Problem Set 14

Problem 1: Convergence and inequalities

Let $(X_n,\, n \geq 1)$ be a sequence of i.i.d. non-negative random variables defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and such that there exists $0 < a < b < +\infty$ with $a < X_n(\omega) \leq b$ for all $n \geq 1$ and $\omega \in \Omega$. Let also $(Y_n,\, n \geq 1)$ be the sequence defined as

$$Y_n = \left(\prod_{j=1}^{n} X_j \right)^{1/n}, \quad n \geq 1$$

a) Show that there exists a constant $\mu > 0$ such that $Y_n \underset{n\to\infty}{\to} \mu$ almost surely.

b) Compute the value of μ in the case where $\mathbb{P}(\{X_1 = a\}) = \mathbb{P}(\{X_1 = b\}) = \frac{1}{2}$ and $a, b > 0$.

c) In this case, look for a good upper bound on $\mathbb{P}(\{Y_n > t\})$ for $n \geq 1$ fixed and $t > \mu$.

Problem 2: Large deviations principle

Let $(X_n,\, n \geq 1)$ be a sequence of i.i.d. $\mathcal{E}(\lambda)$ random variables defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$, i.e., X_1 admits the following pdf:

$$p_{X_1}(x) = \begin{cases} \lambda \exp(-\lambda x) & x \geq 0 \\ 0 & x < 0 \end{cases}$$

Let also $S_n = X_1 + \ldots + X_n$. Using the large deviations principle, find a tight upper bound on

$$\mathbb{P}\left(\{S_n \geq nt\}\right) \quad \text{for } t > \mathbb{E}(X_1) = \frac{1}{\lambda}$$

Problem 3: Moment generating function

Recall that the moment-generating function of a random variable X is defined for every $t \in \mathbb{R}$ as

$$M_X(t) = \mathbb{E}\left(e^{tX}\right).$$

a) Show that if $X \sim \mathcal{N}(0, \sigma^2)$, then

$$M_X(t) = \exp\left(\frac{1}{2} t^2 \sigma^2\right).$$

We now introduce the concept of *sub-gaussianity*. A random variable X is called sub-gaussian if, for every $t > 0$,

$$M_X(t) \leq \exp\left(\frac{1}{2}t^2\eta^2\right)$$

for some $\eta \in \mathbb{R}^+$. (Note that η^2 need not be the variance of X!).

b) Show that if $X \sim \mathcal{U}([-a,a])$ for some $a > 0$, then X is sub-gaussian with $\eta = a$.

Hint: Recall that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.

c) Show that if X is sub-gaussian for some $\eta \in \mathbb{R}^+$, then for every $t > 0$,

$$\mathbb{P}(|X| \geq t) \leq 2\exp\left(-\frac{t^2}{2\eta^2}\right).$$

d) Prove the following generalization of Hoeffding's inequality. Let $X_i, i \in \{1, 2, \ldots, n\}$ be independent random variables, where for each i, $X_i - \mathbb{E}(X_i)$ is sub-gaussian for some $\eta_i \in \mathbb{R}^+$. Let also $S_n = \sum_{i=1}^n X_i$. Show that for every $t > 0$,

$$\mathbb{P}(|S_n - \mathbb{E}(S_n)| \geq t) \leq 2\exp\left(-\frac{t^2}{2\sum_{i=1}^n \eta_i^2}\right).$$

e) Let $X_i, i \in \{1, 2, \ldots, n\}$ be sub-gaussian random variables with the same $\eta \in \mathbb{R}^+$. Show that

$$\mathbb{E}\left(\max_i X_i\right) \leq \eta\sqrt{2\ln n}.$$

Hint: Start by rewriting $\mathbb{E}(\max_i X_i) = \frac{1}{t}\mathbb{E}(\ln\exp(t\max_i X_i))$.

