H_4\mathrm{CNOT}_{2\to3}(\alpha\,|000\rangle + \beta\,|010\rangle + \\
&\qquad\qquad + \gamma\,|101\rangle + \delta\,|111\rangle)\otimes|0\rangle && (31)\\
&= H_4\mathrm{CNOT}_{4\to2}\mathrm{CNOT}_{4\to1}(\alpha\,|000\rangle + \beta\,|011\rangle + \gamma\,|101\rangle + \delta\,|110\rangle)\otimes\frac{|0\rangle+|1\rangle}{\sqrt{2}} && (32)\\
&= \frac{1}{\sqrt{2}}H_4\mathrm{CNOT}_{4\to2}(\alpha\,|000\rangle + \beta\,|011\rangle + \gamma\,|101\rangle + \delta\,|110\rangle)\otimes|0\rangle + \\
&\qquad\qquad + (\alpha\,|100\rangle + \beta\,|111\rangle + \gamma\,|001\rangle + \delta\,|010\rangle)\otimes|1\rangle && (33)\\
&= \frac{1}{\sqrt{2}}H_4(\alpha\,|000\rangle + \beta\,|011\rangle + \gamma\,|101\rangle + \delta\,|110\rangle)\otimes|0\rangle + \\
&\qquad\qquad + (\alpha\,|110\rangle + \beta\,|101\rangle + \gamma\,|011\rangle + \delta\,|000\rangle)\otimes|1\rangle && (34)\\
&= \frac{1}{2}\Big((\alpha+\delta)(|0000\rangle + |1100\rangle) + (\alpha-\delta)(|0001\rangle - |1101\rangle) + \\
&\qquad\qquad + (\beta+\gamma)(|0110\rangle + |1010\rangle) + (\beta-\gamma)(|0111\rangle - |1011\rangle)\Big). && (35)
\end{aligned}$$

g) We want to compute the possible syndromes if the general data state $|\psi\rangle_{q_1,q_2}$ is equal to any of the four Bell states. We look at each Bell state separately.

- $|\psi\rangle_{q_1,q_2} = |\Phi^+\rangle \Rightarrow \alpha = \frac{1}{\sqrt{2}}, \delta = \frac{1}{\sqrt{2}} \Rightarrow |\chi\rangle = \frac{1}{\sqrt{2}}(|0000\rangle + |1100\rangle)$. So we see that the two outcomes are $(m_Z, m_X) = (+1, +1)$,
- $|\psi\rangle_{q_1,q_2} = |\Phi^-\rangle \Rightarrow \alpha = \frac{1}{\sqrt{2}}, \delta = -\frac{1}{\sqrt{2}} \Rightarrow |\chi\rangle = \frac{1}{\sqrt{2}}(|0001\rangle - |1101\rangle)$. So we see that the two outcomes are $(m_Z, m_X) = (+1, -1)$,
- $|\psi\rangle_{q_1,q_2} = |\Psi^+\rangle \Rightarrow \beta = \frac{1}{\sqrt{2}}, \gamma = \frac{1}{\sqrt{2}} \Rightarrow |\chi\rangle = \frac{1}{\sqrt{2}}(|0110\rangle + |1010\rangle)$. So we see that the two outcomes are $(m_Z, m_X) = (-1, +1)$,
- $|\psi\rangle_{q_1,q_2} = |\Psi^-\rangle \Rightarrow \beta = \frac{1}{\sqrt{2}}, \gamma = -\frac{1}{\sqrt{2}} \Rightarrow |\chi\rangle = \frac{1}{\sqrt{2}}(|0111\rangle - |1011\rangle)$. So we see that the two outcomes are $(m_Z, m_X) = (-1, -1)$.