Problem 4: Inequalities

Part I. Let $(X_n,\, n \geq 1)$ be a sequence of i.i.d. random variables such that $\mathbb{P}(\{X_n = +1\}) = p$ and $\mathbb{P}(\{X_n = -1\}) = 1 - p$ for some fixed $0 < p < 1/2$.

Let $S_0 = 0$ and $S_n = X_1 + \ldots + X_n$, $n \geq 1$. Let also $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$, $n \geq 1$.

Preliminary question. Deduce from Hoeffding's inequality that for any $0 < p < 1/2$,

$$\mathbb{P}(\{|S_n - n\,(2p-1)| \geq nt\}) \leq 2\exp\left(-\frac{nt^2}{2}\right) \quad \forall t > 0,\ n \geq 1.$$

This inequality will be useful at some point in this exercise.

Let now $(Y_n,\, n \in \mathbb{N})$ be the process defined as $Y_n = \lambda^{S_n}$ for some $\lambda > 0$ and $n \in \mathbb{N}$.

a) *Using Jensen's inequality only*, for what values of λ can you conclude that the process Y is a submartingale with respect to $(\mathcal{F}_n,\, n \in \mathbb{N})$?

b) Identify now the values of $\lambda > 0$ for which it holds that the process $(Y_n = \lambda^{S_n},\, n \in \mathbb{N})$ is a martingale / submartingale / supermartingale with respect to $(\mathcal{F}_n,\, n \in \mathbb{N})$.

c) Compute $\mathbb{E}(|Y_n|)$ and $\mathbb{E}(Y_n^2)$ for every $n \in \mathbb{N}$ (and every $\lambda > 0$).

d) For what values of $\lambda > 0$ does it hold that $\sup_{n\in\mathbb{N}} \mathbb{E}(|Y_n|) < +\infty$? $\sup_{n\in\mathbb{N}} \mathbb{E}(Y_n^2) < +\infty$?

e) Run the process Y numerically. For what values of $\lambda > 0$ do you observe that there exists a random variable Y_∞ such that $Y_n \underset{n\to\infty}{\to} Y_\infty$ a.s.?

Prove it then theoretically and compute the random variable Y_∞ when it exists (this computation might depend on λ, of course).

f) For what values of $\lambda > 0$ does it hold that $Y_n \overset{L^2}{\underset{n\to\infty}{\to}} Y_\infty$?

g) Finally, for what values of $\lambda > 0$ does it hold that $\mathbb{E}(Y_\infty|\mathcal{F}_n) = Y_n$, $\forall n \in \mathbb{N}$?

Part II. Consider now the (interesting) value λ for which the process Y is a martingale.
(Spoiler: there is a unique such value of λ, and it is greater than 1.)

Let $a \geq 1$ be an integer and consider the stopping time $T_a = \inf\{n \in \mathbb{N} : Y_n \geq \lambda^a \quad \text{or} \quad Y_n \leq \lambda^{-a}\}$.

a) Estimate numerically $\mathbb{P}(\{Y_{T_a} = \lambda^a\})$ for some values of a. Explain your method.

b) Is it true that $\mathbb{E}(Y_{T_a}) = \mathbb{E}(Y_0)$? Justify your answer.

c) If possible, use the previous statement to compute $P = \mathbb{P}(\{Y_{T_a} = \lambda^a\})$ theoretically. How fast does this probability decay with a?

Consider finally the other stopping time $T'_a = \inf\{n \in \mathbb{N} : Y_n \geq \lambda^a\}$.

d) Estimate numerically $\mathbb{P}(\{Y_{T'_a} = \lambda^a\})$ for some values of a. Explain your method.

e) Is it true that $\mathbb{E}(Y_{T'_a}) = \mathbb{E}(Y_0)$? Justify your answer.

f) If possible, use the above statement to compute $P' = \mathbb{P}(\{Y_{T'_a} = \lambda^a\})$ theoretically. Is this probability P' greater or smaller than P?