Now if we prepare an initial state in the Bell state $|\Phi^+\rangle_{q_1,q_2}$, we can predict the outcomes of the measurements upon certain errors:

- Bit-flip error : $|\Phi^+\rangle = \frac{|00\rangle+|11\rangle}{\sqrt{2}} \to \frac{|01\rangle+|10\rangle}{\sqrt{2}} = |\Psi^+\rangle$ means we will measure $m_Z = -1$ and $m_X = +1$.
- Phase-flip error : $|\Phi^+\rangle = \frac{|00\rangle+|11\rangle}{\sqrt{2}} \to \frac{|00\rangle-|11\rangle}{\sqrt{2}} = |\Phi^-\rangle$ means we will measure $m_Z = +1$ and $m_X = -1$.
- Phase flip and bit-flip : $|\Phi^+\rangle = \frac{|00\rangle+|11\rangle}{\sqrt{2}} \to \frac{|00\rangle-|11\rangle}{\sqrt{2}} \to \frac{|01\rangle-|10\rangle}{\sqrt{2}} = |\Psi^-\rangle$, means we will measure $m_Z = -1$ and $m_X = -1$.

Exercise 3 *Shor 9 qubit code implementation*

a) We start by applying the two CNOT gates. To simplify the notation, let's first restrict the calculations on the Hilbert spaces of the 1st, 4th and 5th qubits and drop the tensor product symbol, i.e. $|a\rangle_i \otimes |b\rangle_j \equiv |a\rangle_i |b\rangle_j \equiv |ab\rangle_{ij}$. Applying the CNOTs gives,

$$\begin{aligned}
\mathrm{CNOT}_{1\to7}\mathrm{CNOT}_{1\to4}\,|\psi\rangle_1\,|00\rangle_{47} &= \mathrm{CNOT}_{1\to7}\mathrm{CNOT}_{1\to4}(\alpha\,|000\rangle_{147} + \beta\,|100\rangle_{147}) && (36)\\
&= \alpha\,|000\rangle_{147} + \beta\,|111\rangle_{147} && (37)\\
&= |\phi\rangle_{147}\,. && (38)
\end{aligned}$$

Now applying the three Hadamard gates,

$$\begin{aligned}
H_1H_4H_7\,|\phi\rangle_{147} &= H_1H_4H_7(\alpha\,|000\rangle_{147} + \beta\,|111\rangle_{147}) && (39)\\
&= \frac{1}{2\sqrt{2}}\Big[\alpha(|0\rangle + |1\rangle)_1(|0\rangle + |1\rangle)_4(|0\rangle + |1\rangle)_7 + && (40)\\
&\qquad + \beta(|0\rangle - |1\rangle)_1(|0\rangle - |1\rangle)_4(|0\rangle - |1\rangle)_7\Big]\\
&= \frac{1}{2\sqrt{2}}\Big[(\alpha+\beta)\big(\,|000\rangle + |011\rangle + |101\rangle + |110\rangle\,\big)_{147} + && (41)\\
&\qquad + (\alpha-\beta)\big(\,|001\rangle + |010\rangle + |100\rangle + |111\rangle\,\big)_{147}\Big]\\
&= |\phi'\rangle_{147}\,. && (42)
\end{aligned}$$

Now we want to compute the state after the three first CNOTs are applied, so we must look at Hilbert spaces of qubits 2, 5 and 8. First note the following to compute the effect of CNOTs efficiently,

$$\mathrm{CNOT}_{7\to8}\mathrm{CNOT}_{4\to5}\mathrm{CNOT}_{1\to2}\,|abc\rangle_{147}\,|000\rangle_{258} = |aabbcc\rangle_{124578}\,. \tag{43}$$

For example, you can convince yourself that after applying the CNOTs to $|101\rangle_{147}\,|000\rangle_{258}$, you get $|110011\rangle_{124578}$. Using this identity, one can compute the effect of the three CNOTs,

$$\begin{aligned}
&\mathrm{CNOT}_{7\to8}\mathrm{CNOT}_{4\to5}\mathrm{CNOT}_{1\to2}\,|\phi'\rangle_{147}\,|000\rangle_{258} = && (44)\\
&\quad = \frac{1}{2\sqrt{2}}\Big[(\alpha+\beta)\big(\,|000000\rangle + |001111\rangle + |110011\rangle + |111100\rangle\,\big)_{124578} +\\
&\qquad + (\alpha-\beta)\big(\,|000011\rangle + |001100\rangle + |110000\rangle + |111111\rangle\,\big)_{124578}\Big]\\
&\quad = |\phi''\rangle && (45)
\end{aligned}$$

Finally, we apply the last layer of the circuit, namely the three CNOTs on qubits 3, 6, and 9:

$$
\begin{aligned}
&\text{CNOT}_{7\to8}\text{CNOT}_{4\to5}\text{CNOT}_{1\to2}\,|\phi''\rangle_{124578}\,|000\rangle_{369} = \qquad (46)\\
&= \frac{1}{2\sqrt{2}}\Big[(\alpha+\beta)\big(\,|000000000\rangle + |000111111\rangle + |111000111\rangle + |111111000\rangle\,\big)_{1-9} + \\
&\quad + (\alpha-\beta)\big(\,|000000111\rangle + |000111000\rangle + |111000000\rangle + |111111111\rangle\,\big)_{1-9}\Big].\\
&= \frac{1}{2\sqrt{2}}\Big[\alpha\big(\,|000000000\rangle + |000111111\rangle + |111000111\rangle + |111111000\rangle + \qquad (47)\\
&\quad + |000000111\rangle + |000111000\rangle + |111000000\rangle + |111111111\rangle\,\big)_{1-9} + \\
&\quad + \beta\big(\,|000000000\rangle + |000111111\rangle + |111000111\rangle + |111111000\rangle + \\
&\quad - |000000111\rangle - |000111000\rangle - |111000000\rangle - |111111111\rangle\,\big)_{1-9}\Big]\\
&= \alpha\frac{(|000\rangle+|111\rangle)^{\otimes 3}}{2\sqrt{2}} + \beta\frac{(|000\rangle-|111\rangle)^{\otimes 3}}{2\sqrt{2}} = \alpha\,|0\rangle_L + \beta\,|1\rangle_L\,. \qquad (48)
\end{aligned}
$$

We conclude that applying this circuit to an arbitrary state and 8 other physical qubits yields a logical qubit.

b) For the logical $\overline{X}$ gate, we observe that the difference between the two logical states $|0\rangle_L$ and $|1\rangle_L$ is a negative sign in $|1\rangle_L$ for each term with an odd number of "$|1\rangle$". So, we deduce that a combination of Z gates

$$\overline{X} = Z_1 \otimes ... \otimes Z_9 \tag{49}$$

must be used to compute $\overline{X}$, where each Z_i-gate probes if the i-th qubit is in a 0 or 1 state.

For the logical $\overline{Z}$ gate, we observe that, in $|1\rangle_L$, each term $|aaabbbccc\rangle$ with negative sign has its complementary $\left|\overline{aaabbbccc}\right\rangle$ with positive signs. So, flipping every bit does not change $|0\rangle_L$ but flips the sign of $|1\rangle_L$, and therefore

$$\overline{Z} = X_1 \otimes ... \otimes X_9. \tag{50}$$

The logical $\overline{Y}$ shoud verify $\overline{Y} = i\overline{X}\cdot\overline{Z}$, so we have

$$\overline{Y} = i(Z_1 \otimes ... \otimes Z_9)(X_1 \otimes ... \otimes X_9) = i^{10}(Y_1 \otimes ... \otimes Y_9) = -(Y_1 \otimes ... \otimes Y_9). \tag{51}$$

For the Hadamard gate $\overline{H}$, we can use the fact that $HXH = Z$ and $HZH = X$. If we choose

$$\overline{H} = H_1 \otimes ... \otimes H_9. \tag{52}$$

we obtain the logical operations $\overline{HXH} = \overline{Z}$ and $\overline{HZH} = \overline{X}$, as required. However, $H^{\otimes 9}$ does not map $|0\rangle_L$ to $\frac{|0\rangle_L+|1\rangle_L}{\sqrt{2}}$ and $|1\rangle_L$ to $\frac{|0\rangle_L-|1\rangle_L}{\sqrt{2}}$, so it is not a logical Hadamard gate in that sense. A Hadamard gate that satisfies this property cannot be implemented transversally in the Shor code. A simple way to implement such a logical Hadamard gate is to decode the logical qubit, apply a physical Hadamard gate, and re-encode the logical qubit.